Lecture 3
Good Triangulations

Jean-Daniel Boissonnat

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Outline

1. Nets
2. Optimality of Delaunay triangulations
3. Delaunay refinement and angle bounds
4. Thickness in higher dimensions
5. The Local Lovasz Lemma and Thick Triangulations

Definition and existence of nets

Definition

Let Ω be a bounded subset of $\mathbb{R}^d$ and $\bar{\eta}$ a constant. A finite set of points P is called an $(\varepsilon, \bar{\eta})$-net of Ω iff

Density : $\forall x \in \Omega, \exists p \in P : \|x - p\| \leq \varepsilon$

Separation : $\forall p, q \in P : \|p - q\| \geq \bar{\eta}\, \varepsilon$

Lemma Ω admits an $(\varepsilon, 1)$-net.

Proof. While there exists a point $p \in \Omega$, $d(p, P) \geq \varepsilon$, insert p in P

Size of a net

Lemma The number of points of an $(\varepsilon, \bar{\eta})$-net is at most

$$n(\varepsilon, \bar{\eta}) \leq \frac{\text{vol}_d(\Omega^{+\frac{\eta}{2}})}{\text{vol}_d(B(\frac{\eta}{2}))} = O\left(\frac{1}{\varepsilon^d}\right)$$

where the constant in the O depends on the geometry of Ω and on $\bar{\eta}^d$.

Proof. Consider the balls $B(p, \frac{\eta}{2})$ of radius $\frac{\eta}{2}$ that are centered at the points $p \in$ P. These balls are disjoint by definition of an $(\varepsilon, \bar{\eta})$-sample and they are all contained in $\Omega^{+\frac{\eta}{2}}$

Delaunay complex of a net

Lemma Let Ω be a bounded subset of $\mathbb{R}^d$, P an $(\varepsilon, \bar{\eta})$-net of Ω, and assume that d and $\bar{\eta}$ are positive constants. The restriction of the Delaunay triangulation of P to Ω has linear size $O(n)$ where $n = |\mathsf{P}| = O(\frac{1}{\varepsilon^d})$

Proof

1. The number of neighbours of p is $n_p = O(1)$ using a volume argument

2. The number of simplices incident on a vertex is at most the number of faces of the convex hull of n_p points of $\mathbb{R}^d$

$$n_p^{\lfloor \frac{d}{2} \rfloor} = O(1)$$

3. For the construction, use a grid G_ε of resolution ε and compute, for each $p \in P$, the subset $N(p) \subset P$ of points that lie at distance at most 2ε from the cell that contains p. We have

$$|N(p)| = O(1) \quad \text{and} \quad \mathrm{star}(p, \mathrm{Del}_{|\Omega}(\mathsf{P})) = \mathrm{star}(p, \mathrm{Del}_{|\Omega}(N(p)))$$

Optimizing the angular vector ($d = 2$)

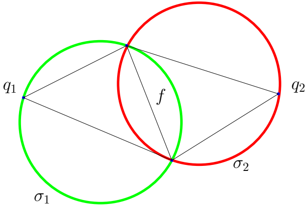
Angular vector of a triangulation $T(\mathcal{P})$

$$\text{ang}\,(T(\mathcal{P})) = (\alpha_1, \ldots, \alpha_{3t}), \qquad \alpha_1 \leq \ldots \leq \alpha_{3t}$$

Optimality Any triangulation of a given point set $\mathcal{P}$ whose angular vector is maximal (for the lexicographic order) is a Delaunay triangulation of $\mathcal{P}$

Good for matrix conditioning in FE methods

Local characterization of Delaunay complexes



Pair of regular simplices

$$\sigma_2(q_1) \geq 0 \quad \text{and} \quad \sigma_1(q_2) \geq 0$$

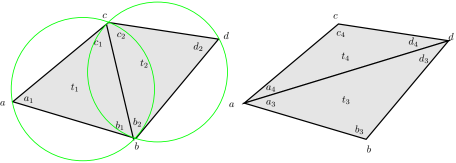
$\Leftrightarrow \hat{c}_1 \in h^+_{\sigma_2}$ and $\hat{c}_2 \in h^+_{\sigma_1}$

Theorem A triangulation $T(P)$ such that all pairs of simplexes are regular is a Delaunay triangulation $\mathrm{Del}_((P))$

Proof The PL function whose graph G is obtained by lifting the triangles is locally convex and has a convex support

$$\Rightarrow \quad G = \mathrm{conv}^-(\hat{Q}) \quad \Rightarrow \quad T(Q) = \mathrm{Del}(Q)$$

Lawson’s proof using flips



While $\exists$ a non regular pair (t_3, t_4)

/* $t_3 \cup t_4$ is convex */

replace (t_3, t_4) by (t_1, t_2)

Regularize $\Leftrightarrow$ improve $\text{ang}\,(T(\mathcal{P}))$

$\text{ang}\,(t_1, t_2) \geq \text{ang}\,(t_3, t_4)$

$$a_1 = a_3 + a_4,\ d_2 = d_3 + d_4,$$
$$c_1 \geq d_3,\ \ b_1 \geq d_4,\ \ b_2 \geq a_4,\ \ c_2 \geq a_3$$

- The algorithm terminates since the number of triangulations of $\mathcal{P}$ is finite and $\text{ang}(T(\mathcal{P}))$ cannot decrease
- The obtained triangulation is a Delaunay triangulation of $\mathcal{P}$ since all its edges are regular

Some optimality properties of Delaunay triangulations

Among all possible triangulations of $\mathcal{P}$, $\mathrm{Del}_((\mathcal{P}))$

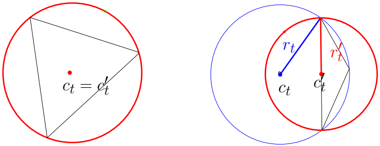
1. (2d) maximizes the smallest angle [Lawson]

2. (2d) Linear interpolation of $\{(p_i, f(p_i))\}$ that minimizes [Rippa]

$$R(T) = \sum_i \int_{T_i} \left(\left(\frac{\partial \phi_i}{\partial x}\right)^2 + \left(\frac{\partial \phi_i}{\partial y}\right)^2 \right) dx\, dy \qquad \text{(Dirichlet energy)}$$

$\phi_i =$ linear interpolation of the $f(p_j)$ over triangle $T_i \in T$

3. (any d) minimizes the radius of the maximal smallest ball enclosing a simplex) [Rajan]



Triangle mesh generation by Delaunay refinement

[Chew 1993, Ruppert 1995, Shewchuk 2002]

Domain : $\Omega = \mathbb{R}^2/\mathbb{Z}^2$ (periodic plane)

Sizing field : $\phi : \Omega \to \mathbb{R}$ = α-Lipschitz function, $\alpha < 1$

$$\forall x \in \Omega,\ 0 < \phi_0 \leq \phi(x)$$

Bad triangle $t = pqr$: $\mathrm{Vor}(p,q,r) = c_t \in \Omega$ and $\|c_t - p\| > \phi(c_t)$

Algorithm

INIT compute an initial (small) sample $\mathcal{P}_0 \subset \Omega$

REPEAT WHILE ($\exists$ a bad triangle t)

insert_in_Del(c_t)

UNTIL no bad element remains

Non uniform nets

A finite set of points $P \subset \Omega$ is a net of Ω wrt ϕ if there exists two constants c and c' such that

Density: $\forall x \in \Omega, \exists p \in P : \quad \|x - p\| \leq c\, \phi(x)$

Separation: $\forall p, q \in P : \quad \|p - q\| \geq c' \, \max(\phi(p), \phi(q))$

The algorithm outputs a net of Ω wrt ϕ

Separation : $\forall p, q \in P : \|p - q\| \geq \frac{1}{1+\alpha}\, \phi_0$

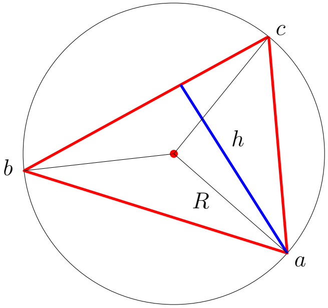
$$\begin{aligned} \forall p \in \mathcal{P}, d(p, \mathcal{P} \setminus \{p\}) &= \|p - q\| \\ &\geq \min(\phi(p), \phi(q)) \\ &\geq \phi(p) - \alpha\, \|p - q\| \end{aligned}$$

(ϕ is α-Lipschitz)

$\Rightarrow \quad \|p - q\| \geq \frac{1}{1+\alpha}\, \phi(p) \geq \frac{1}{1+\alpha}\, \phi_0 > 0$ $\quad \Rightarrow$ the algorithm terminates

Density : $\forall x \in \Omega : d(x, P) \leq \frac{\phi(x)}{1-\alpha}$

Angle bound and density



$\|a-c\| \geq \phi(a)$ (a inserted after c)

$R \leq \phi(c_{abc})$ (c_{abc} not inserted)

$$\sin b = \frac{\|a-c\|}{2R} \geq \frac{\phi(a)}{2\phi(c_{abc})}$$

$$\forall x \in [abc] : \phi(c_{abc}) \leq \phi(x) + \alpha\, \|c_{abc} - x\| \leq \phi(x) + \alpha\, \phi(c_{abc})$$

$\Rightarrow \quad \phi(c_{abc}) \leq \frac{\phi(x)}{1-\alpha} \qquad \Rightarrow \quad \sin b \geq \frac{1-\alpha}{2}$

$\Rightarrow$ for some $p \in \{a, b, c\} : \|x-p\| \leq R \leq \phi(c_t) \leq \frac{\phi(x)}{1-\alpha}$

Size of the sample $= \Theta\left(\int_\Omega \frac{dx}{\phi^2(x)}\right)$

Upper bound

$B_p = B(p, \frac{\phi(p)}{2(1+\alpha)}),\ p \in \mathcal{P}$

$$\int_\Omega \frac{dx}{\phi^2(x)} \geq \sum_p \int_{B_p \cap \Omega} \frac{dx}{\phi^2(x)} \qquad \text{(the } B_p \text{ are disjoint)}$$

$$\geq \left(\frac{2+2\alpha}{2+3\alpha}\right)^2 \sum_p \frac{\mathrm{area}(B_p \cap \Omega)}{\phi^2(p)} \qquad (\phi(x) \leq \phi(p) + \alpha\,\|p-x\|$$

$$\leq \phi(p) + \frac{\alpha\phi(p)}{2(1+\alpha)} = \frac{(2+3\alpha)\,\phi(}{2+2\alpha}$$

$$\geq \left(\frac{2+2\alpha}{2+3\alpha}\right)^2 \frac{\pi}{4(1+\alpha)^2}\,|\mathcal{P}|$$

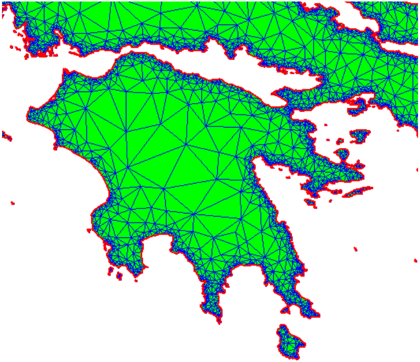
$$= \frac{\pi}{4(2+3\alpha)^2}\,|\mathcal{P}|$$

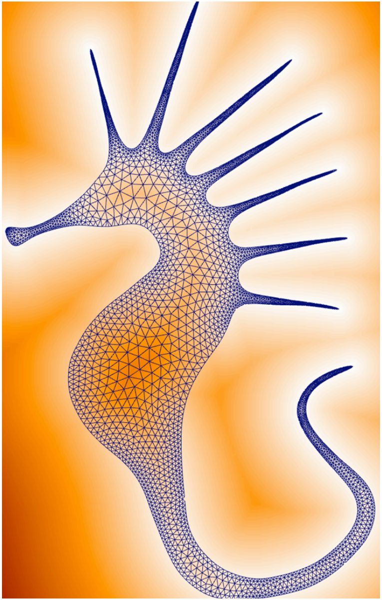
Lower bound

- Observe that the balls $B'_p(p, \frac{\phi(p)}{1-\alpha})$ cover Ω
- Use a covering instead of a packing

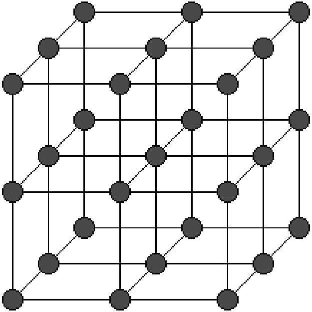
Results

$$\phi(x) = \phi_0 + \alpha d(x, \partial\Omega)$$





3D Delaunay Triangulations are not thick even if the vertices are well-spaced



- Each square face can be circumscribed by an empty sphere
- This remains true if the grid points are slightly perturbed therefore creating thin simplices

The long quest for thick triangulations

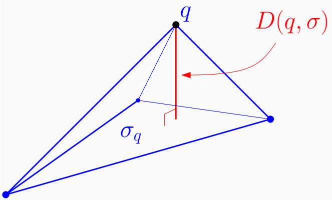
Differential Topology [Cairns], [Whitehead], [Whitney], [Munkres]

Differential Geometry [Cheeger et al.]

Geometric Function Theory [Peltonen], [Saucan]

Simplex quality

Altitudes



If σ_q, the face opposite q in σ is protected, The *altitude* of q in σ is

$$D(q,\sigma) = d(q, \mathrm{aff}(\sigma_q)),$$

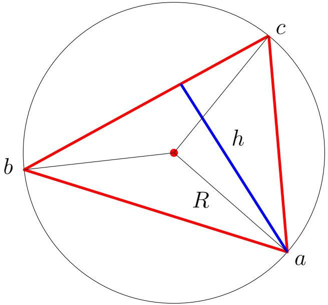
where σ_q is the face opposite q.

Definition (Thickness [Cairns, Whitney, Whitehead et al.])

The *thickness* of a j-simplex σ with diameter $\Delta(\sigma)$ is

$$\Theta(\sigma) = \begin{cases} 1 & \text{if } j = 0 \\ \min_{p\in\sigma} \frac{D(p,\sigma)}{j\Delta(\sigma)} & \text{otherwise.} \end{cases}$$

Thickness and angle bounds for Delaunay **triangles**



$$h = |ab| \times sinb = \frac{|ab|\times|ac|}{2R}$$

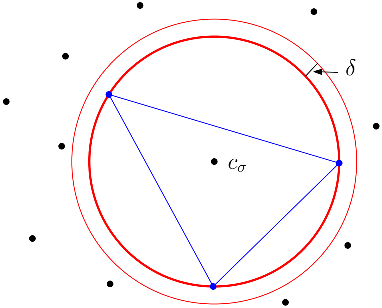
$$\Theta(abc) = \frac{h}{2\Delta} \geq \frac{h}{4R}$$

If P is an $(\varepsilon, \bar{\eta})$-net and σ a Delaunay triangle, then :

$$|ab|,\ |ac| \geq \frac{\bar{\eta}\varepsilon}{2} \quad \text{and} \quad R \leq \varepsilon$$

$$\Rightarrow \quad \Theta(abc) \geq \frac{\bar{\eta}^2}{32} \quad \text{and} \quad \sin b \geq \frac{\bar{\eta}}{2}$$

Protection



δ-protection We say that a Delaunay simplex $\sigma \subset L$ is δ-protected if

$$\|c_\sigma - q\| > \|c_\sigma - p\| + \delta \qquad \forall p \in \sigma \ \text{ and } \ \forall q \in L \setminus \sigma.$$

Protection implies separation and thickness

Let P be a $(\varepsilon, \bar{\eta})$-net, i.e.

- $\forall x \in \Omega, \quad d(x, P) \leq \varepsilon$
- $\forall p, q \in P, \quad \|p - q\| \geq \bar{\eta}\varepsilon$

if all d-simplices of $\mathrm{Del}(P)$ are $\bar{\delta}\varepsilon$-protected, then

- **Separation** of P : $\bar{\eta} \geq \bar{\delta}$
- **Thickness :** $\forall \sigma \in \mathrm{Del}(P), \quad \Theta(\sigma) \geq \frac{\bar{\delta}^2}{8d}$

The Lovász Local Lemma

Motivation

Given: A set of (bad) events $A_1, ..., A_N$,
each happens with $\text{proba}(A_i) \leq p < 1$

Question : what is the probability that none of the events occur?

The case of independent events

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) \geq (1-p)^N > 0$$

What if we allow a limited amount of dependency among the events?

LLL : symmetric version [Lovász & Erdös 1975]

If, for $i = 1, ..., N$,

1. A_i is independent of all other events except $\leq \Gamma$ ones
2. $\text{proba}(A_i) \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then

$$\text{proba}(\neg A_1 \wedge ... \wedge \neg A_N) > 0$$

Moser and Tardos' constructive proof of the LLL [2010]

$\mathcal{P}$ a finite set of mutually independent random variables

$\mathcal{A}$ a finite set of events that are determined by the values of some variables of $\mathcal{P}$

Two events are independent iff they share no variable

Algorithm

for all $p \in \mathcal{P}$ **do**
  $v_p \leftarrow$ a random evaluation of p;

while some events in $\mathcal{A}$ occur when $(p = v_p, p \in \mathcal{P})$ **do**

  pick an arbitrary such event $A \in \mathcal{A}$;

  for all $p \in \text{variables}(A)$ **do**
    $v_p \leftarrow$ a new random evaluation of p;

return $(v_p)_{p \in \mathcal{P}}$;

Moser and Tardos' theorem

if for all $i \in [1, N]$

1. A_i is independent of all other events except $\leq \Gamma$ other ones
2. $\text{proba}(A_i) \leq \frac{1}{e\,(\Gamma+1)}$ $\qquad e = 2.718...$

then $\exists$ an assignment of values to the variables $\mathcal{P}$ such that there is no event in $\mathcal{A}$ occurs

The randomized algorithm resamples an event $A \in \mathcal{A}$ at most $\boxed{\frac{1}{\Gamma}}$ expected times before it finds such an evaluation

The expected total number of resampling steps is at most $\boxed{\frac{N}{\Gamma}}$

Home work

- Read the proof of Moser & Tardos (or Spencer's nice note)
- Listen to a talk by Aravind Srinivasan on further extensions

  `https://video.ias.edu/csdm/2014/0407-AravindSrinivasan`

Protecting Delaunay simplices via perturbation

P is an $(\varepsilon, \bar{\eta})$-net

Picking regions : pick p' in $B(p, \rho)$ Hyp. $\rho \leq \frac{\eta}{4} \;\; (\leq \frac{\varepsilon}{2})$

Sampling parameters of a perturbed point set

P′ is an $(\varepsilon', \bar{\eta}')$-net, where

$$\varepsilon' = \varepsilon(1 + \bar{\rho}) \text{ and } \bar{\eta}' = \frac{\bar{\eta} - 2\bar{\rho}}{1 + \bar{\rho}} \geq \frac{\bar{\eta}}{3}$$

Notation : $\bar{x} = \frac{x}{\varepsilon}$

The LLL framework

Random variables : P' a set of random points $\{p',\ p' \in B(p,\rho), p \in \mathsf{P}\}$

Event: $\exists \phi' = (\sigma', p')$ (Bad configuration)

σ' is a d simplex with $R_{\sigma'} \leq \varepsilon + \rho$

$p' \in Z_\delta(\sigma') \qquad Z_\delta(\sigma') = B(c_{\sigma'}, R_{\sigma'} + \delta) \setminus B(c_{\sigma'}, R_{\sigma'})$

Algorithm

Input: P, ρ, δ

while a bad configration $\phi' = (\sigma', p')$ occurs **do**

resample the points of ϕ'

update $\mathrm{Del}(\mathsf{P}')$

Output: P' and $\mathrm{Del}(\mathsf{P}')$

Bounding Γ

Lemma : An event is independent of all but at most Γ other bad events where Γ depends on $\bar{\eta}$, $\bar{\rho}$, $\bar{\delta}$ and d

Proof :

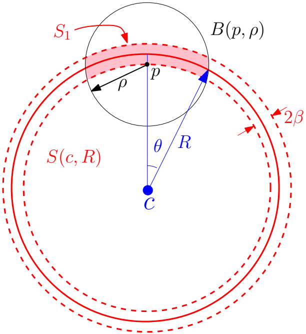
- Let $\phi' = (\sigma', p')$ be a bad configuration.

$$\forall p' \in \phi', \quad \|p' - c_{\sigma'}\| \leq R_{\sigma'} + \delta = R = \varepsilon + \rho + \delta = \varepsilon\,(1 + \bar{\rho} + \bar{\delta})$$

- the number of events that may not be independent from an event (σ', p') is at most the number of subsets of $(d+2)$ points in $B(c_{\sigma'}, 3R)$.
- Since P′ is η'-sparse,

$$\Gamma = \left(\frac{3R + \frac{\eta'}{2}}{\frac{\eta'}{2}}\right)^{d(d+2)} = \left(1 + 6\,\frac{(1+\bar{\rho}+\bar{\delta})\,(1+\bar{\rho})}{\bar{\eta} - 2\bar{\rho}}\right)^{d(d+2)}$$

Bounding proba(σ, p) be a bad configuration



$S(c, R)$ a hypersphere of $\mathbb{R}^d$

$T_\delta = B(c, R+\delta) \setminus B(c, R)$

B_ρ any d-ball of radius $\rho < R$

$$\mathrm{vol}_d(T_\delta \cap B_\rho) \leq U_{d-1}\,(R\theta)^{d-1}\,\delta,$$

$$\frac{2}{\pi}\theta \leq \sin\theta \leq \frac{\rho}{R} \qquad (\theta < \frac{\pi}{2} \Leftarrow \; \rho < R)$$

$$\Rightarrow \quad R\theta \leq \frac{\pi}{2}\rho$$

$$\mathrm{proba}(p' \in Z_\delta(\sigma')) \leq \varpi = \frac{U_{d-1}}{U_d}\,\left(\frac{\pi}{2}\right)^{d-1}\,\frac{\delta}{\rho} \leq \frac{C}{\sqrt{d}}\,\frac{\delta}{\rho}$$

Main result

Under the condition

$$\frac{eC}{\sqrt{d}}\,(\Gamma+1)\,\delta \leq \rho \leq \frac{\eta}{4}$$

the algorithm terminates.

Guarantees on the output

- $d_H(P, P') \leq \rho$
- the d-simplices of $\mathrm{Del}(P')$ are δ-protected
- and therefore have a positive thickness

Bound on the number of events

$\Sigma(p')$: number of d-simplices that can possibly make a bad configuration with $p' \in P'$ for some perturbed set P'

$R = \varepsilon + \rho + \delta$

$$\begin{aligned}\sum_{p' \in P'} \Sigma(p') &\leq n \times |P' \cap B(p', 2R)|^{d+1} \\ &\leq n \left(\frac{2\left(1 + \bar{\rho} + \bar{\delta} + \frac{\bar{\eta}'}{2}\right)}{\frac{\bar{\eta}'}{2}} \right)^{d(d+1)} \\ &= C' n\end{aligned}$$

Complexity of the algorithm

- The number of resamplings executed by the algorithm is at most

$$\frac{C'n}{\Gamma} \leq C''\,n$$

where C'' depends on $\bar{\eta}$, $\bar{\rho}$, $\bar{\delta}$ and (exponentially) d

- Each resampling consists in perturbing $O(1)$ points
- Updating the Delaunay triangulation after each resampling takes $O(1)$ time
- The expected complexity is linear in the number of points

Thickness and stability of Delaunay triangulations

- If a simplex is thick, its circumcenter is not much affected by a small perturbation of the position of the points or of the metric
- Applications to the triangulation of manifolds