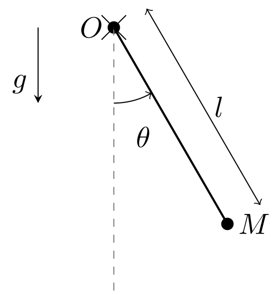
Practical 1 - Implicit and explicit schemes

Florence Bertails-Descoubes [*], Mélina Skouras [†], Mickaël Ly [‡]

Exercise 1 - Simple pendulum - explicit Euler scheme

The goal of this exercise is to observe the behavior of a mechanical system integrated using the simple explicit Euler scheme.

We start by considering the following simple pendulum: a mass M tied with an inextensible/incompressible massless string of length l to a point O, and submitted to gravity.



(Q1) During morning class, you have already derived its equation of motion. Discretize in time these equations using the explicit Euler scheme.

(Q2) Implement the resulting model.

Remark 1: Have a look at the file `scenes.py` and the class `DummyDynamicSystem` to see how the proposed framework is working.

(Q3) Compare the behavior of the system for different timesteps with the following parameters: $g = 9.81$, $l_0 = 1$, $\theta_0(t = 0) = \frac{\pi}{5}$ and $m_1 = 1$.

Remark 2: You can use `matplotlib` to plot $\theta(t)$.

(Q4) For small angle deviation, verify the condition on the timestep for guaranteeing stability.

[*] florence.bertails@inria.fr
[†] melina.skouras@inria.fr
[‡] mickael.ly@inria.fr

Remark 3: Short reminder for the three Euler schemes :

Explicit Euler	Implicit Euler	Semi-implicit Euler
$\dot{q}^{n+1} = \dot{q}_n + f(q^{\mathbf{n}}, \dot{q}^{\mathbf{n}})$	$\dot{q}^{n+1} = \dot{q}_n + f(q^{\mathbf{n+1}}, \dot{q}^{\mathbf{n+1}})$	$\dot{q}^{n+1} = \dot{q}_n + f(q^{\mathbf{n}}, \dot{q}^{\mathbf{n+1}})$
$q^{n+1} = q_n + \delta t \dot{q}^{\mathbf{n}}$	$q^{n+1} = q_n + \delta t \dot{q}^{\mathbf{n+1}}$	$q^{n+1} = q_n + \delta t \dot{q}^{\mathbf{n+1}}$

To retrieve the notations of the morning class, you can note $x \equiv \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ and $\delta t \equiv h$.

Exercise 2 - Simple pendulum - (Semi-)implicit Euler scheme

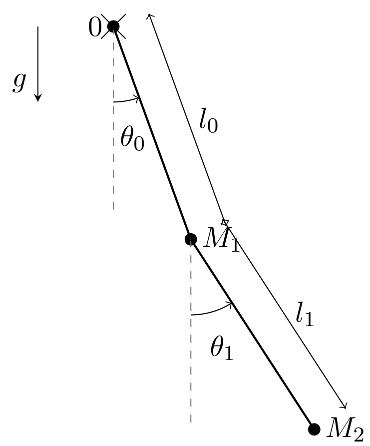
(Q1) Do questions 3 to 5 of the previous exercise with the implicit Euler scheme and the semi-implicit Euler scheme (also called symplectic Euler), **both linearized** (see the remark below).

Remark 4: Linearizing a nonlinear equation. Instead of solving the true (nonlinear) equation $x^{n+1} = f(x^{n+1})$, we consider the linear equation where f is developped at the first order around x^n: $x^{n+1} = f(x^n) + D_x f(x^n) \underbrace{(x^{n+1} - x^n)}_{\delta x}$. This equation is actually a *linear* equation in δx.

(Q2) What can you say about the stability of these schemes ? (Plot $\theta(t)$).

(Q3) Compare the behavior of the three numerical schemes.

Exercise 3 - 2-arm pendulum



(Q1) We now consider a 2-arm pendulum, with still massless inextensible/incompressible strings between the masses. Extend your implementation to this system.

Remark 5: NB : Studying here the implicit scheme is optional.

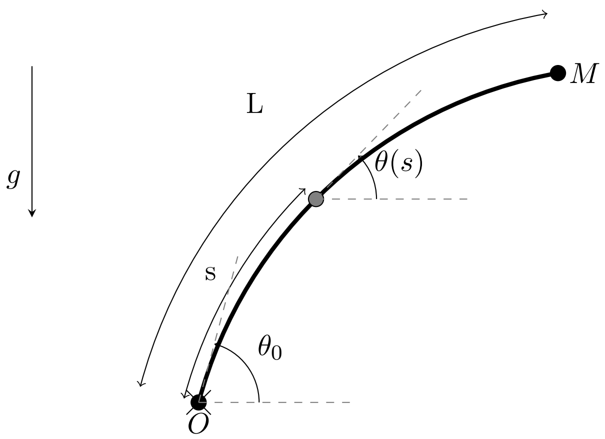
Remark 6: You will have a 2×2 matrix, so it is not a big deal to invert it. You can do it by hand, or by taking a look at the solvers of `Numpy` or `Scipy`.

(Q2) *(Optional)*Assuming that now bending forces apply at joint M_1, implement the model with such a bending force (you can take it linear with respect to the deviation angle).

Exercise 4 - *(Optional)*To go further - Euler elastica

This exercise is a preview of the third class on slender structures and Euler elastica. This also well illustrates the use of yet another type of degree of freedom (curvature instead of angle).

We consider a 2D inextensible/incompressible rod of length L, clamped at the origin O $\Big(\theta(s=0)=\theta_0\Big)$, and a mass M located on its other end. We assume the mass of the rod to be negligible compared to M (massless rod approximation).



The elastic energy of this system is $E_{el} = \displaystyle\int_{s=0}^{L} \left(\frac{d\theta}{ds}(s)\right)^2 ds$. To simplify the problem, we consider the rod to have a *uniform* curvature $\kappa(s) = \dfrac{d\theta}{ds}(s) = C^{te}$, i.e., κ is independent of the space variable s (note however that κ varies with time). This choice leads to a Lagrangian system with a single degree of freedom, and in the following we call this model the *super-circle*, due to its circular shape.

(Q1) For $s \in [0, L]$, express the tangent vector $t(s)$ at the point of curvilinear abscissa s in function of κ.

Remark 7: For later : write $t(s)$ as the product of a constant rotation matrix by a vector.

(Q2) Write the total potential energy of the system.

(Q3) Show that when $\kappa \to 0$, we retrieve the energy of the simple pendulum model.

(Q4) Derive the kinematic and dynamic equations of the super-circle model.

(Q5) Write a time-integration scheme for the model, and implement it.