Uniformity of point samples using gap ratio

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Outline

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Motivation

Generating uniformly distributed points over a specific domain has applications in

- digital halftoning(Asano, 2008; Teramoto, Asano, Katoh, Doerr, 2006; Zhang et. al., 2011)
- numerical integration (Chazelle: The Discrepancy Method; Matousek: Geometric Discrepancy)
- computer graphics (Chazelle: The Discrepancy Method)
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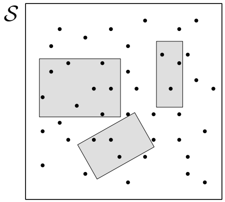
The Problem

Given a metric space $(\mathcal{M}, \delta)$, we want to choose a set $P \subsetneq \mathcal{M}$ of k points such that it is uniformly distributed in $\mathcal{M}$.

Measures of Uniformity: Discrepancy



- The discrepancy $D(P)$ of a point set P is,
  $$D(P) = \sup_{B \in \mathcal{R}} \left| n \cdot \mathrm{vol}(B) - |P \cap B| \right|,$$
  where $\mathcal{R}$ is the set of rectangles contained in the unit square.

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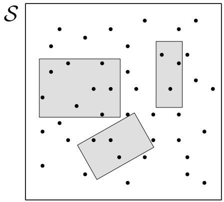

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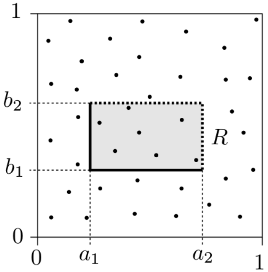
Computing the measure

How to compute the above discrepancy measure? Seems difficult.

Measures of Uniformity: Discrepancy (contd.)

- What if we restrict R to be the set of all axis-parallel rectangles inside $\mathcal{S}$?



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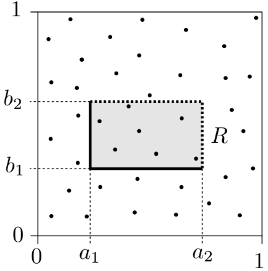

Solution

For n points in a square, $O\left(n^2 \log n\right)$ time is required to compute this value (Dobkin and Gunopulos, 1994).

The problem definition (Generalization)

Definition of *Gap Ratio* (A generalization of Teramoto, et. al., 2006)

- Let $(\mathcal{M}, \delta)$ be a metric space and let P be a set of n points sampled from $\mathcal{M}$.

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The problem

We look to sample a set of points P from $\mathcal{M}$ such that GR_P is minimised.

Making sense of the definition

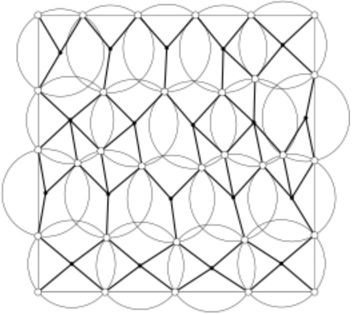


Figure: Maximum Gap = Covering Radius

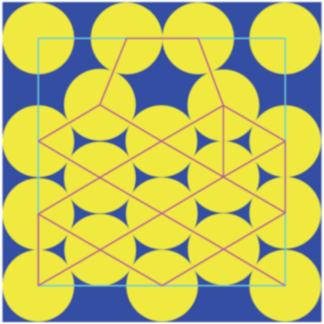


Figure: Minimum Gap = Packing Radius

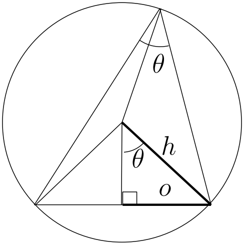
The problem definition

The optimization problem

Given a metric space $(\mathcal{M}, \delta)$, an integer k and a parameter g, find a set $P \subset \mathcal{M}$ such that $|P| = k$ and $GR_P \leq g$.

Gap Ratio and Delaunay

Consider the Delaunay graph of a point set with gap ratio g. Let θ be the minimum angle in the delaunay triangulation.



Clearly $R \geq h$ and $r \leq o \implies \sin\theta = \frac{o}{h} \geq \frac{r}{R} = \frac{1}{g}$

Outline

Previous Works

Up until now the problem has only been studied in an online setting where the gap ratio is minimised at every stage.

- At least $2^{\frac{\lfloor n/2 \rfloor}{\lfloor n/2 \rfloor + 1}}$ where n points are inserted in the interval $[0, 1]$
- At most 2 where n integral points are inserted in the interval $[0, m]$ where m is a positive integer and $0 < n < m$
- No online algorithm can achieve strictly less than 2.5 for a 3×3 grid (discrete space)

Previous results (Algorithms)

Algorithms:

- Linear time optimal algorithm for inserting n points in the interval $[0, 1]$
- An insertion strategy that achieves a gap ratio of at most 2 in a unit square
- An insertion strategy that achieves a gap ratio of at most $2\sqrt{2}$ in a bounded two dimensional grid

Outline

A Summary of Results

Metric space		Lower Bounds	Hardness	Approximation Factor
General		none	Yes	2-approx. hard
Discrete	connected Graph	$\frac{2}{3}$	Yes	approx. factor 3
				$\frac{3}{2}$-approx. hard
	Euclidean	-	-	$(1+\epsilon)$-approx. algo.
Continuous	Path Connected	1	Yes	approx. factor 2
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No lower bound for gap ratio

Note the maximum gap and minimum gap are not the maximum and minimum of the same parameter.

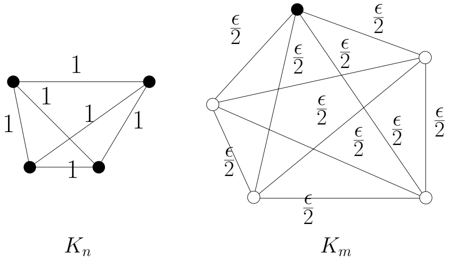


Figure: Discrete space (The black points form the sampled set.)

- Let $\mathcal{M} = V[K_n] \cup V[K_m]$ and the metric $\delta =$ edge weights of $K_n \cup K_m$ and ∞, when there is no edge.
- $R = \epsilon/2$ and $r = 1/2$. Thus, $GR = \epsilon$.

No lower bound for gap ratio

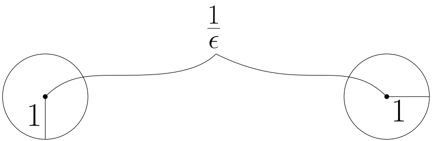
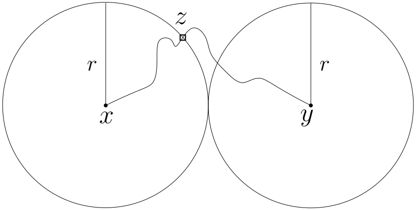


Figure: Continuous Space (The black points form the sampled set).

- Let $\mathcal{M}$ be the union of two balls A and B in $\mathbb{R}^d$ of diameter 1 with distance between their centres being $\frac{1}{\epsilon}$, where $0 < \epsilon < 1$.
- P is any two points, one in A and the other in B.
- $R \leq 1$, and $r \geq \frac{1}{2\epsilon} - \frac{1}{2}$. Thus, $GR \leq \frac{2\epsilon}{1+\epsilon}$.

Lower bound for Path-connected space is 1



- Let x and y be the closest pair of points in a sampled set P from a path connected space $(\mathcal{M}, \delta)$. $\delta(x, y) = 2r$.

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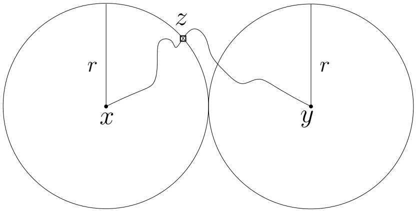

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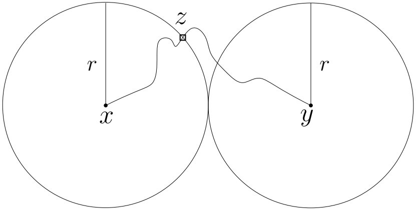

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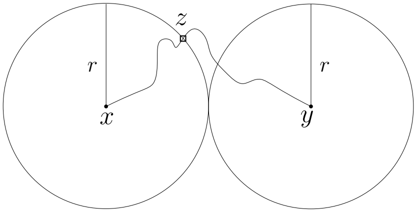

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- z must be at least at a distance r from P. Thus, $R \geq r$.

Lower Bounds: Graph (Discrete space)

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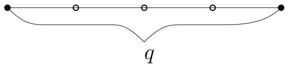

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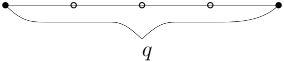

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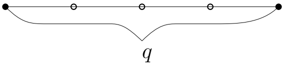

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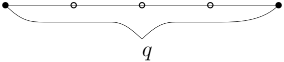

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- This gives us $\frac{R}{r} \geq f(q) = \frac{2}{q} \left\lfloor \frac{q}{2} \right\rfloor$.

For $q \geq 2$, $f(q) = \begin{cases} 1 & q \text{ is even} \\ \frac{q-1}{q} & q \text{ is odd} \end{cases}$.
$q = 1$ is an exception as R is always at least 1.
Thus $\frac{R}{r} \geq f(q) \geq \frac{2}{3}$.

Lower Bounds: Unit square

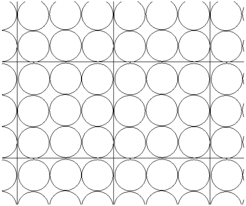
Definition (Packing and covering densities)

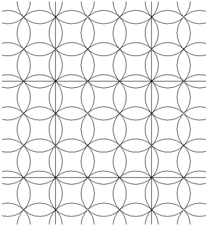
The density of a family $\mathcal{S}$ of sets with respect to a set C of Lebesgue measure $\mu(C) < \infty$, is defined as $d(\mathcal{S}, C) = \left(\sum_{S \in \mathcal{S}, S \cap C \neq \emptyset} \mu(S) \right) / \mu(C)$.

Basically, Density $= \frac{\text{Total area covered}}{\text{Area of space}}$.

For an unbounded area we may just take the limit over an unbounded area.

Lower Bounds: Unit Square (contd..)





Bounds on packing and covering density of plane by congruent disks. (Kuperberg, 1987)

Packing density: $d_1 \leq \frac{\pi}{\sqrt{12}}$ and Covering density: $D_1 \geq \frac{2\pi}{\sqrt{27}}$.

Lower Bounds: Unit Square (contd..)

Suppose we choose k points. Then, we have

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Using the upper bound on packing density

$$\frac{k\pi r^2}{(1+2r)^2} \leq \frac{\pi}{\sqrt{12}} \implies r \leq \left(\sqrt{k\sqrt{12}} - 2\right)^{-1}$$

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Using the lower bound on covering density

$$\frac{k\pi R^2}{1} \geq \frac{2\pi}{\sqrt{27}} \implies R \geq \sqrt{\frac{2}{k\sqrt{27}}}.$$

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Take the ratio

Thus we have $\frac{R}{r} \geq \frac{2}{\sqrt{3}} - o_k(1)$.

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Hardness Results

Theorem

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Theorem

Let $\mathcal{M}$ be a continuous metric space. Then, it is NP-hard to find a finite set $P \subset \mathcal{M}$ of cardinality k such that the gap ratio of P is at most $\frac{2}{q}$ for some $q > 2$.

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Theorem

It is NP-hard to find a set of k points in a path connected space such that gap ratio is 1.

Outline

$\frac{3}{2}$-Approximation Hardness on graphs

Theorem

In an unweighted graph, it is NP-hard to approximate the gap ratio better than a factor of $\frac{3}{2}$.

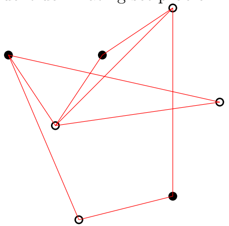
2-Approximation Hardness

Theorem

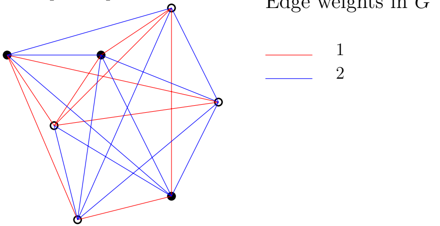
In a general metric space, it is NP-hard to approximate the gap ratio better than a factor of 2.

2-Approximation Hardness

G : Instance for the
independent dominating set problem



G : Instance for the
Gap ratio problem



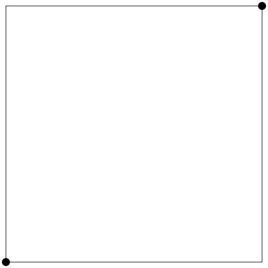
r \ R	1	2
$\frac{1}{2}$	2	4
1	1	2

Outline

Farthest point Method

Start with a pair of farthest points in $\mathcal{M}$. Call this set P_2.

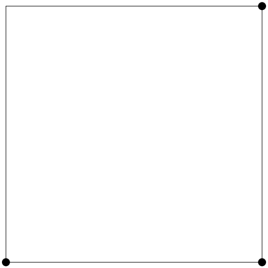
P_2



Farthest point Method

$$P_i \leftarrow P_{i-1} \cup \{p_i\} \text{ s.t., } \delta\left(p_i, P_{i-1}\right) = \sup_{x \in \mathcal{M}} \delta\left(x, P_{i-1}\right)$$

P_3



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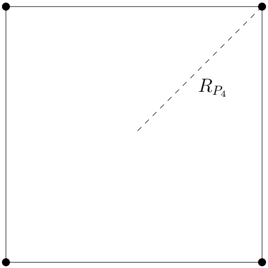
P_4



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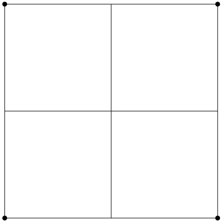


$$GR_{P_i} = \frac{R_{P_i}}{r_{P_i}} = \frac{2R_{P_i}}{R_{P_{i-1}}} \leq 2$$

The approximation result

Without loss of generality, let $P = \{p_1, \ldots, p_k\}$ be the set with optimal gap ratio, and let $GR = \alpha$.

Theorem

Farthest point insertion gives the following approximation guarantees: (i) if $\alpha \geq 1$, then the approximation ratio is $\frac{2}{\alpha} \leq 2$, (ii) if $\frac{2}{3} \leq \alpha < 1$, the approximation ratio is $\frac{2}{\alpha} \leq 3$, and (iii) if $\alpha < \frac{2}{3}$, the approximation ratio is $\frac{4}{2-\alpha} < 3$.

Outline

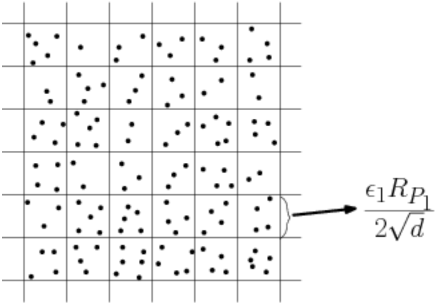
(k, ϵ)-coreset for set of points in $\mathbb{R}^d$

Coreset

- A $(k, \epsilon) - \textit{coreset}$ of $(\mathcal{M}, \delta)$ for the gap ratio problem is a set $(\mathcal{C}, \delta)$, $\mathcal{C} \subset \mathcal{M}$, such that for any $P \subsetneq \mathcal{C}$, $(|P| = k)$,
- $(1 - \epsilon)\, GR_P^{\mathcal{M}} \leq GR_P^{\mathcal{C}} \leq (1 + \epsilon)\, GR_P^{\mathcal{M}}$.

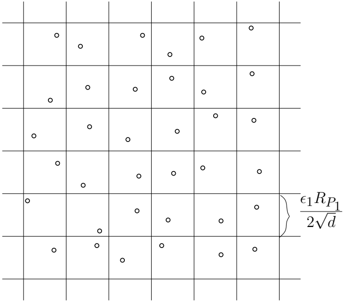
ϵ-coreset for set of points in $\mathbb{R}^d$

Assume $\epsilon \in \left(0, \frac{1}{2}\right)$ and $\epsilon_1 := \frac{\epsilon}{3+2\epsilon} < \epsilon$.
Sample set P_1 ($|P_1| = k$) by the farthest point algorithm take a grid as below.



ϵ-coreset for set of points in $\mathbb{R}^d$

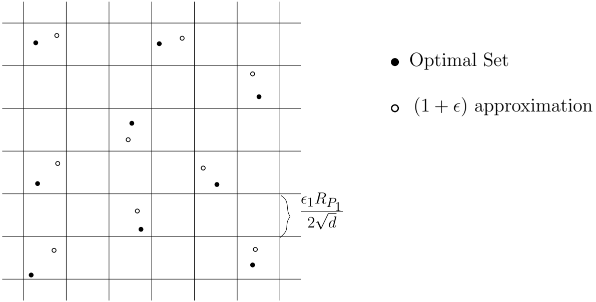
For each cell of the grid choose one point of $\mathcal{M}$.



$R_{\min} \leq R_{P_1} \leq 2R_{\min} \implies O\left(k\left\lceil\frac{1}{\epsilon_1}\right\rceil^d\right)$ points are chosen.

$(1+\epsilon)$-algorithm

This set has k points "close" to the optimal set. So brute force on it gives an $(1+\epsilon)$-approximation of the gap ratio.



Running time of $(1+\epsilon)$-algorithm

- We look at $O\left(\left(k\left\lceil\frac{1}{\epsilon_1}\right\rceil^d\right)^k\right)$ sets.
- Finding closest pair takes $O(k \log k)$.
- Computing the maximum gap takes $O((n-k)k)$ time
- Total runtime: $O\left(k^k\left\lceil\frac{1}{\epsilon_1}\right\rceil^{kd}(k \log k + (n-k)k)\right)$ time.

The runtime is comparable to the best known algorithms of k-center.

$(1+\epsilon)$-algorithm (Streaming)

- In the streaming case the idea is to to maintain a coreset similar to the above.
- The key to the coreset selection: farthest point algorithm gave a constant approximation of the covering radius, i.e., $R_{\min} \leq R_{P_1} \leq 2R_{\min}$
- Doubling algorithm for k-centre has this property in the streaming case. To be precise, $R_{\min} \leq R_{P_1} \leq 8R_{\min}$, which allows us again to choose a grid with cell size $O\left(\frac{R_{P_1}}{2\sqrt{d}}\right)$.
- Thus we again get a core-set by choosing one point per grid cell.

To Sum up

Metric space		Lower Bounds	Hardness	Approximation Factor
General		none	Yes	2-approx. hard
Discrete	connected Graph	$\frac{2}{3}$	Yes	approx. factor 3
				$\frac{3}{2}$-approx. hard
	Euclidean	-	-	$(1+\epsilon)$-approx. algo.
Continuous	Path Connected	1	Yes	approx. factor 2
	Unit Square in $\mathbb{R}^2$	$\frac{2}{\sqrt{3}} - o(1)$	-	approx. factor $\sqrt{3} + o(1)$

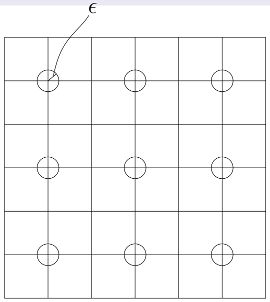
Outline

Randomised methods

Theorem

When n points are sampled using jittered sampling in a unit square, then the gap ratio is $O\left(\sqrt[2d]{n}\right)$ with probability at least $\frac{1}{2}$.

- $m \times m$ grid
- Ball of radius ϵ, centred at alternate grid points
- At most $\left(\frac{m}{d}\right)^d$ balls are considered.
- A_i denotes the event that the i-th ball is hit by two points.



$$
\begin{aligned}
& \mathbb{P}(2r \leq 2\epsilon) \geq \mathbb{P}(\cup A_i) \\
\geq\ & \sum \mathbb{P}(A_i) - \sum \mathbb{P}(A_i \cap A_j) \\
=\ & \sum \binom{2^d}{2} \left(\frac{c_d \epsilon^d}{2^d} / \frac{1}{m^d} \right)^2 - \sum \binom{2^d}{2}^2 \left(\frac{c_d \epsilon^d}{2^d} / \frac{1}{m^d} \right)^4 \\
=\ & \left(\frac{m}{d}\right)^d \binom{2^d}{2} \left(\frac{c_d \epsilon^d}{2^d} / \frac{1}{m^d} \right)^2 - \binom{\left(\frac{m}{d}\right)^d}{2} \binom{2^d}{2}^2 \left(\frac{c_d \epsilon^d}{2^d} / \frac{1}{m^d} \right)^4 \\
=\ & \left(\frac{m}{d}\right)^d \binom{2^d}{2} \left(\frac{c_d \epsilon^d m^d}{2^d} \right)^2 \\
& - \frac{1}{2} \left(\frac{m}{d}\right)^d \left(\left(\frac{m}{d}\right)^d - 1 \right) \binom{2^d}{2}^2 \left(\frac{c_d \epsilon^d m^d}{2^d} \right)^4
\end{aligned}
$$

Set $\epsilon^{2d} = 2^{2d} \binom{2^d}{2}^{-1} m^{-2d} \left(\left(\frac{m}{d}\right)^d - 1 \right)^{-1} c_d^{-2}$.

Thus, $\mathbb{P}\left(2r \leq 2\epsilon\right) \geq \frac{1}{2} \frac{\left(\frac{m}{d}\right)^d}{\left(\left(\frac{m}{d}\right)^d - 1\right)} > \frac{1}{2}$

Use $R \geq \frac{\sqrt{d}}{2m}$

Randomised methods

Theorem

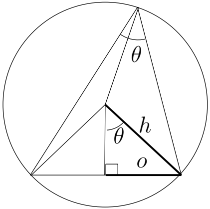
Suppose n points are sampled uniformly at random in a d-dimensional unit cube. Then the gap ratio $\to \infty$ as $n \to \infty$ with probability 1.

A tight lower bound for the unit square

Theorem

Let the point set P be the vertex set of a constrained Delaunay triangulation of the square with $|P|$ points such that the maximum angle is at least $\frac{\pi}{2}$. Then $GR_P \geq \sqrt{2}$.

A tight lower bound for the unit square



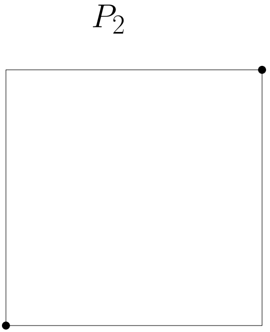
Consider any triangular face Δ of the Delaunay mesh of P and any angle θ of it. Let c be the radius of the circumcircle of Δ.

Proof

- $s \geq r$, $c \leq R$; where r: minimum gap and R: maximum gap.
- $\sin\theta = \frac{s}{c} \geq \frac{r}{R} = \frac{1}{g}$.
- Let $\theta_{\max}$: largest angle; then, $\theta_{\max} \geq \frac{\pi}{2}$.
- Then, $g^{-1} \leq \sin\theta_{\min} \leq \sin\frac{\pi-\theta_{\max}}{2} \leq \sin\frac{\pi}{4} \leq \frac{1}{\sqrt{2}}$

Farthest Point algorithm achieves the lower bound

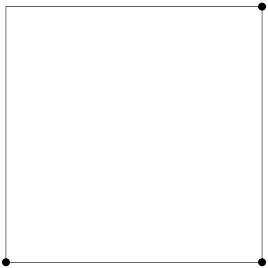
Initialisation: Sample a farthest pair of points.



Farthest Point algorithm achieves the lower bound

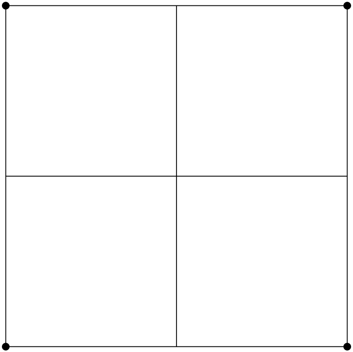
$$P_i \leftarrow P_{i-1} \cup \{p_i\} \text{ s.t., } \delta(p_i, P_{i-1}) = \sup_{x \in \mathcal{M}} \delta(x, P_{i-1})$$

P_3



Farthest Point algorithm achieves the lower bound

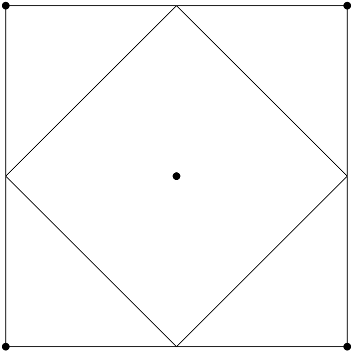
We continue for a few iterations.



Gap ratio $= \sqrt{2}$.

Farthest Point algorithm achieves the lower bound

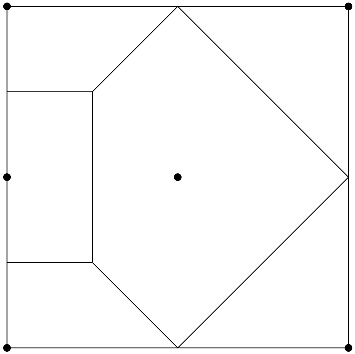
We continue for a few iterations.



Gap ratio $= \sqrt{2}$.

Farthest Point algorithm achieves the lower bound

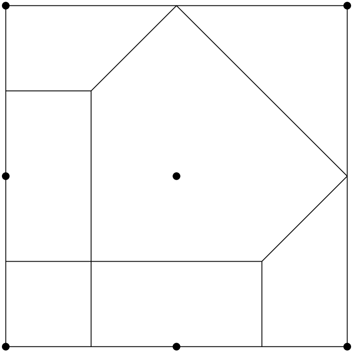
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

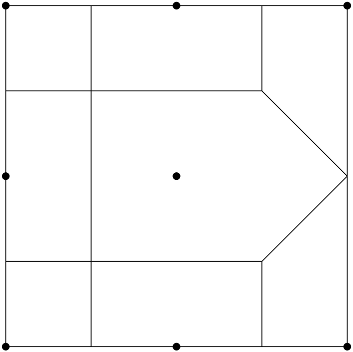
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

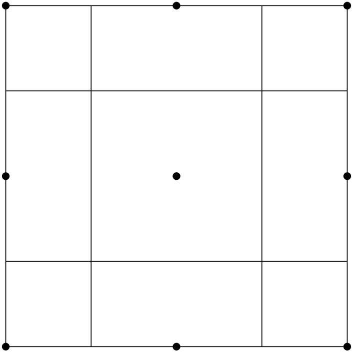
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

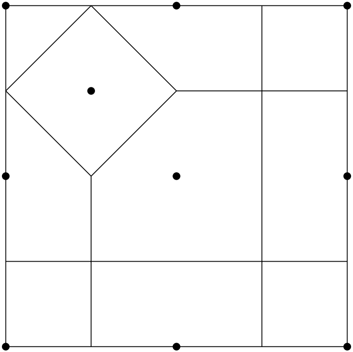
We continue for a few iterations.



Gap ratio $= \sqrt{2}$.

Farthest Point algorithm achieves the lower bound

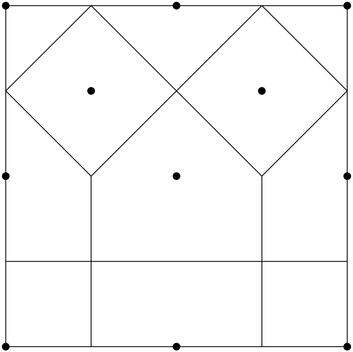
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

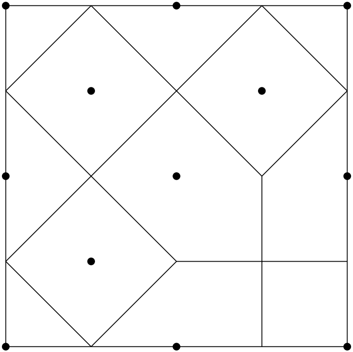
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

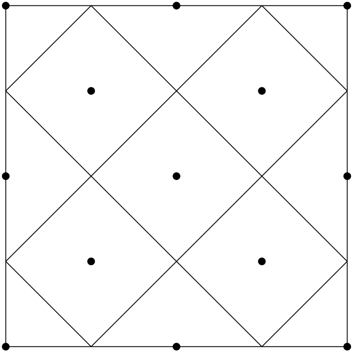
We continue for a few iterations.



Gap ratio $= 2$.

Farthest Point algorithm achieves the lower bound

We continue for a few iterations.



Gap ratio $= \sqrt{2}$.

Farthest Point algorithm achieves the lower bound

Note that

- Every new point is a Voronoi vertex of the point set from the previous iteration.
- Gap ratio becomes $\sqrt{2}$ when the Voronoi digram tiles the unit square with congruent square cells.
- Point samples of size n, where $n = k^2$ or $n = k^2 + (k-1)^2$ for some k achieve $\sqrt{2}$.

Theorem

If the set P of size $n \geq 4$ points is sampled from the unit square using the farthest point algorithm then the gap ratio is $\sqrt{2}$ if and only if $n = G_j$ where,

$$G_j = \begin{cases} 4^{i-1} + 2^i + 1 & j = 2i - 1 \\ 2 \cdot 4^{i-1} + 2^i + 1 & j = 2i \end{cases}$$

for $i \in \mathbb{N}$.

Other point sets which achieve the lower bound

Not all the grid patterns occur in the farthest point pattern, for eg.:- the 4×4 grid does not occur in the farthest point method. Also grid patterns which do not have the corners of the square do not occur in the farthest point algorithm.

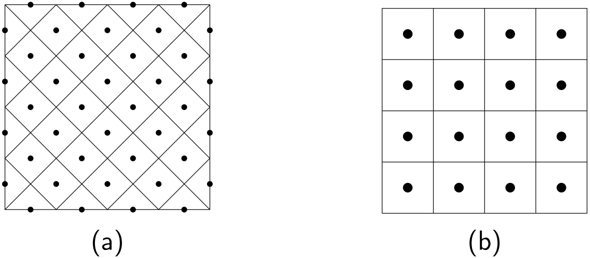


Figure: (a) Point sample of size $40 = 2 \cdot 5 \cdot 4 = 2 \cdot k \cdot (k-1)$ with gap ratio $= \sqrt{2}$; (b) A 4×4 sample with no point on the boundary

Lattices: 1-parameter family

$$\Delta : \mathbb{R}^d \to \mathbb{R}$$
$$\Delta(x) = \langle x, \mathbf{1} \rangle = \sum_{i=1}^{d} x_i$$
$$D = \frac{\delta - 1}{d}$$
$$T_\delta : \mathbb{R}^d \to \mathbb{R}^d$$
$$x \mapsto x + D\Delta(x) . \mathbf{1}$$
$$L_\delta := T_\delta\left(\mathbb{Z}^d\right)$$

Lattices: 1-parameter family

$$g^2 = \begin{cases} \frac{(d^2-1)+(d^2+2)\delta^2+(d^2-1)\delta^4}{3d^2\delta^2} & 0 \le \delta \le \frac{1}{\sqrt{d+1}} \\ \frac{(d^2-1)+(d^2+2)\delta^2+(d^2-1)\delta^4}{3(d+\delta^2-1)} & \frac{1}{\sqrt{d+1}} \le \delta \le 1 \\ \begin{cases} \frac{d^2-1+\delta^2}{d+\delta^2-1} & d \text{ is odd} \\ \frac{d^2-2+\delta^2}{d+\delta^2-1} + \frac{1}{\delta^2(d+\delta^2-1)} & d \text{ is even} \end{cases} & 1 \le \delta \le \sqrt{d+1} \\ \begin{cases} \frac{d^2-1+\delta^2}{2d} & d \text{ is odd} \\ \frac{d^2-2+\delta^2}{2d} + \frac{1}{2d\delta^2} & d \text{ is even} \end{cases} & \sqrt{d+1} \le \delta \end{cases}$$

Lattices: Random construction

Theorem

Suppose the fundamental region of a lattice is obtained by using two unit vectors an an angle chosen uniformly at random from $\left[0, \frac{\pi}{2}\right]$. Then,

$$\mathbb{P}\left(\textit{Gap Ratio of lattice} \geq g\right) = \begin{cases} \frac{1}{2} - \frac{2}{\pi}\arccos\left(\frac{2}{g^2} - 1\right) + \frac{2}{\pi}\arcsin\left(\frac{1}{g}\right) & g \in \left(\frac{2}{\sqrt{3}}, \sqrt{2}\right) \\ \frac{2}{\pi}\arcsin\left(\frac{1}{g}\right) & g \geq \sqrt{2} \end{cases}$$

Outline

Gap Ratio and discrepancy (1D)

Discrepancy in 1 dimension

$$D_N = \sup_{0 \leq a < b \leq 1} \left| b - a - \frac{A\left([a, b), P\right)}{N} \right|$$

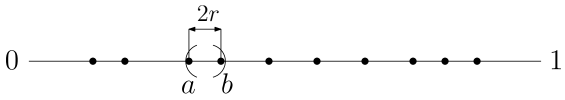
Theorem

For N points in the interval $[0, 1]$ with discrepancy $D_N < \frac{2}{n}$, $GR \leq \frac{D_N}{\frac{1}{N} - \frac{D_N}{2}} = \frac{4}{2 - ND_N} - 2$.

Gap Ratio and discrepancy (1D)

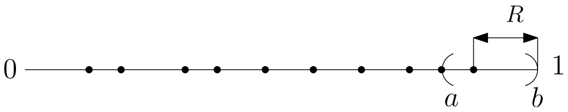
Proof

We can bound r by looking at the smallest interval with exactly 2 points i.e., $-D_N \leq 2r - \frac{2}{N}$. Thus, $r \geq \frac{1}{N} - \frac{D_N}{2}$.



Proof

We can bound the largest interval $[a, b)$ which contains exactly no point as, $b - a \leq D_N$. Thus, we have $R \leq D_N$.



Thanks !