Class 1: Lagrange mechanics
and first steps into numerical integration

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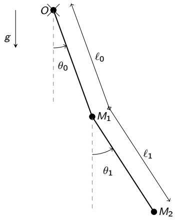
Example: the double pendulum



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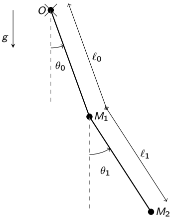

Questions

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- How to solve these equations in a “safe” way ?

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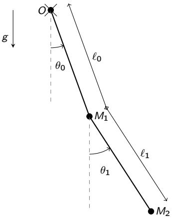

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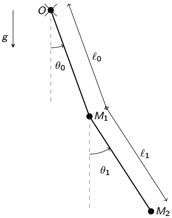

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- II. Finite differences: how to solve the equations of motion

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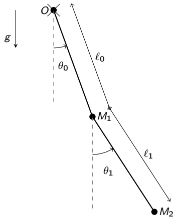

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NB: Once these concepts are known, more complex systems can be considered

Keywords and bibliography

Part I: Lagrange mechanics

- **Analytical Mechanics**

  Landau and Lifshitz, Mechanics Vol 1; J. Fereira, Mécanique analytique

- **Calculus of Variations**

  J.-P. Bourguignon, Calcul variationnel ("Variational calculus")

Part II: Finite differences

- **Numerical Analysis**

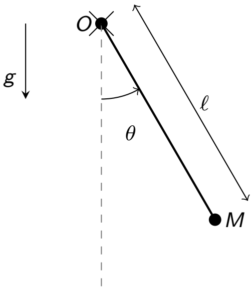
  G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization")

- **Energy conservation properties**

  E. Hairer et al., Geometric numerical integration

Part I: Lagrange mechanics

The simple pendulum



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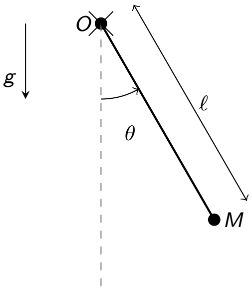

Exercise

- Write the equations of motion of the simple pendulum...

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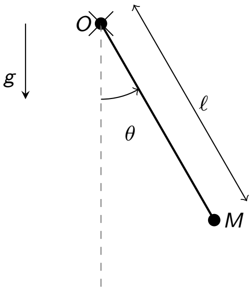

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- ...using Newton's second law

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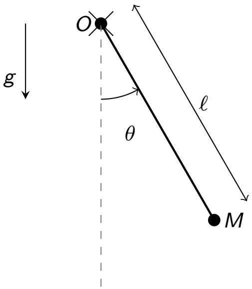

Exercise

- Write the equations of motion of the simple pendulum...
- ...using Newton's second law
- ...using the principle of virtual work

Generalized coordinates

Definition

Generalized coordinates, denoted q_i, are n independent variables (functions of time) which allow to characterize the configuration of a system possessing n degrees of freedom. The generalized velocities of the system are defined by $\frac{\mathrm{d}}{\mathrm{d}t} q_i = \dot{q}_i$.

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Remark: A generalized velocity $\dot{q}_i$ does not necessarily correspond to the velocity $\mathbf{v}_M$ of a given material point M. Reconstructing the configuration and the material velocities of the system corresponds to writing the kinematics of the system.

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Example: For the simple pendulum, the position of the mass M can be computed as $\mathbf{OM} = \ell\, \mathbf{e}_r$ and its velocity $\mathbf{v}_M$ as $\mathbf{v}_M = \ell \dot{\theta}\, \mathbf{e}_\theta$ (in the basis $(\mathbf{e}_r, \mathbf{e}_\theta)$ defined by θ).

Equations of motion

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- Principle of virtual work (just derived previously)
- Principle of least action: more general settings

General method: Least action principle

Let $q = \{q_0, \ldots, q_i, \ldots, q_{n-1}\}$ and $\dot{q} = \frac{\mathrm{d}}{\mathrm{d}t} q$.
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Principle of least action (Hamilton principle)

The actual trajectory $q(t)$ followed by the system between two instants a and $b > a$ should be such that the action of the system,

$$\mathcal{S}(q, \dot{q}) = \int_a^b \underbrace{\mathcal{L}(q(t), \dot{q}(t), t)}_{T-U} \mathrm{d}t,$$

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$$\mathcal{L} : \left| \begin{array}{ccc} \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ (x, u, t) & \longmapsto & \mathcal{L}(x, u, t), \end{array} \right.$$

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Remark: $\mathcal{S}$ is a *functional*, as it takes as arguments the two functions q and $\dot{q}$.

Euler-Lagrange equations

We assume $\mathcal{L}$ is differentiable, and that its partial derivatives $\mathcal{L}_{x_i} = \frac{\partial \mathcal{L}}{\partial x_i}$ and $\mathcal{L}_{u_i} = \frac{\partial \mathcal{L}}{\partial u_i}$ are continuous.

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A necessary condition for the action $\mathcal{S}$ to be minimal is that q satisfies for all $t \in [a, b]$ the so-called Euler-Lagrange equations,

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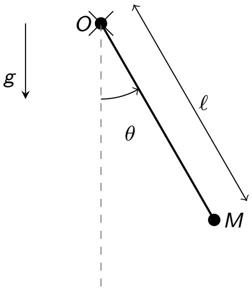
Remark: This is a variational principle (minimum condition on a functional). It can be applied beyond dynamics, for instance to find object shapes with minimal weight, or to compute shapes at static equilibrium (see Class 3).

Back to the simple pendulum



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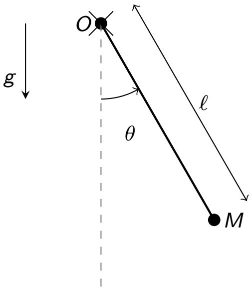

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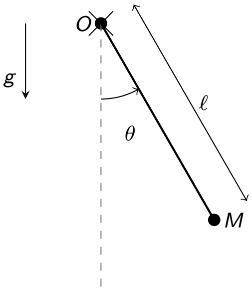

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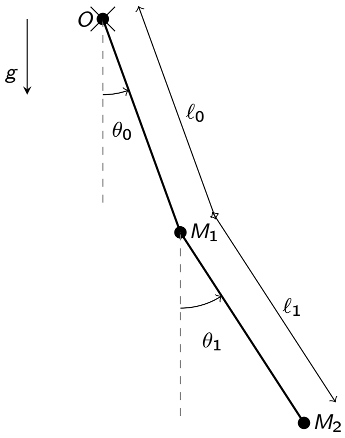

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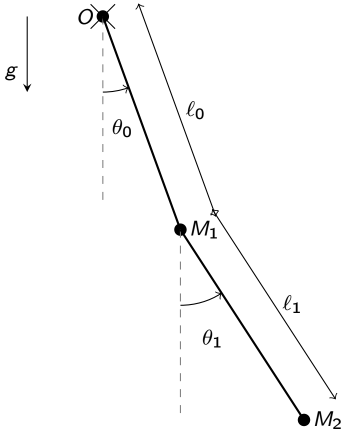

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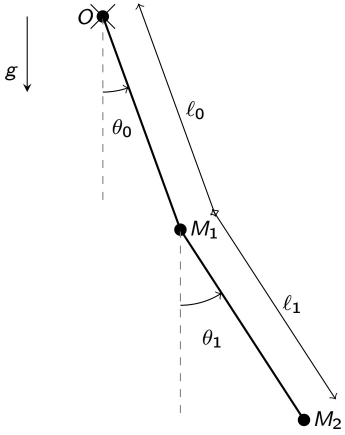

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Part II: Finite differences

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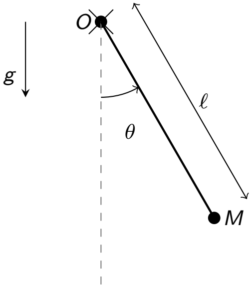

Computing the dynamics

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0$$

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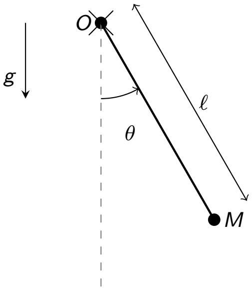

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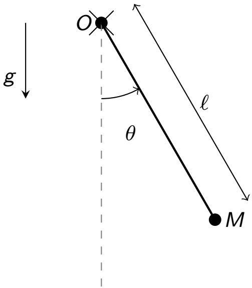

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- $\rightarrow$ Recourse to numerical integration

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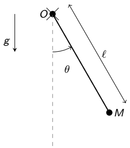
Exercise

Show that the simple pendulum equation of motion enters the formalism above.



Equation of motion (reminder)

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0$$

Existence and uniqueness of a solution

Definition

A function $g : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz continuous if there exists a real constant $K \geq 0$ such that

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- A Lipshitz continuous function is necessarily continuous (opposite is not true).

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A function $g : \mathbb{R}^p \to \mathbb{R}^p$ is Lipschitz continuous if there exists a real constant $K \geq 0$ such that

$$\forall (x, y) \in \mathbb{R}^p, \|g(y) - g(x)\| \leq K \|y - x\|.$$

Remark:

- K is called the Lipschitz constant. If $K < 1$, the function is said to be a contraction.
- A Lipshitz continuous function is necessarily continuous (opposite is not true).
- A differentiable function g is Lipshitz continuous on E if and only if its first derivative is bounded (and $\sup_E \|g'(x)\|$ is the smallest Lipshitz constant).

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If f is locally Lipschitz continuous with respect to its first variable $x(t)$, then there exists a unique solution $\bar{x}$ to the Cauchy problem, and $\bar{x}$ is C^1 continuous.

Numerical scheme

Recall that the Cauchy problem $\begin{cases} \dot{x} = f(x(t), t) & t \in [a, b] \\ x(0) = x_0 \end{cases}$

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$\rightarrow$ We subdivide the interval $[a, b]$ in N segments of length h ($h =$ the timestep), and we aim at constructing a sequence of points $x_0, x_1, \cdots, x_k, \cdots, x_N$ at discrete times $t_0 + h, t_0 + 2h, \cdots, t_0 + k\,h, \cdots, t_0 + n\,h$, which approximate well the exact solution.

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→ The greater m, the faster the method converges to the exact solution.

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Definition

A single-step numerical scheme is defined as the recurrence

$$x_0 \text{ given} \qquad x_{k+1} = x_k + h\Phi(x_k, h, t_k)$$

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Remark: Convergence is generally hard to prove directly. Instead, one usually prefers to prove consistency + stability w.r.t. errors instead.

Consistency of a numerical scheme

Definition

We define $R_k = \frac{\bar{x}_{k+1} - \bar{x}_k}{h} - \Phi(\bar{x}_k, t_k, h)$ as the consistency error of the scheme at t_k.

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Exercise

Show that `Explicit Euler` is a consistent scheme of first order.

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$$\tilde{x}_0 = x_0 + \varepsilon_0 \qquad \tilde{x}_{k+1} = \tilde{x}_k + h\left(\Phi(\tilde{x}_k, t_k, h) + \varepsilon_{k+1}\right).$$

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- If f is regular enough (Lipshitz continuous w.r.t. x), `Explicit Euler` is stable w.r.t. errors.

Convergence

Theorem

Convergence = consistency + stability

Stability of the numerical solution

Exercise

Derive the `Explicit Euler` scheme for the simple scalar Cauchy problem

$$x(0) = x_0 \quad \dot{x}(t) = -\lambda\, x(t) \quad \forall t \in [a, b] \quad \text{with } \lambda > 0.$$

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$\rightarrow$ To keep in mind: stability w.r.t. errors is useful for proving convergence. Stability as defined above is useful when considering integration of systems on moderate or large time intervals, or when using a large timestep (which is often useful in practice!).

Explicit vs. Implicit Euler

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Exercise 1

Consider the linearized pendulum problem (valid for small angle θ),

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0,$$

and express the condition on the time step h for `Explicit Euler` to be stable.

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Stability of `Explicit Euler`

`Explicit Euler` is conditionally stable.

Stability of `Implicit Euler`

`Implicit Euler` is inconditionally stable.

Exercise 1

Consider the linearized pendulum problem (valid for small angle θ),

$$\ddot{\theta} + \frac{g}{\ell}\theta = 0 \qquad \text{with } \theta(0) = \theta_0 \text{ and } \dot{\theta}(0) = \lambda_0,$$

and express the condition on the time step h for `Explicit Euler` to be stable.

Exercise 2

For the linearized pendulum problem, verify that `Implicit Euler` is inconditionally stable.

Conclusion

In a nutshell

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- References: G. Allaire, Analyse numérique et optimisation ("Numerical analysis and optimization"), E. Hairer et al., Geometric numerical integration.