Lecture 3
Randomized Algorithms

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Outline

1. Combinatorial complexity
2. Deterministic incremental construction of convex hulls
3. Randomized incremental algorithm
4. Fast point location
5. k-order Voronoi Diagrams

Combinatorial complexity

The combinatorial complexity of the Voronoi diagram of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of the intersection of n half-spaces of $\mathbb{R}^{d+1}$

The combinatorial complexity of the Delaunay triangulation of n points of $\mathbb{R}^d$ is the same as the combinatorial complexity of the convex hull of n points of $\mathbb{R}^{d+1}$

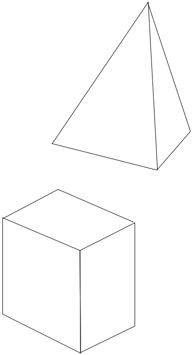
Both complexities are the same by duality

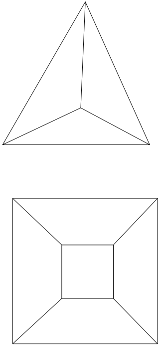
Euler formula for 3-polytopes

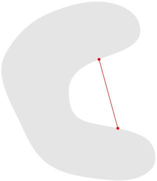
The numbers of vertices s, edges a and facets f of a polytope of $\mathbb{R}^3$ satisfy

$$s - a + f = 2$$

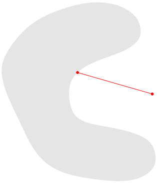
Schlegel diagram







$s = s'$
$a' = a + 1$
$f' = f + 1$



$s' = s + 1$
$a' = a + 1$
$f' = f$

Euler formula for 3-polytopes : $s - a + f = 2$

Incidences edges-facets

$$2a \geq 3f \quad \Longrightarrow \quad \begin{array}{l} a \leq 3s - 6 \\ f \leq 2s - 4 \end{array}$$

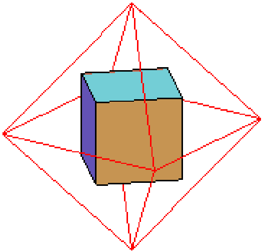
with equality when all facets are triangles

Beyond the 3rd dimension

Upper bound theorem [McMullen 1970]

If $\mathcal{H}$ is the intersection of n half-spaces of $\mathbb{R}^d$

$$\text{nb faces of } \mathcal{H} = \Theta(n^{\lfloor}$$



Hyperplanes in general position

- any k-face is the intersection of $d-k$ hyperplanes defining $\mathcal{H}$
- all vertices of $\mathcal{H}$ are incident to d edges and have distinct x_d
- the affine hull of $k < d$ edges incident to a vertex p contains a k-face of $\mathcal{H}$

Proof of the upper bound theorem

Bounding the number of vertices

1. $\geq \lceil \frac{d}{2} \rceil$ edges incident to a vertex p are in $h_p^+ : x_d \geq x_d(p)$ or in h_p^-
   $\Rightarrow$ p is a x_d-max or x_d-min vertex of at least one $\lceil \frac{d}{2} \rceil$-face of $\mathcal{H}$
   $\Rightarrow$ $\#$ vertices of $\mathcal{H} \leq 2 \times \#$ $\lceil \frac{d}{2} \rceil$-faces of $\mathcal{H}$

2. A k-face is the intersection of $d-k$ hyperplanes defining $\mathcal{H}$

   $\Rightarrow$ $\#$ k-faces $= \begin{pmatrix} n \\ d-k \end{pmatrix} = O(n^{d-k})$
   $\Rightarrow$ $\#$ $\lceil \frac{d}{2} \rceil$-faces $= O(n^{\lfloor \frac{d}{2} \rfloor})$

Bounding the total number of faces

The number of faces incident to p depends on d but not on n

Computing the convex hull of n points of $\mathbb{R}^d$

Adjacency graph (AG) of the facets

In general position, all the facets are $(d-1)$-simplexes

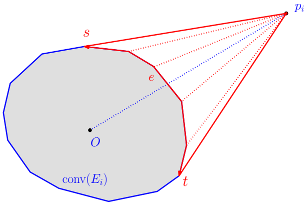
Adjacency graph (V, E)

- $V =$ set of $(d-1)$-faces (facets)
- $(f, f') \in E$ iff $f \cap f'$ share a $(d-2)$-face

Incremental algorithm

$\mathcal{P}_i$: set of the i points that have been inserted first

conv($\mathcal{P}_i$) : convex hull at step i



$f = [p_1, ..., p_d]$ is a red facet iff its supporting hyperplane separates p_i from conv($\mathcal{P}_i$)

$\iff \texttt{orient}(p_1, ..., p_d, p_i) \times \texttt{orient}(p_1, ..., p_d, O) < 0$

$$\text{orient}(p_0, p_1, ..., p_d) = \begin{vmatrix} 1 & 1 & ... & 1 \\ p_0 & p_1 & ... & p_d \end{vmatrix} = \begin{vmatrix} 1 & 1 & ... & 1 \\ x_{01} & x_{11} & ... & x_{d1} \\ \vdots & \vdots & ... & \vdots \\ x_{0d} & x_{1d} & ... & x_{dd} \end{vmatrix}$$

Update of conv($\mathcal{P}_i$)

red facet = facet whose supporting hyperplane separates o and p_{i+1}

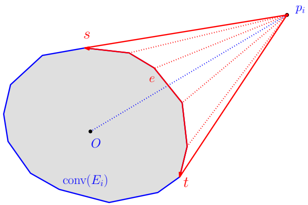
horizon : $(d-2)$-faces shared by a blue and a red facet

Update conv($\mathcal{P}_i$) :

1. find the red facets
2. remove them and create the new facets

   $$[p_{i+1}, g],\ \forall g \in \text{horizon}$$

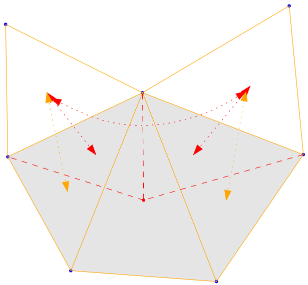
3. create the new adjacencies



Complexity

proportional to the number of red facets

Updating the adjacency graph



We look at the d-simplices to be removed and at their neighbors

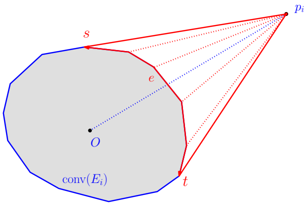
The number of times a removed d-simplex is considered is equal to the number of its $(d-2)$-faces

$$\leq \begin{pmatrix} d+1 \\ d-1 \end{pmatrix} = \frac{d(d+1)}{2}$$

Update cost $= O(\#$ created and deleted simplices $)$
$= O(\#$ created simplices$)$

Complexity analysis

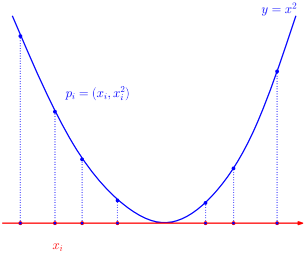
- update proportional to the number of red facets
- # new facets = $|\text{conv}(i, d-1)|$
  $= O(i^{\lfloor \frac{d-1}{2} \rfloor})$
- fast locate : insert the points in lexicographic order and search a 1st red facet in $\text{star}(p_{i-1})$
  (which necessarily exists)



$$T(n,d) = O(n \log n) + \sum_{i=1}^{n} i^{\lfloor \frac{d-1}{2} \rfloor})$$
$$= O(n \log n + n^{\left\lfloor \frac{d+1}{2} \right\rfloor})$$

Worst-case optimal in even dimensions

Lower bound



conv($\{p_i\}$) $\Longrightarrow$ tri($\{x_i\}$)

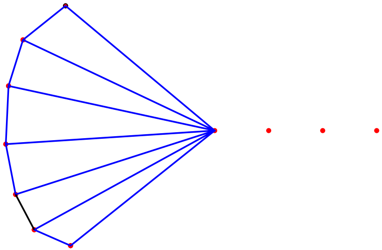
the orientation test reduces to 3 comparisons

$$\text{orient}(p_i, p_j, p_k) = \begin{vmatrix} x_i - x_j & x_i - x_k \\ x_i^2 - x_j^2 & x_i^2 - x_k^2 \end{vmatrix}$$

$$= (x_i - x_j)(x_j - x_k)(x_k - x_i)$$

$\Longrightarrow$ Lower bound : $\Omega(n \log n)$

Lower bound for the incremental algorithm



No incremental algorithm can compute the convex hull of n points of $\mathbb{R}^3$ in less than $\Omega(n^2)$

Constructing $\mathrm{Del}(P), \quad P = \{p_1, ..., p_n\} \subset \mathbb{R}^d$

Algorithm

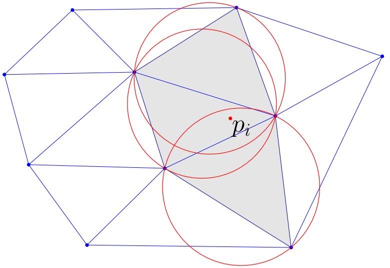
1. Lift the points of P onto the paraboloid $x_{d+1} = x^2$ of $\mathbb{R}^{d+1}$:
   $p_i \rightarrow \hat{p}_i = (p_i, p_i^2)$
2. Compute conv($\{\hat{p}_i\}$)
3. Project the lower hull conv$^-$($\{\hat{p}_i\}$) onto $\mathbb{R}^d$

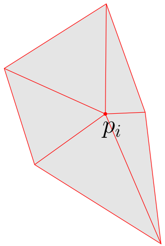
Complexity : $\Theta(n \log n + n^{\lfloor \frac{d+1}{2} \rfloor})$

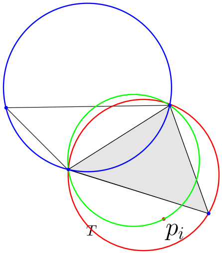
A pedestrian view : insertion of a new point p_i

1. Location : find all the d-simplices that conflict with p_i
   i.e. whose circumscribing ball contains p_i

2. Update : construct the new d-simplices





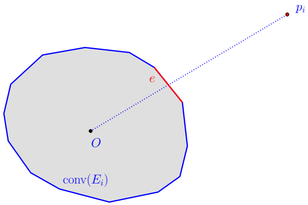


Randomized incremental algorithm

o : a point inside conv($\mathcal{P}$)

$\mathcal{P}_i$: the set of the first i inserted points

conv($\mathcal{P}_i$) : convex hull at step i



Conflict graph

bipartite graph $\{p_j\} \times \{$facets of conv($\mathcal{P}_i$)$\}$

$$p_j \dagger f \iff j > i \quad (p_j \text{ not yet inserted}) \quad \wedge \quad f \cap op_j \neq \emptyset$$

Randomized analysis

Hyp. : points are inserted in random order

Conflict : †

Notations R : random sample of size r of $\mathcal{P}$

$F(R) = \{$ subsets of d points of $R\}$
$F_0(R) = \{$ elements of $F(R)$ with 0 conflict in $R\}$

(i.e. $\in$ conv(R))

$F_1(R) = \{$ elements of $F(R)$ with 1 conflict in $R\}$
$C_i(r, \mathcal{P}) = E(|F_i(R)|)$

(expectation over all random samples $R \subset \mathcal{P}$ of size r)

Lemma

$$C_i(r, \mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor}), \quad i = 1, 2$$

Proof of the lemma : $C_1(r,\mathcal{P}) = C_0(r,\mathcal{P}) = O(r^{\lfloor \frac{d}{2} \rfloor})$

$R' = R \setminus \{p\}$

$f \in F_0(R')$ if $f \in F_1(R)$ and $p \dagger f$ (proba $= \frac{1}{r}$)
or $f \in F_0(R)$ and $R' \ni$ the d vertices of f (proba $= \frac{r-d}{r}$)

Taking the expectation,

$$\begin{aligned}
C_0(r-1,R) &= \frac{1}{r}\,|F_1(R)| + \frac{r-d}{r}\,|F_0(R)| \\
C_0(r-1,\mathcal{P}) &= \frac{1}{r}\,C_1(r,\mathcal{P}) + \frac{r-d}{r}\,C_0(r,\mathcal{P}) \\
C_1(r,\mathcal{P}) &= d\,C_0(r,\mathcal{P}) - r\,\left(C_0(r,\mathcal{P}) - C_0(r-1,\mathcal{P})\right) \\
&\leq d\,C_0(r,\mathcal{P}) \\
&= O(r^{\lfloor \frac{d}{2} \rfloor})
\end{aligned}$$

Randomized analysis 1

Updating the convex hull + memory space

Expected number $N(i)$ of facets created at step i

$$
\begin{aligned}
N(i) &= \sum_{f \in F(\mathcal{P})} \mathsf{Proba}(f \in F_0(\mathcal{P}_i)) \times \frac{d}{i} \\
&= \frac{d}{i}\, O\left(i^{\lfloor \frac{d}{2} \rfloor}\right) \\
&= O(n^{\lfloor \frac{d}{2} \rfloor - 1})
\end{aligned}
$$

Expected total number of created facets $= O(n^{\lfloor \frac{d}{2} \rfloor})$

$O(n)$ if $d = 2, 3$

Randomized analysis2

Updating the conflict graph

Cost proportional to the number of faces of conv($\mathcal{P}_i$) in conflict with p_{i+1} and some p_j, $j > i$
$N(i,j)$ = expected number of faces of conv($\mathcal{P}_i$) in conflict with p_{i+1} and p_j, $j > i$

$\mathcal{P}_i^+ = \mathcal{P}_i \cup \{p_{i+1}\} \cup \{p_j\}$: a random subset of $i+2$ points of $\mathcal{P}$

$$N(i,j) = \sum_{f \in F(\mathcal{P})} \text{Proba}(f \in F_2(\mathcal{P}_i^+)) \times \begin{pmatrix} i+2 \\ 2 \end{pmatrix}^{-1} = \frac{2\,C_2(i+1)}{(i+1)(i+2)} = O(i^{\lfloor \frac{d}{2} \rfloor - 2})$$

Expected total cost of updating the conflict graph

$$\sum_{i=1}^{n} \sum_{j=i+1}^{n} N(i,j) = \sum_{i=1}^{n} (n-i)\, O(i^{\lfloor \frac{d}{2} \rfloor - 2}) = O(n \log n + n^{\lfloor \frac{d}{2} \rfloor})$$

Theorem

- The convex hull of n points of $\mathbb{R}^d$ can be computed in time $O(n \log n + n^{\lfloor \frac{d}{2} \rfloor})$ using $O(n^{\lfloor \frac{d}{2} \rfloor})$ space
- The same bounds hold for computing the intersection of n half-spaces of $\mathbb{R}^d$
- The randomized algorithm can be derandomized

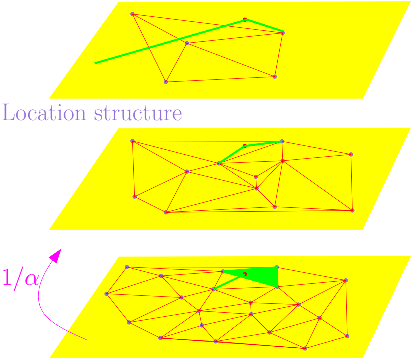
[Chazelle 1992]

The Delaunay hierarchy

A location data structure

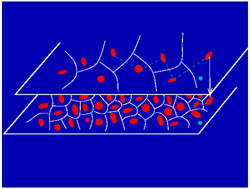
Level 0 is $\mathrm{Del}(\mathcal{P})$

Each data point p in level l is introduced in level $l+1$ with probability $\beta = \frac{1}{\alpha}$



Point location in the Delaunay Triangulation

Location of point q:
find the nearest neighbor of q in $\mathcal{P}$
$n_l(q)$: nearest neighbor of q in $\mathcal{P}_l$



Locate q in the highest level
From $n_{l+1}(q)$ to $n_l(q)$:
- use the pointer of $n_{l+1}(q)$ to level l
- walk in level l from $n_{l+1}(q)$ to $n_l(q)$

The number of steps performed at $level(l)$: m_l
$m_l \leq k$ if $n_{l+1}(p)$ is the kth neighbor of q in $\mathcal{P}_l$

$$\begin{aligned} \text{Exp}(m_l) &\leq \sum_{k=1}^{n_l} k(1-\beta)^{k-1}\beta \\ &\leq \beta\Big[-\frac{\partial}{\partial\beta}\sum_k (1-\beta)^k\Big] = \frac{1}{\beta} \end{aligned}$$

Expected total number of steps: $O(\log n)$.

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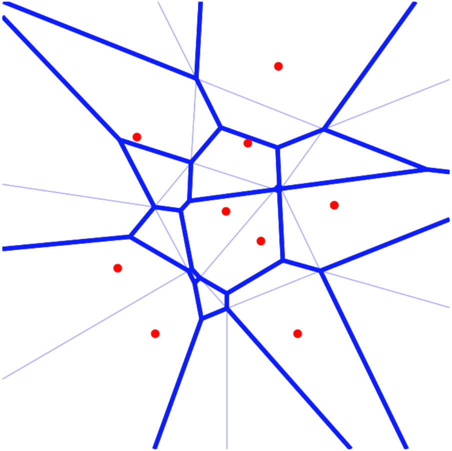
5 k-order Voronoi Diagrams

k-order Voronoi Diagrams



Let P be a set of sites.
Each cell in the k-order Voronoi diagram $\text{Vor}_k(P)$ is the locus of points in $\mathbb{R}^d$ that have the same subset of P as k-nearest neighbors.

k-order Voronoi diagrams are power diagrams

Let $S_1, S_2, \dots$ denote the subsets of k points of P.
The k-order Voronoi diagram is the minimization diagram of $\delta(x, S_i)$:

$$\begin{aligned} \delta(x, S_i) &= \frac{1}{k} \sum_{p \in S_i} (x-p)^2 \\ &= x^2 - \frac{2}{k} \sum_{p \in S_i} p \cdot x + \frac{1}{k} \sum_{p \in S_i} p^2 \\ &= \pi(b_i, x) \end{aligned}$$

where b_i is the ball

1. centered at $c_i = \frac{1}{k} \sum_{p \in S_i} p$
2. with $s_i = \pi(o, b_i) = c_i^2 - r_i^2 = \frac{1}{k} \sum_{p \in S_i} p^2$
3. and radius $r_i^2 = c_i^2 - \frac{1}{k} \sum_{p \in \mathbb{S}_i} p^2$.

Combinatorial complexity of k-order Voronoi diagrams

Theorem

If P be a set of n points in $\mathbb{R}^d$, the number of vertices and faces in all the Voronoi diagrams $\text{Vor}_j(P)$
of orders $j \leq k$ is:

$$O\left(k^{\lceil \frac{d+1}{2} \rceil} n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$

Proof

uses :

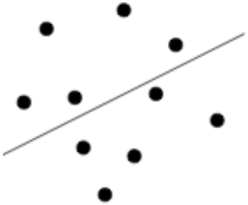
- bijection between k-sets and cells in k-order Voronoi diagrams
- the sampling theorem (from randomization theory)

k-sets and k-order Voronoi diagrams

P a set of n points in $\mathbb{R}^d$

k-sets

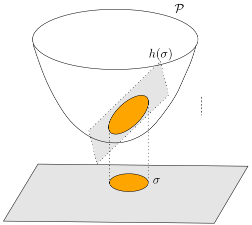
A k-set of P is a subset P' of P with size k that can be separated from $P \setminus P'$ by a hyperplane



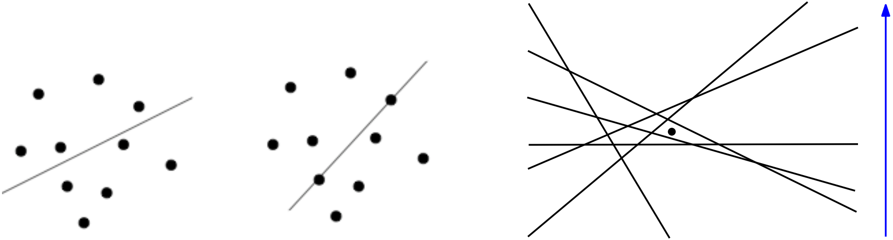
k-order Voronoi diagrams

k points of P have a cell in $\text{Vor}_k(P)$ iff there exists a ball that contains those points and only those

$\Rightarrow$ each cell of $\text{Vor}_k(P)$ corresponds to a k-set of $\phi(P)$

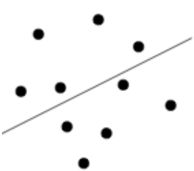


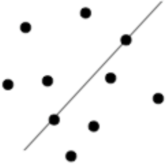
k-sets and k-levels in arrangements of hyperplanes

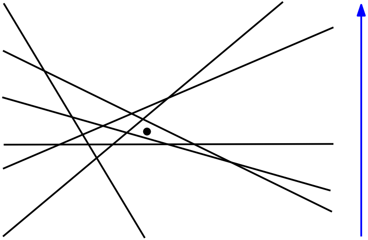


- For a set of points $P \in \mathbb{R}^d$, we consider the arrangement of the dual hyperplanes $\mathcal{A}(P^*)$
- h defines a k set P' $\Rightarrow$ h separates P' (below h) from $P \setminus P'$ (above h) $\Rightarrow$ h^* is below the k hyperplanes of P'^* and above those of $P^* \setminus P'^*$
- k-sets of P are in 1-1 correspondance with the cells of $\mathcal{A}(P^*)$ of level k, i.e. with k hyperplanes of P^* above it.

Bounding the number of k-sets







$c_k(P)$: Number of k-sets of P = Number of cells of level k in $\mathcal{A}(P^*)$
$c_{\leq k}(P) = \sum_{l \leq k} c_l(P)$
$c'_{\leq k}(P)$: Number of vertices of $\mathcal{A}(P^*)$ with level at most k
$c_{\leq k}(n) = \max_{|P|=n} c_{\leq k}(P)$ $c'_{\leq k}(n) = \max_{|P|=n} c'_{\leq k}(P)$

Hyp. in general position : each vertex $\in d$ hyperplanes incident to 2^d cells

Vertices of level k are incident to cells with level $\in [k, k+d]$

Cells of level k have incident vertices with level $\in [k-d, k]$

$$c_{\leq k}(n) = O\left(c'_{\leq k}(n)\right)$$

Regions, conflicts and the sampling theorem

O a set of n objects.

$\mathcal{F}(O)$ set of configurations defined by O

- each configuration is defined by a subset of b objects
- each configuration is in conflict with a subset of O

$\mathcal{F}_j(O)$ set of configurations in conflict with j objects
$|\mathcal{F}_{\leq k}(O)|$ number of configurations defined by O
in conflict with at most k objects of O
$f_0(r) = \text{Exp}(|\mathcal{F}_0(R|)$ expected number of configurations
defined and without conflict on a random r-sample of O.

The sampling theorem [Clarkson & Shor 1992]

For $2 \leq k \leq \frac{n}{b+1}$, $|\mathcal{F}_{\leq k}(O)| \leq 4\,(b+1)^b\,k^b\,f_0(\lfloor \frac{n}{k} \rfloor)$

Proof of the sampling theorem

$$f_0(r) = \sum_j |\mathcal{F}_j(O)| \frac{\binom{n-b-j}{r-b}}{\binom{n}{r}} \geq |\mathcal{F}_{\leq k}(O)| \frac{\binom{n-b-k}{r-b}}{\binom{n}{r}}$$

then,we prove that
for $r = \frac{n}{k}$

$$\frac{\binom{n-b-k}{r-b}}{\binom{n}{r}} \geq \frac{1}{4(b+1)^b k^b}$$

$$\frac{\binom{n-b-k}{r-b}}{\binom{n}{r}} = \underbrace{\frac{r!}{(r-b)!} \frac{(n-b)!}{n!}}_{\geq \frac{1}{(b+1)^b k^b}} \underbrace{\frac{(n-r)!}{(n-r-k)!} \frac{(n-b-k)!}{(n-b)!}}_{\geq \frac{1}{4}}$$

Proof of the sampling theorem
end

$$\frac{(n-r)!}{(n-r-k)!}\frac{(n-b-k)!}{(n-b)!} = \prod_{j=1}^{k}\frac{n-r-k+j}{n-b-k+j} \geq \left(\frac{n-r-k+1}{n-b-k+1}\right)^k$$

$$\geq \left(\frac{n-n/k-k+1}{n-k}\right)^k$$

$$\geq (1-1/k)^k \geq 1/4 \text{ pour } (2 \leq k),$$

$$\frac{r!}{(r-b)!}\frac{(n-b)!}{n!} = \prod_{l=0}^{b-1}\frac{r-l}{n-l} \geq \prod_{l=1}^{b}\frac{r+1-b}{n}$$

$$\geq \prod_{l=1}^{b}\frac{n/k-b}{n}$$

$$\geq 1/k^b(1-\frac{bk}{n})^b \geq \frac{1}{k^b(b+1)^b} \text{ pour } (k \leq \frac{n}{b+1}).$$

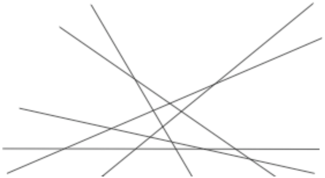
Bounding the number of k-sets

$c_k(P)$: Number of k-sets of P = Number of cells of level k in $\mathcal{A}(P^*)$.
$c_{\leq k}(P) = \sum_{l \leq k} c_l(P)$
$c'_{\leq k}(P)$: Number of vertices of $\mathcal{A}(P^*)$ with level at most k.

Objects O: n hyperplanes of $\mathbb{R}^d$

Configurations : vertices in $\mathcal{A}(O)$, $b = d$

Conflict between v and h : $v \in h^+$



Sampling th: $c'_{\leq k}(P) \leq 4(d+1)^d k^d f_0\left(\left\lfloor \frac{n}{k} \right\rfloor\right)$

Upper bound th: $f_0(\left\lfloor \frac{n}{k} \right\rfloor) = O\left(\frac{n^{\lfloor \frac{d}{2} \rfloor}}{k^{\lfloor \frac{d}{2} \rfloor}}\right)$

$\Rightarrow c'_{\leq k}(n) = O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$

Combinatorial complexities

- Number of vertices of level $\leq k$ in an arrangement of n hyperplanes in $\mathbb{R}^d$

  Number of cells of level $\leq k$ in an arrangement of n hyperplanes in $\mathbb{R}^d$

  Total number of $j \leq k$ sets for a set of n points in $\mathbb{R}^d$

$$O\left(k^{\lceil \frac{d}{2} \rceil} n^{\lfloor \frac{d}{2} \rfloor}\right)$$

- Total number of faces in the Voronoi diagrams of order $j \leq k$ for a set of n points in $\mathbb{R}^d$

$$O\left(k^{\lceil \frac{d+1}{2} \rceil} n^{\lfloor \frac{d+1}{2} \rfloor}\right)$$