Lecture 1
Discrete Geometric Structures

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Outline

1. Convex Polytopes
2. Voronoi diagrams and Delaunay triangulations
3. Weighted Voronoi diagrams
4. Union of balls
5. Discrete metric spaces
Convex Polytopes

Two ways of defining convex polytopes
Intersection of $n$ half-planes

**Input:** $n$ half-planes $H_i : a_i x + b_i y + c_i \leq 0, \ i = 1, \ldots, n$

**Output:** the convex polygon (possibly empty) $H = \bigcap_{i=1}^{n} H_i$
Convex hull of \( n \) points

**Input:** \( n \) points \( p_1, \ldots, p_n \) in \( \mathbb{R}^2 \)

**Output:** the convex polygon \( \text{conv}(P) = \sum_{i}^{n} \lambda_i p_i \)

\[
\begin{align*}
\lambda_i &\geq 0 \\
\sum_{i}^{n} \lambda_i & = 1
\end{align*}
\]

Equivalently, \( \text{conv}(P) \) is the intersection of the halfplanes that

- are bounded by lines passing through 2 points of \( P \)
- contain all the points of \( P \)
Space of lines and duality

Duality point/line
(non vertical) line \( h = \{(x, y) : y = ax - b\} \rightarrow \text{dual point } h^* = (a, b) \)
point \( p = (a, b) \rightarrow \text{dual line } p^* = \{(x, y) : y = ax - b\} \).

The mapping \( * \)
- is an involution and is thus bijective: \( h^{**} = h \) and \( p^{**} = p \)
- preserves incidences:

\[
p = (x, y) \in h \iff y = ax - b \iff b = xa - y \iff h^* \in p^*.
\]

- preserves inclusions

\[
p \in h^+ \iff h^* \in p^{**+} \quad (h^+ = \{(x, y) : y > ax - b\})
\]
Intersections of halfplanes and convex hulls

Let $h_1, \ldots, h_n$ be $n$ lines and let $H = \cap h_i^+$

A vertex $s$ of $H$ is the intersection point of two lines $h_i$ and $h_j$
lying above all other lines $h_k$, $k \neq i, j$

$$s^* = (h_i^* h_j^*)$$
$s^*$ is a line supporting $\text{conv}^- (\{h_i^*\})$

From a computational point of view, the two problems are equivalent
Facial structure of a polytope

Convex polytopes and duality can be defined in very much the same in any fixed dimension.

Supporting hyperplane $h$:

$H \cap \mathcal{P} \neq \emptyset$

$\mathcal{P}$ on one side of $h$

Faces: $\mathcal{P} \cap h$, $h$ supp. hyp.

Dimension of a face: the dim. of its affine hull
General position

Points in general position

- $P$ is in general position iff no subset of $k + 2$ points lie in a $k$-flat
  \[\Rightarrow\] any $k$-face is the convex hull of $k + 1$ point of $P$, i.e. a $k$-simplex
  \[
  \text{conv}(P) \text{ is called a simplicial polytope}
  \]

Hyperplanes in general position

- $H$ is in general position iff the intersection of any subset of $d - k$ hyperplanes intersect in a $k$-flat
  \[\Rightarrow\] any $k$-face is the intersection of $d - k$ hyperplanes
  \[
  \bigcap H \text{ is called a simple polytope}
  \]
1 Convex Polytopes

2 Voronoi diagrams and Delaunay triangulations

3 Weighted Voronoi diagrams

4 Union of balls

5 Discrete metric spaces
Voronoi diagrams in nature
The solar system (Descartes)
Growth of meristem
Euclidean Voronoi diagrams

Voronoi cell

\[ V(p_i) = \{ x : \| x - p_i \| \leq \| x - p_j \|, \forall j \} \]

Voronoi diagram \((P)\)

\[ = \{ \text{collection of all cells} V(p_i), p_i \in P \} \]
Voronoi diagrams and convex polyhedra

Convex Polyhedron
The intersection of a finite collection of half-spaces:

\[ V = \bigcap_{i \in I} h_i^+ \]

- Each Voronoi cell is a convex polyhedron
- The Voronoi diagram has the structure of a cell complex: two cells either do not intersect or intersect in a common face
- The Voronoi diagram of \( P \) is the projection of a convex polyhedron of \( \mathbb{R}^{d+1} \)
Voronoi diagrams and convex polyhedra

- \( \text{Vor}(p_1, \ldots, p_n) \) is the minimization diagram of the \( n \) functions \( \delta_i(x) = (x - p_i)^2 \)

- \( \arg \min(\delta_i) = \arg \max(h_i) \)
  where \( h_{p_i}(x) = 2p_i \cdot x - p_i^2 \)

- The minimization diagram of the \( \delta_i \) is also the maximization diagram of the affine functions \( h_{p_i}(x) \)

- The faces of \( \text{Vor}(P) \) are the projections of the faces of \( \mathcal{V}(P) = \bigcap_i h_{p_i}^+ \)

  \[
  h_{p_i}^+ = \{ x : x_{d+1} > 2p_i \cdot x - p_i^2 \}
  \]

Note!

the graph of \( h_{p_i}(x) \) is the hyperplane tangent to \( Q \) : \( x_{d+1} = x^2 \) at \( (x, x^2) \)
Lifting map
The faces of $\text{Vor}(P)$ are the projection of the faces of the polytope

$$\mathcal{V}(P) = \bigcap_i h_{p_i}^+$$

where $h_{p_i}$ is the hyperplane tangent to paraboloid $Q$ at the lifted point $(p_i, p_i^2)$

Corollaries
- The size of $\text{Vor}(P)$ is the same as the size of $\mathcal{V}(P)$
- Computing $\text{Vor}(P)$ reduces to computing $\mathcal{V}(P)$
Voronoi diagram and Delaunay complex

Finite set of points \( P \in \mathbb{R}^d \)

- The Delaunay complex is the nerve of the Voronoi diagram
- It is not always embedded in \( \mathbb{R}^d \)
Empty circumballs

An (open) $d$-ball $B$ circumscribing a simplex $\sigma \subset \mathcal{P}$ is called empty if

1. $\text{vert}(\sigma) \subset \partial B$
2. $B \cap \mathcal{P} = \emptyset$

$\text{Del}(\mathcal{P})$ is the collection of simplices admitting an empty circumball
Point sets in general position wrt spheres

$P = \{p_1, p_2 \ldots p_n\}$ is said to be in general position wrt spheres if there do not exist $d + 2$ points of $P$ lying on a same $(d-1)$-sphere.

**Theorem [Delaunay 1936]**

If $P$ is in general position wrt spheres, $\text{Del}(P)$ has a natural realization in $\mathbb{R}^d$ called the Delaunay triangulation of $P$. 
Proof of Delaunay’s theorem 1

Linearization

\[ S(x) = x^2 - 2c \cdot x + s, \quad s = c^2 - r^2 \]

\[ S(x) < 0 \iff \begin{cases} z < 2c \cdot x - s \\ z = x^2 \end{cases} \quad (h_S^-) \quad (P) \]

\[ \iff \hat{x} = (x, x^2) \in h_S^- \]
Proof of Delaunay’s theorem 2

Proof of Delaunay’s th.

$P$ general position wrt spheres
$\iff \hat{P}$ in general position

$\sigma$ a simplex, $S_\sigma$ its circumscribing sphere

$\sigma \in \text{Del}(P) \iff S_\sigma$ empty

$\iff \forall i, \hat{p}_i \in h^{+}_{S_\sigma}$

$\iff \hat{\sigma}$ is a face of $\text{conv}^{-}(\hat{P})$

$\text{Del}(P) = \text{proj}(\text{conv}^{-}(\hat{P}))$
VD and DT in the space of spheres

\[ h_{p_i} : x_{d+1} = 2p_i \cdot x - p_i^2 \]

\[ \hat{p}_i = (p_i, p_i^2) = h_{p_i}^* \]

\[ V(P) = h_{p_1}^+ \cap \ldots \cap h_{p_n}^+ \]

Voronoi Diagram of \( P \)

\[ \mathcal{D}(P) = \text{conv}^{-}\left(\{\hat{p}_1, \ldots, \hat{p}_n\}\right) \]

Delaunay Complex of \( P \)
If $P$ is in general position wrt spheres:

\[ \mathcal{V}(P) = h_{p_1}^{+} \cap \ldots \cap h_{p_n}^{+} \quad \xrightarrow{\text{duality}} \quad \mathcal{D}(P) = \text{conv}^{-}(\{\hat{p}_1, \ldots, \hat{p}_n\}) \]

\[ \uparrow \quad \xrightarrow{\text{nerve}} \quad \downarrow \]

Voronoi Diagram of $P$ \quad \xrightarrow{\text{nerve}} \quad Delaunay Complex of $P$
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3 Weighted Voronoi diagrams

4 Union of balls

5 Discrete metric spaces
Foams, molecules and weighted Voronoi diagrams
Sites: $n$ balls $B = \{b_i(p_i, r_i), i = 1, \ldots n\}$

Power distance: $\pi(x, b_i) = (x - p_i)^2 - r_i^2$

It is not a distance

Power Diagram: $\text{Vor}(B)$

One cell $V(b_i)$ for each site

$V(b_i) = \{x : \pi(x, b_i) \leq \pi(x, b_j). \forall j \neq i\}$

- Each cell is a convex polyhedron
- $V(b_i)$ may be empty
- $p_i$ may not belong to $V(b_i)$
Power diagrams are maximization diagrams

Cell of $b_i$ in the weighed Voronoi diagram $Vor(B)$

$$V(b_i) = \{ x \in \mathbb{R}^d : \pi(x, b_i) \leq \pi(x, b_j). \forall j \neq i \}$$

$$= \{ x \in \mathbb{R}^d : 2p_ix - s_i = \max_{j \in [1,...,n]} \{ 2p_jx - s_j \} \}$$

$Vor(B)$ is the maximization diagram of the set of affine functions

$$\{ h_i(x) = 2p_ix - s_i, i = 1, \ldots, n \}$$

Geometrically: the vertical projection of $\mathcal{P} \cap h_i^-$ is the ball $b_i$
Weighted Delaunay triangulations

\[ B = \{ b_i(p_i, r_i) \} \] a set of balls

\[ \text{Del}(B) = \text{nerve of Vor}(B) \]

Let \( B_\tau = \{ b_i(p_i, r_i), i = 0, \ldots , k \} \subset B : \]

\[ B_\tau \in \text{Del}(B) \iff \bigcap_{b_i \in B_\tau} V(b_i) \neq \emptyset \]
Weighted VD and DT in the space of spheres

\[ h_{b_i} : x_{d+1} = 2p_i \cdot x - p_i^2 + r_i^2 \]

\[ \phi(b_i) = (p_i, p_i^2 - r_i^2) = h_{b_i}^* \]

\[ \mathcal{V}(B) = h_{b_1}^+ \cap \ldots \cap h_{b_n}^+ \quad \overset{\text{duality}}{\longrightarrow} \quad \mathcal{D}(B) = \text{conv}^{-}(\{\phi(b_1), \ldots, \phi(b_n)\}) \]

Voronoi Diagram of $B$ \quad \overset{\text{nerve}}{\longrightarrow} \quad \text{Delaunay Complex of } B
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Union of balls

- What is the combinatorial complexity of the boundary of the union $U$ of $n$ balls of $\mathbb{R}^d$?

- Compare with the complexity of the arrangement of the bounding hyperspheres

- How can we compute $U$?

- What is the image of $U$ in the space of spheres?
Weighted Voronoi Diagram

\[ \sigma_i(x) = (x - p_i)^2 - r_i^2 \]
Restriction of $\text{Del}(B)$ to $U = \bigcup_{b \in B} b$

- $U = \bigcup_{b \in B} b \cap V(b)$ and $\partial U \cap \partial b = V(b) \cap \partial b$.
- The nerve of $\mathcal{C}$ is the restriction of $\text{Del}(B)$ to $U$, i.e. the subcomplex $\text{Del}|_U(B)$ of $\text{Del}(B)$ whose faces have a circumcenter in $U$.
- $\forall b$, $b \cap V(b)$ is convex and thus contractible.
- $\mathcal{C} = \{b \cap V(b), b \in B\}$ is a good cover of $U$.
- The nerve of $\mathcal{C}$ is a deformation retract of $U$.

homotopy equivalent (Nerve theorem)
Interfaces between proteins

Interface antigen-antibody
Fonctions distance et modèles de croissance
Möbius Diagrams

\[ W_i = (p_i, \lambda_i, \mu_i) \]

\[ \delta_M(x, W_i) = \lambda_i \| x - p_i \|^2 - \mu_i \]

bisectors are \textbf{hyperspheres} (hyperplanes or \emptyset)

\[ \text{Mob}(W_i) = \{ x, \delta(x, \sigma_i) \leq \delta(x, \sigma_j) \} \]
Linearization Lemma

We can associate to each weighted point $W_i$ a hypersphere $\Sigma_i$ of $\mathbb{R}^{d+1}$ so that

the faces of the Möbius diagram of the $W_i$ are obtained by

1. computing the Laguerre Diagram of the $\Sigma_i$
2. intersecting this diagram with the paraboloid $\mathcal{P}$
3. projecting vertically the faces of this intersection onto $\mathbb{R}^d$
Proof

\[
\lambda_i (x - p_i)^2 - \mu_i \leq \lambda_j (x - p_j)^2 - \mu_j
\]

\[\iff\]

\[
(x - \lambda_i p_i)^2 + (x^2 + \frac{\lambda_i}{2})^2 - \lambda_i^2 p_i^2 - \frac{\lambda_i^2}{4} + \lambda_i p_i^2 - \mu_i
\]

\[
\leq (x - \lambda_j p_j)^2 + (x^2 + \frac{\lambda_j}{2})^2 - \lambda_j^2 p_j^2 - \frac{\lambda_j^2}{4} + \lambda_j p_j^2 - \mu_j
\]

\[\iff\]

\[
(X - C_i)^2 - \rho_i^2 \leq (X - C_j)^2 - \rho_j^2
\]

where \(X = (x, x^2) \in \mathbb{R}^{d+1},\)

\[
C_i = (\lambda_i p_i, -\frac{\lambda_i}{2}) \in \mathbb{R}^{d+1} \text{ and } \rho_i^2 = \lambda_i^2 p_i^2 + \frac{\lambda_i^2}{4} - \lambda_i p_i^2 + \mu_i
\]

The Möbius diagram of the doubly weighted points \(W_i\) is the projection onto \(\mathbb{R}^d\) of the restriction to the paraboloid \(Q\) of the Laguerre diagram of the \(\Sigma_i = B(C_i, \rho_i)\)
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Let $\sigma$ be a (abstract) simplex with vertices in $L$, and let $w \in W$. We say that $w$ is a witness of $\sigma$ if

$$
\|w - p\| \leq \|w - q\| \quad \forall p \in \sigma \text{ and } \forall q \in L \setminus \sigma
$$

The witness complex $\text{Wit}(L, W)$ is the complex consisting of all simplexes $\sigma$ such that for any simplex $\tau \subseteq \sigma$, $\tau$ has a witness in $W$.
Easy consequences of the definition

- The witness complex can be defined for any metric space and, in particular, for discrete metric spaces

- If $W' \subseteq W$, then $\text{Wit}(L, W') \subseteq \text{Wit}(L, W)$

- $\text{Del}(L) \subseteq \text{Wit}(L, \mathbb{R}^d)$
Identity of $\text{Del}_{\Omega}(L)$ and $\text{Wit}(L, \mathbb{R}^d)$

[de Silva 2008]

**Theorem:** \( \text{Wit}(L, W) \subseteq \text{Wit}(L, \mathbb{R}^d) = \text{Del}(L) \)

**Remarks**

- Faces of all dimensions have to be witnessed

- \( \text{Wit}(L, W) \) is *embedded* in \( \mathbb{R}^d \) if \( L \) is in general position wrt spheres
Proof of de Silva’s theorem

\( \tau = [p_0, ..., p_k] \) is a \( k \)-simplex of \( \text{Wit}(L) \) witnessed by a ball \( B_\tau \) (i.e. \( B_\tau \cap L = \tau \))

We prove that \( \tau \in \text{Del}(L) \) by induction on \( k \)

Clearly true for \( k = 0 \)

Hyp. : true for \( k' \leq k - 1 \)

\[ B := B_\tau \]

\[ \sigma := \partial B \cap \tau, \quad l := |\sigma| \]

\[ \quad // \sigma \in \text{Del}(L) \text{ by the hyp.} \]

\[ \text{while } l + 1 = \text{dim} \sigma < k \text{ do} \]

\[ B \leftarrow \text{the ball centered on } [cw] \text{ s.t.} \]

- \( \sigma \subset \partial B \),
- \( B \) witnesses \( \tau \)
- \( |\partial B \cap \tau| = l + 1 \)

\( (B \text{ witnesses } \tau) \land (\tau \subset \partial B) \Rightarrow \tau \in \text{Del}(L) \)
Case of sampled domains: \( \text{Wit}(L, W) \neq \text{Del}(L) \)

\( W \) a finite set of points \( \subset \Omega \subset \mathbb{R}^d \)

\( \text{Wit}(L, W) \neq \text{Del}(L) \), even if \( W \) is a dense sample of \( \Omega \)

\[ [ab] \in \text{Wit}(L, W) \iff \exists p \in W, \ Vor_2(a, b) \cap W \neq \emptyset \]
Relaxed witness complex

**Alpha-witness** Let $\sigma$ be a simplex with vertices in $L$. We say that a point $w \in W$ is an $\alpha$-witness of $\sigma$ if

\[ \|w - p\| \leq \|w - q\| + \alpha \quad \forall p \in \sigma \quad \text{and} \quad \forall q \in L \setminus \sigma \]

**Alpha-relaxed witness complex** The $\alpha$-relaxed witness complex $\text{Wit}^\alpha(L, W)$ is the maximal simplicial complex with vertex set $L$ whose simplices have an $\alpha$-witness in $W$

$\text{Wit}^0(L, W) = \text{Wit}(L, W)$

**Filtration** $\alpha \leq \beta \implies \text{Wit}^\alpha(L, W) \subseteq \text{Wit}^\beta(L, W)$
Lemma Assume that $W$ is $\varepsilon$-dense in $\Omega$ and let $\alpha \geq 2\varepsilon$. Then

$$\text{Wit}(L, W) \subseteq \text{Del}|_{\Omega}(L) \subseteq \text{Wit}^{\alpha}(L, W)$$

Proof

$\sigma$: a $d$-simplex of $\text{Del}(L)$, $c_{\sigma}$ its circumcenter

$W$ $\varepsilon$-dense in $\Omega$  $\exists w \in W$ s.t. $\|c_{\sigma} - w\| \leq \varepsilon$

For any $p \in \sigma$ and $q \in L \setminus \sigma$, we then have

$$\forall p \in \sigma \text{ and } q \in L \setminus \sigma \quad \|w - p\| \leq \|c_{\sigma} - p\| + \|c_{\sigma} - w\|$$

$$\leq \|c_{\sigma} - q\| + \|c_{\sigma} - w\|$$

$$\leq \|w - q\| + 2\|c_{\sigma} - w\|$$

$$\leq \|w - q\| + 2\varepsilon$$