Discrete Controller Synthesis for Logico-numerical Systems with ReaX

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Ctrl-A Seminar

Outline

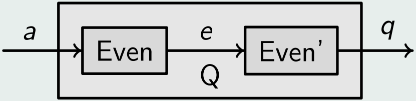
- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
- Discrete Controller Synthesis Principles
- ReaX: Technical Choices, Implementation & Evaluations
- Conclusions

Outline

- Discrete Controller Synthesis for Logico-numerical Programs
  - Boolean Reactive/Synchronous Programs
  - Logico-numerical Reactive/Synchronous Programs
  - Discrete Controller Synthesis Problem Statement
- State of the Art
- Discrete Controller Synthesis Principles
- ReaX: Technical Choices, Implementation & Evaluations
- Conclusions

Example Finite Reactive/Synchronous Program

Synchronous Circuit



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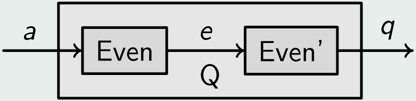
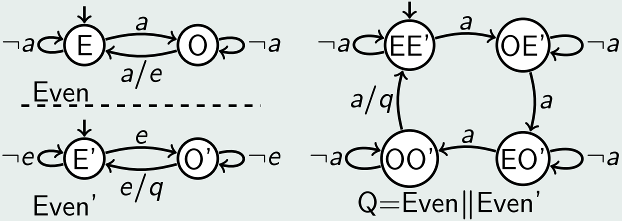


Automata



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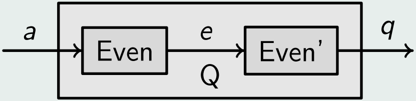

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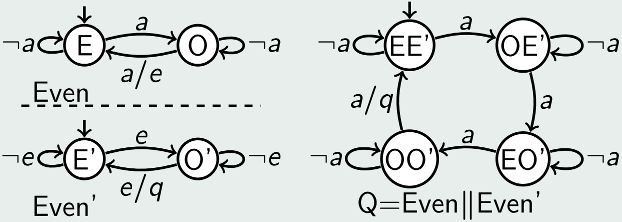


Equations

$$\rightsquigarrow \quad \begin{cases} x' = (\neg x \wedge a) \vee (x \wedge \neg a) & e = x \wedge a & \neg x_0 & \leftarrow \text{Even} \\ y' = (\neg y \wedge e) \vee (y \wedge \neg e) & q = y \wedge e & \neg y_0 & \leftarrow \text{Even'} \end{cases}$$

Example Finite Reactive/Synchronous Program (cont'd)

Equations

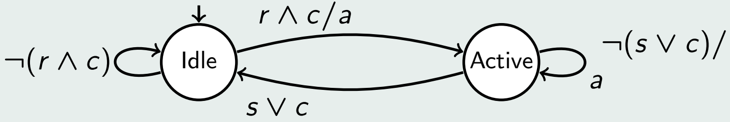
$$\begin{cases} x' = (\neg x \wedge a) \vee (x \wedge \neg a) & e = x \wedge a & \neg x_0 \\ y' = (\neg y \wedge e) \vee (y \wedge \neg e) & q = y \wedge e & \neg y_0 \end{cases}$$

Textual Form (nbac Format)

```
state x, y: bool;                              (* State Variables *)
      q: bool;                                        (* (Output) *)
input a: bool;
local e: bool;          (* Substituted in Expressions Below *)
definition
  e = x and a;
transition (* Transition Function, Assigning Every State *)
  x' = (not x and a) or (x and not a);       (* ... Variable *)
  y' = (not y and e) or (y and not e);
  q' = y and e;
initial not x and not y;      (* Initial State(s) Predicate *)
assertion true;                          (* Assertion Predicate *)
```

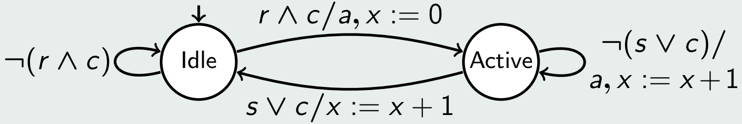
Example Logico-numerical Reactive/Synchronous Program

Example Logico-numerical Task Model



Example Logico-numerical Reactive/Synchronous Program

Example Logico-numerical Task Model



Expressing Reactive/Synchronous Programs

Definition (Symbolic Transition System — STS)

$S = \langle X, I, T, A, \Theta_0 \rangle$ where:

- $X = \langle x_1, \ldots, x_n \rangle$ ← Vector of State Variables
- $I = \langle i_1, \ldots, i_m \rangle$ ← Vector of Input Variables
- $T = \langle t_1, \ldots, t_n \rangle$, t_i: Expression on $X \cup I$ ← Transition Function
- A: Predicate on $X \cup I$ ← Assertion
- Θ_0: Predicate on X ← Initial State(s)

Definition (Arithmetic Symbolic Transition System — ASTS)

STS S $(= \langle X, I, T, A, \Theta_0 \rangle)$ where:

- $X = \mathbb{B}^s \cup \mathbb{R}^{s'} \cup \mathbb{Z}^{s''}$ $(s + s' + s'' = n)$
- $I = \mathbb{B}^t \cup \mathbb{R}^{t'} \cup \mathbb{Z}^{t''}$ $(t + t' + t'' = m)$
- T, A and Θ_0 Involve Arithmetic Expressions

Reactive/Synchronous Program Invariant

Definition (Invariant Property for an STS)

Given an ASTS $S = \langle X, I, T, A, \Theta_0 \rangle$, a Predicate Φ over X is an *Invariant* of S (Noted $S \models \Phi$) iff:

- "All Executions of S Remain in States Satisfying Φ" *i.e.,*

$\forall x_0 \in \mathcal{D}_X$ ← Initial State

$\forall (\iota_0, \ldots, \iota_p) \in \mathcal{D}_I^p$ ← Sequence of p Vectors of Inputs

$$\Theta_0(x_0) \wedge \forall i \in [0, p], A(T(\ldots T(x_0, \iota_0) \ldots, \iota_i))$$

$$\Rightarrow \forall i \in [0, p], \Phi(T(\ldots T(x_0, \iota_0) \ldots, \iota_i))$$

Discrete Controller Synthesis (DCS) Problem

- Initiated by Ramadge and Wonham 1989[1] (Formal Languages Setting)

Controller Synthesis Problem for Invariant Enforcement in ASTSs

Given an ASTS $S = \langle X, I_{uc} \uplus I_c, T, A, \Theta_0 \rangle$ and an Invariant Φ over X, Compute a Predicate A_Φ such that:

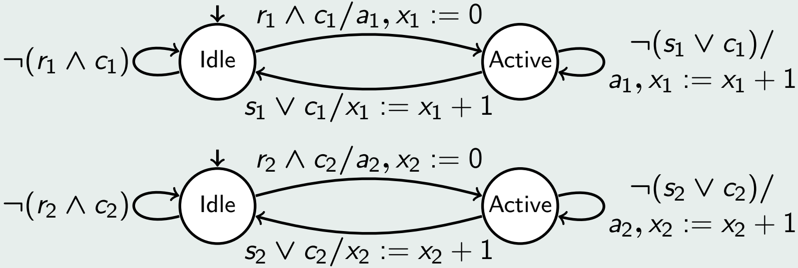
$$S' = \langle X, I_{uc} \uplus I_c, T, A_\Phi, \Theta_0 \rangle \models \Phi$$

- I_{uc} $\leftarrow$ *Non-controllable Input Variables*
- I_c $\leftarrow$ *Controllable Input Variables*

[1] Peter J. G. Ramadge and W. Murray Wonham. "The control of discrete event systems". In: *Proceedings of the IEEE* 77.1 (Jan. 1989), pp. 81–98.

Discrete Controller Synthesis (DCS) Problem (cont'd)

Example DCS Problem for a Logico-numerical Program



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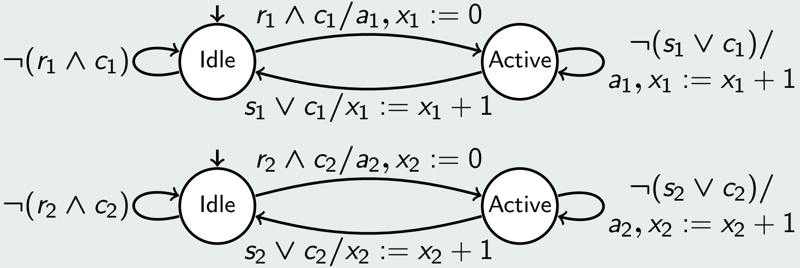
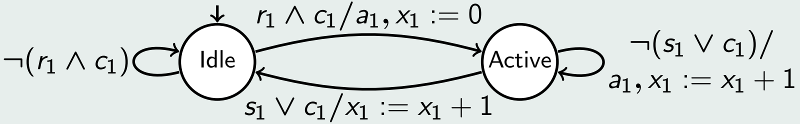
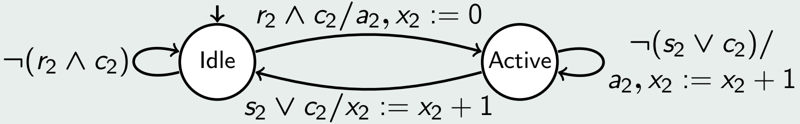


- $X = \langle t_1, t_2, x_1, x_2, a_1, a_2 \rangle$ $\qquad \mathcal{D}_X = \{\text{Idle}, \text{Active}\}^2 \times \mathbb{Z}^2 \times \mathbb{B}^2$
- $I_{uc} = \langle r_1, r_2, s_1, s_2 \rangle, I_c = \langle c_1, c_2 \rangle$ $\qquad \mathcal{D}_{I_{uc}} = \mathbb{B}^4,\ \mathcal{D}_{I_c} = \mathbb{B}^2$

Discrete Controller Synthesis (DCS) Problem (cont'd)

Example DCS Problem for a Logico-numerical Program





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- $I_{uc} = \langle r_1, r_2, s_1, s_2 \rangle, I_c = \langle c_1, c_2 \rangle$ $\mathcal{D}_{I_{uc}} = \mathbb{B}^4, \mathcal{D}_{I_c} = \mathbb{B}^2$
- Enforcing Mutual Exclusion Between Active States:
$$\Phi(\langle t_1, t_2, x_1, x_2, a_1, a_2 \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle})$$
- Enforcing Bounds on x_i's:
$$\Phi(\langle t_1, t_2, x_1, x_2, a_1, a_2 \rangle) = (x_1 \leqslant 10 \wedge x_2 \leqslant 10)$$

Outline

- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
  - For Other Kinds of Models
  - For Synchronous Languages
  - For Infinite-State Systems
- Discrete Controller Synthesis Principles
- ReaX: Technical Choices, Implementation & Evaluations
- Conclusions

State of the Art: Actual Tools

- Petrify[2]
- SynthKro[3]
- Supremica[4]
- Uppaal-tiga[5]
- ...

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[2] Jordi Cortadella et al. "Petrify: a tool for manipulating concurrent specifications and synthesis of asynchronous controllers". In: *IEICE Transactions on Information and Systems* 80.3 (1997), pp. 315–325.

[3] Karine Altisen and Stavros Tripakis. "Tools for controller synthesis of timed systems". In: *Proceedings of the 2nd Workshop on Real-Time Tools (RT-TOOLS'02)*. 2002, pp. 2002–025.

[4] Knut Akesson et al. "Supremica: an integrated environment for verification, synthesis and simulation of discrete event systems". In: *8th International Workshop on Discrete Event Systems*. IEEE. 2006, pp. 384–385.

[5] Gerd Behrmann et al. "Uppaal-tiga: Time for playing games!" In: *Computer Aided Verification*. Springer. 2007, pp. 121–125.

State of the Art: Actual Tools for Synchronous Languages

- Sigali[6]
  - Polynomial Dynamical Systems ($\mathbb{Z}/3\mathbb{Z}$)
  - Symbolic Computations using *Ternary Decision Diagrams*
  - Finite Systems
- $\ldots\!\!\!/\,.$

[6] Hervé Marchand et al. "Synthesis of Discrete-Event Controllers based on the Signal Environment". In: *Discrete Event Dynamic System: Theory and Applications* 10.4 (Oct. 2000), pp. 325–346.

State of the Art: Actual Tools for Infinite-State Systems

- Smacs[7]
  - *Explicit* Control Flow Graph (*e.g.*, $X = \mathit{Locations} \cup \mathbb{Z}^n$)
    - $\rightsquigarrow$ Boolean State Space Explosion
  - *Partial Observability* of the Variables
  - Using *Abstract Interpretation Techniques*

[7] Gabriel Kalyon et al. "Symbolic Supervisory Control of Infinite Transition Systems under Partial Observation using Abstract Interpretation". In: *Discrete Event Dynamic Systems : Theory and Applications* 22.2 (2011), pp. 121–161.

Outline

- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
- Discrete Controller Synthesis Principles
  - Infinite Transition Systems
  - Algorithmic Principle
  - Finite Case
  - Infinite Case
- ReaX: Technical Choices, Implementation & Evaluations
- Conclusions

Infinite Transition Systems

- Reasoning about State Spaces

Definition (Controllable Infinite Transition System of an ASTS)

One Associates to an ASTS $S = \langle X, I_{uc} \uplus I_c, T, A, \Theta_0 \rangle$ a *Controllable Infinite Transition System* $[S] = \langle \mathcal{X}, \mathcal{I}, \mathcal{T}_S, \mathcal{A}_S, \mathcal{X}_0 \rangle$ where:

- $\mathcal{X} = \mathcal{D}_X$ ← State Space
- $\mathcal{I} = \mathcal{U} \times \mathcal{C}$
  - $\mathcal{U} = \mathcal{D}_{I_{uc}}$ ← Non-controllable Input Space
  - $\mathcal{C} = \mathcal{D}_{I_c}$ ← Controllable Input Space
- $\mathcal{T}_S \subseteq \mathcal{X} \times \mathcal{I} \to \mathcal{X}$ ← Transition Function

  $= \lambda(x, \iota).\,(t_j(x, \iota))_{j \in [1,n]}$
- $\mathcal{A}_S \subseteq \mathcal{X} \times \mathcal{I}$ ← Assertion on Environment

  $= \{(x, \iota) \mid A(x, \iota) = true\}$
- $\mathcal{X}_0 \subseteq \mathcal{X}$ ← Initial States

  $= \{x \in \mathcal{X} \mid \Theta_0(x) = true\}$

Discrete Controller Synthesis: Algorithmic Principle

- Let $Bad \stackrel{\text{def}}{=} \{x \in \mathcal{X} \mid \Phi(x) = \mathit{false}\}$ $\leftarrow$ States to Avoid
- Computing States I_{Bad} "Uncontrollably" Reaching Bad $\leftarrow$ Fix-point
  - Using Pre-image Function $\mathcal{T}_S^{-1} : \wp(\mathcal{X}) \to \wp(\mathcal{X} \times \mathcal{U} \times \mathcal{C})$

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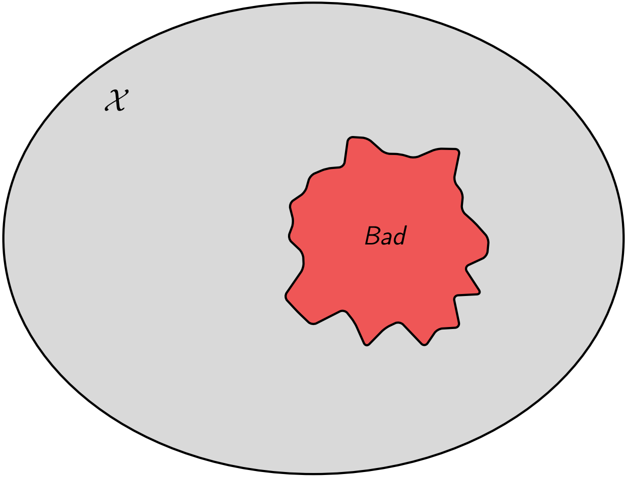
Finite Case

- State Variables on *Finite Domains* (*e.g.*, $\mathcal{X} = \mathbb{B}^n$)

Discrete Controller Synthesis Principle in the Finite Case



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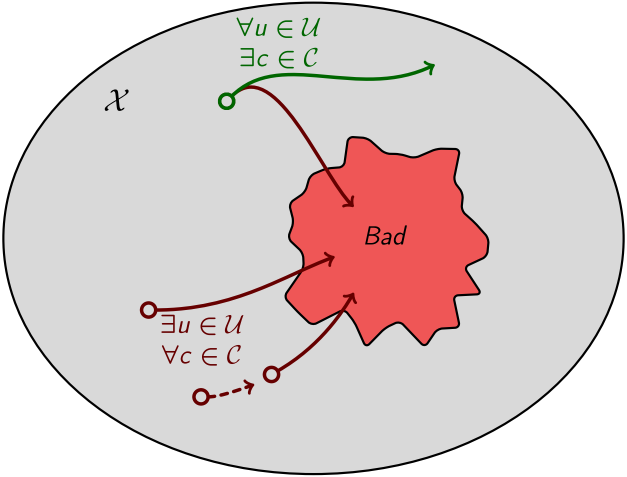

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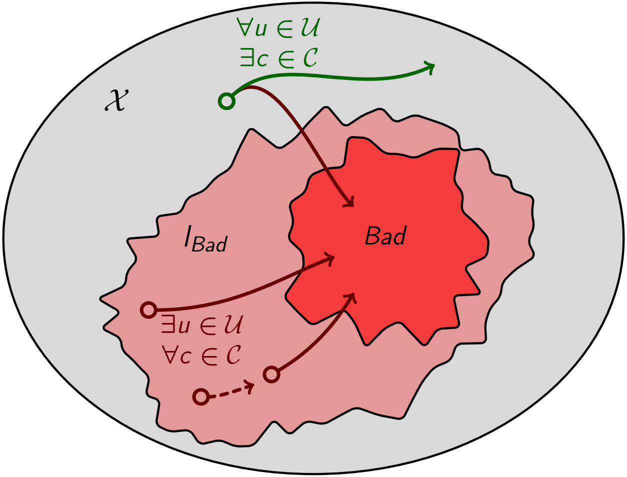

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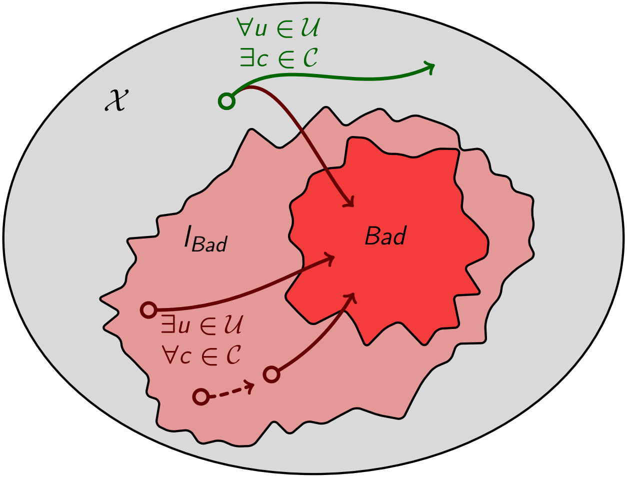

$I_{Bad} = \mathrm{coreach}_u(Bad)$ $\qquad$ $\mathrm{coreach}_u(B) \stackrel{\text{def}}{=} \mathrm{lfp}(\lambda\beta.B \cup \mathrm{pre}_u(\beta))$

$$\mathrm{pre}_u(B) \stackrel{\text{def}}{=} \left\{x \in \mathcal{X} \,\middle|\, \exists u \in \mathcal{U}, \forall c \in \mathcal{C}, (x, u, c) \in \mathcal{T}_S^{-1}(B) \cap \mathcal{A}_S\right\}$$

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Finite Case

- State Variables on *Finite Domains* $(e.g.,\ \mathcal{X} = \mathbb{B}^n)$

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  - Using Pre-image Function $\mathcal{T}_S^{-1} : \wp(\mathcal{X}) \to \wp(\mathcal{X} \times \mathcal{U} \times \mathcal{C})$
- Success iff $\mathcal{X}_0 \cap I_{Bad} = \varnothing$

  $\rightsquigarrow$ New Predicate ← Solution to the DCS Problem

Finite Case

- State Variables on *Finite Domains* (*e.g.*, $\mathcal{X} = \mathbb{B}^n$)

$\rightsquigarrow$ *Maximally Permissive Controller*

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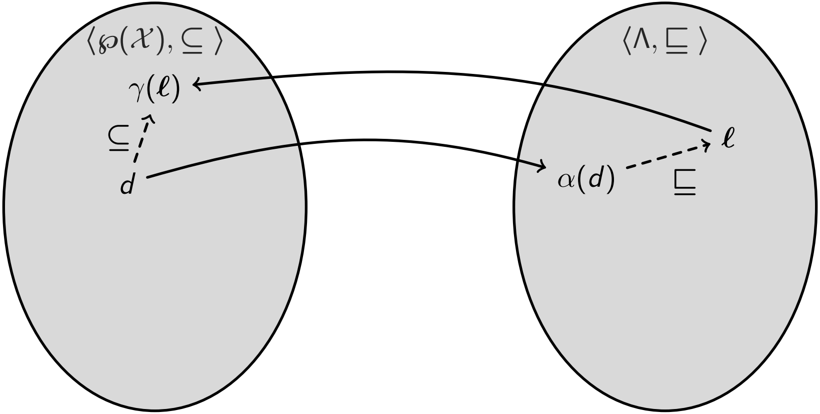
$\rightsquigarrow$ *Maximally Permissive Controller*

Infinite Case

- Allowing *Numerical* State Variables $(e.g., \mathcal{X} = \mathbb{B}^n \times \mathbb{Z}^m)$
- Undecidability Problem $\rightsquigarrow$ "Over-approximating Synthesis"

Abstract Interpretation Principles



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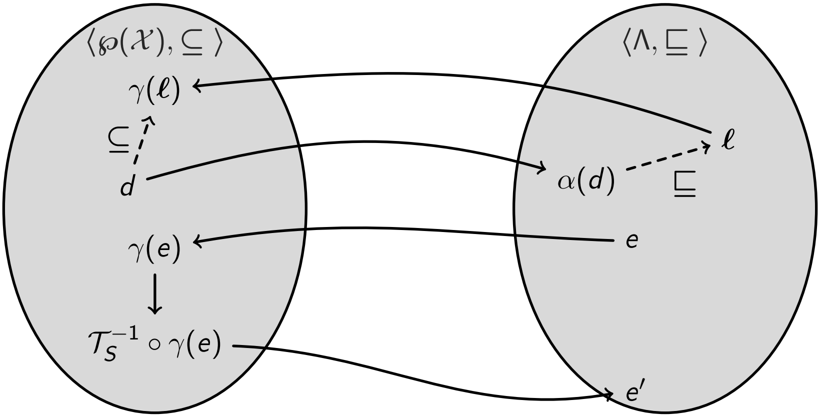

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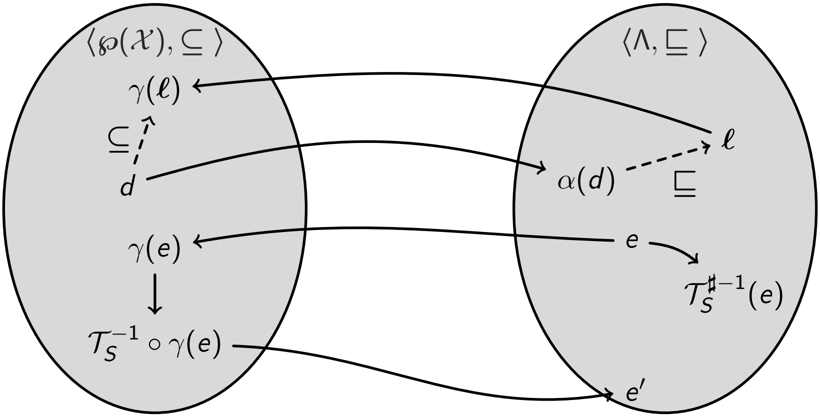

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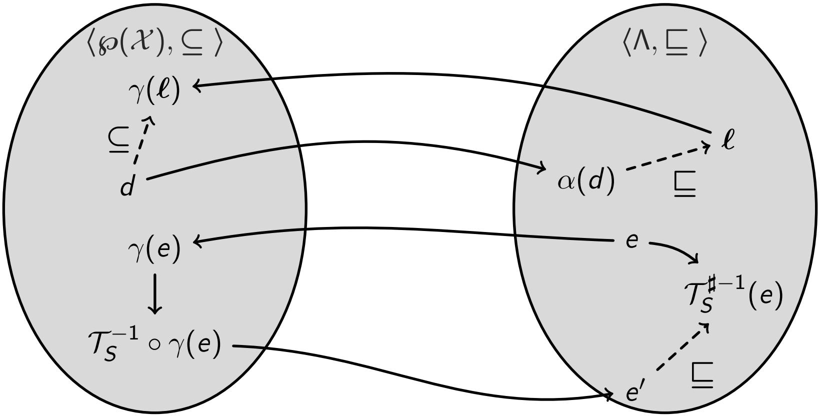

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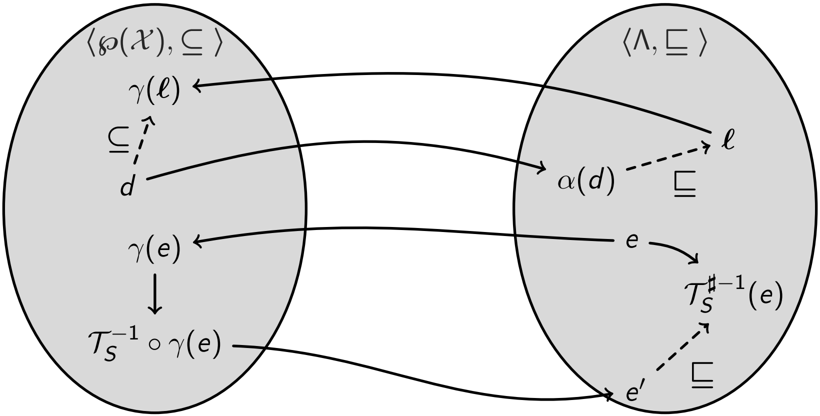
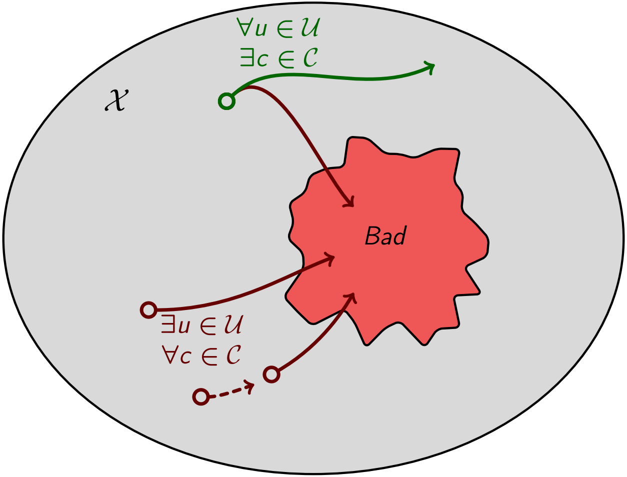


Requirements

- $\langle \Lambda, \sqsubseteq, \sqcup, \sqcap, \top, \bot \rangle$, α and γ such that: $\wp(\mathcal{X}) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \Lambda$; $+\ \mathcal{T}_S^{\sharp-1}$

\+ Widening Operator $\nabla : \Lambda \times \Lambda \rightarrow \Lambda$ $\leftarrow$ Forcing Convergence

Over-approximating Discrete Controller Synthesis



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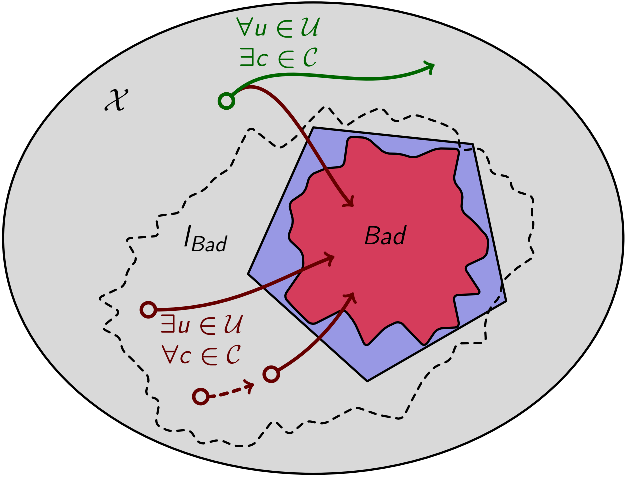
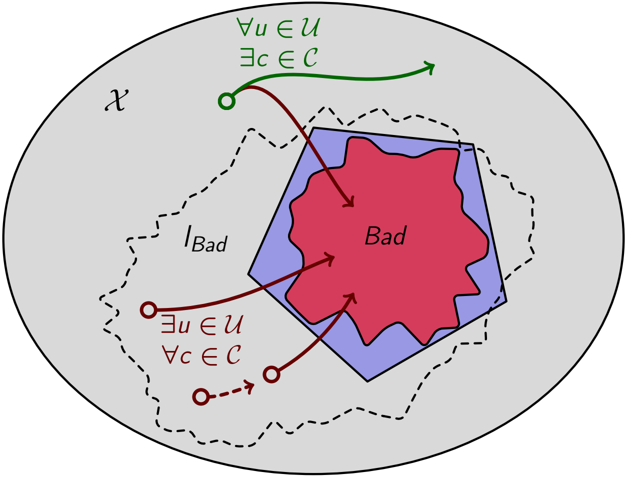


- Computing $I'_{Bad} \supseteq I_{Bad}$

Over-approximating Discrete Controller Synthesis



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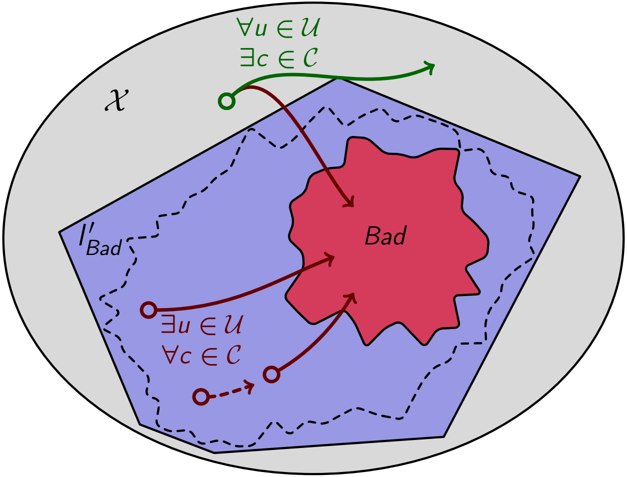
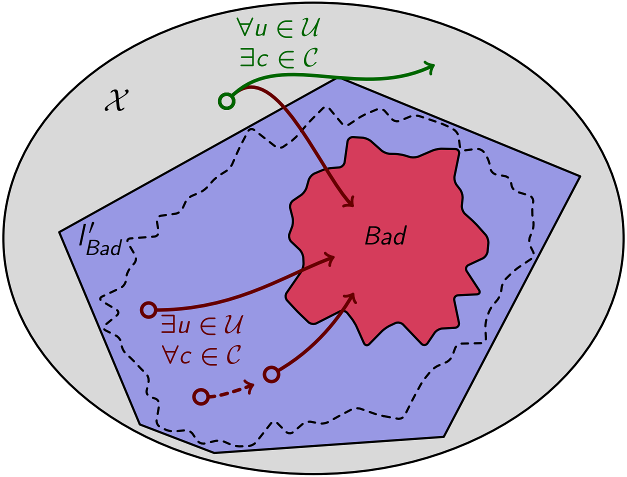


$I'_{Bad} = \gamma \circ \mathrm{coreach}^\sharp_u \circ \alpha(Bad)$ $\qquad$ $\mathrm{coreach}^\sharp_u(B) \stackrel{\text{def}}{=} \mathrm{lfp}(\lambda\beta.B \sqcup \mathrm{pre}^\sharp_u(\beta))$

$$\mathrm{pre}^\sharp_u(B) \stackrel{\text{def}}{=} \exists^\sharp_{\mathcal{U}}\left(\forall^\sharp_{\mathcal{C}}\left(\mathcal{T}_S^{\sharp-1}(B) \sqcap \alpha(\mathcal{A}_S)\right)\right)$$

Over-approximating Discrete Controller Synthesis



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- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
- Discrete Controller Synthesis Principles
- ReaX: Technical Choices, Implementation & Evaluations
  - Over-approximating Logico-numerical State Spaces
  - Further Technical Choices
  - Implementation Details
  - Performance Comparison with Sigali
  - Evaluations of Over-approximating Synthesis
- Conclusions

Over-approximating Logico-numerical State Spaces

Over-approximating Numerical Spaces (*e.g.*, $\mathcal{X} = \mathbb{Z}^n \times \mathbb{R}^m$)

Numerical Domains Provide α, γ and $\mathcal{N}$ such that $\wp(\mathbb{Z}^n \times \mathbb{R}^m) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{N}$

[8] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

Over-approximating Logico-numerical State Spaces

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- *Intervals*:

$$\bigwedge_{i \in [1, n+m]} (a_i \leqslant v_i \leqslant b_i)$$

- *Convex Polyhedra*: Conjunction of k Linear Constraints of the Form:

$$\left(\sum_{i \in [1, n+m]} a_i v_i\right) \leqslant b$$

[8] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

Over-approximating Logico-numerical State Spaces

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- *Convex Polyhedra*: Conjunction of k Linear Constraints of the Form:

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Over-approximating Logico-numerical State Spaces (*e.g.*, $\mathcal{X} = \mathbb{B}^n \times \mathbb{Z}^m$)

Composing: One Numerical Domain $\mathcal{N}$ & One Boolean Domain[8]

[8] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

Over-approximating Logico-numerical State Spaces

Over-approximating Numerical Spaces (*e.g.*, $\mathcal{X} = \mathbb{Z}^n \times \mathbb{R}^m$)

Numerical Domains Provide α, γ and $\mathcal{N}$ such that $\wp(\mathbb{Z}^n \times \mathbb{R}^m) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathcal{N}$

- *Intervals*:

$$\bigwedge_{i\in[1,n+m]}(a_i \leqslant v_i \leqslant b_i)$$

- *Convex Polyhedra*: Conjunction of k Linear Constraints of the Form:

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- *Product*: $\wp(\mathbb{B}^n \times \mathbb{Z}^m \times \mathbb{R}^o) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \wp(\mathbb{B}^n) \times \mathcal{N}$

[8] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

Over-approximating Logico-numerical State Spaces

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Over-approximating Logico-numerical State Spaces (*e.g.*, $\mathcal{X} = \mathbb{B}^n \times \mathbb{Z}^m$)

Composing: One Numerical Domain $\mathcal{N}$ & One Boolean Domain[8]

- *Product*: $\wp(\mathbb{B}^n \times \mathbb{Z}^m \times \mathbb{R}^o) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \wp(\mathbb{B}^n) \times \mathcal{N}$
- *Power*: $\wp(\mathbb{B}^n \times \mathbb{Z}^m \times \mathbb{R}^o) \underset{\alpha}{\overset{\gamma}{\leftrightarrows}} \mathbb{B}^n \to \mathcal{N}$ $(= \mathcal{N}^{\mathbb{B}^n})$

[8] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

Further Technical Choices

Restricting to $\mathcal{C} = \mathbb{B}^p$

- Computing the Universal Quantification in the Concrete Domain

$$\mathrm{pre}_u^{\sharp\prime}(B) \stackrel{\text{def}}{=} \exists^\sharp_{\mathcal{U}} \alpha \left\{ (x,u) \,\middle|\, \forall c \in \mathcal{C}, (x,u,c) \in \gamma\left(\mathcal{T}_S^{\sharp-1}(B) \sqcap \alpha(\mathcal{A}_S) \right) \right\}$$

- Permits *Triangularization* of the Controller[9]

[9] Gwenaël Delaval, Hervé Marchand, and Éric Rutten. "Contracts for modular discrete controller synthesis". In: *Proceedings of the Conference on Languages, Compilers, and Tools for Embedded Systems*. Stockholm, Sweden, 2010, pp. 57–66.

ReaX: Implementation Details

Extension of ReaVer[10], Itself Exploiting BddApron[11]

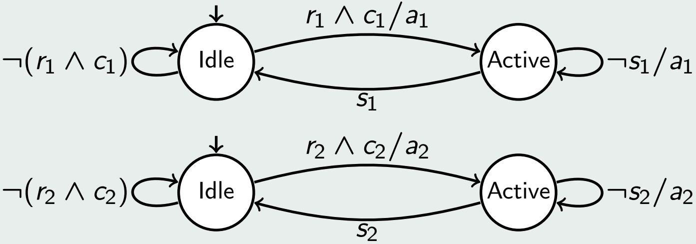
- Decision Diagrams: CUDD
- Numerical Abstract Domains of APRON; *e.g.*,
  - Intervals
  - Convex Polyhedra
- Combining *Binary Decision Diagrams* with *Numerical Abstract Domains*
  - $\rightsquigarrow$ Product $(\mathbb{B}^n \times \mathcal{N})$, Power $(\mathbb{B}^n \to \mathcal{N})$
- Pre-processing Features
  - Partial Control Flow Graph Generation
  - $\rightsquigarrow$ Improving Precision
- BZR Backend

[10] Peter Schrammel. "Logico-Numerical Verification Methods for Discrete and Hybrid Systems". PhD thesis. University of Grenoble, 2012.

[11] Bertrand Jeannet. *BddApron: A logico-numerical abstract domain library*. 2009. url: http://pop-art.inrialpes.fr/~bjeannet/bjeannet-forge/bddapron/.

Performance Comparison with SIGALI

Benchmark



- $X = \langle t_1, t_2, a_1, a_2 \rangle$ $\qquad \mathcal{D}_X = \{\text{Idle}, \text{Active}\}^2 \times \mathbb{B}^2$
- $I_{uc} = \langle r_1, r_2, s_1, s_2 \rangle$, $I_c = \langle c_1, c_2 \rangle$ $\qquad \mathcal{D}_{I_{uc}} = \mathbb{B}^4$, $\mathcal{D}_{I_c} = \mathbb{B}^2$
- Enforcing Mutual Exclusion Between Active States

$$\Phi(\langle t_1, t_2, \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle})$$

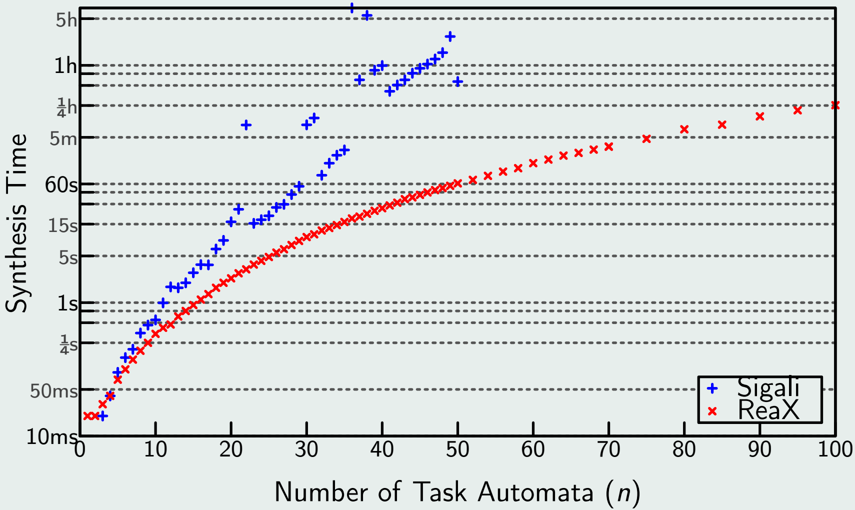
Performance Comparison with SIGALI

Benchmark

```
typedef State = enum {Idle, Active};
state t1, t2: State;                          (* State Variables *)
input r1, r2, s1, s2: bool;       (* Non-controllable Inputs *)
controllable c1, c2: bool;            (* Controllable Inputs *)
transition
  t1' = if t1 = Idle   and r1 and c1 then Active else
        if t1 = Active and s1        then Idle   else t1;
  t2' = if t2 = Idle   and r2 and c2 then Active else
        if t2 = Active and s2        then Idle   else t2;
initial t1 = Idle and t2 = Idle;
invariant t1 <> t2 or t1 = Idle;       (* To be Enforced (Φ) *)
```

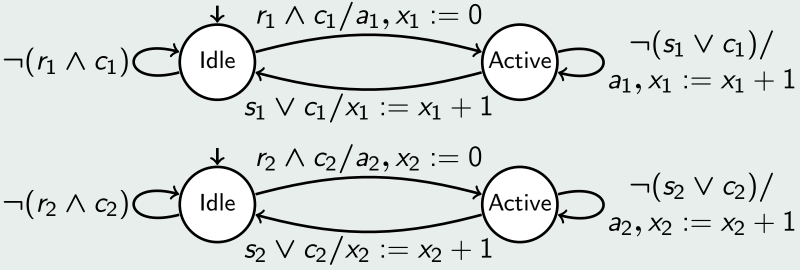
Performance Comparison with Sigali (cont'd)

Comparing Execution Times



Example Logico-numerical DCS Problem

Example Logico-numerical Program & Invariant Properties



- $X = \langle t_1, t_2, x_1, x_2, a_1, a_2 \rangle$ $\quad\mathcal{D}_X = \{\text{Idle}, \text{Active}\}^2 \times \mathbb{Z}^2 \times \mathbb{B}^2$
- $I_{uc} = \langle r_1, r_2, s_1, s_2 \rangle, I_c = \langle c_1, c_2 \rangle$ $\quad\mathcal{D}_{I_{uc}} = \mathbb{B}^4, \mathcal{D}_{I_c} = \mathbb{B}^2$
- Enforcing Mutual Exclusion Between Active States

$$\Phi(\langle t_1, t_2, x_1, x_2 \dots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle})$$

Example Logico-numerical DCS Problem

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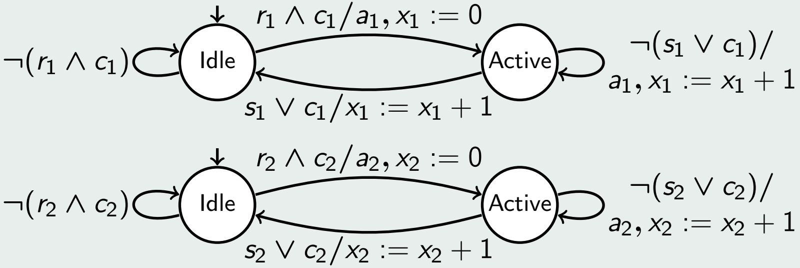

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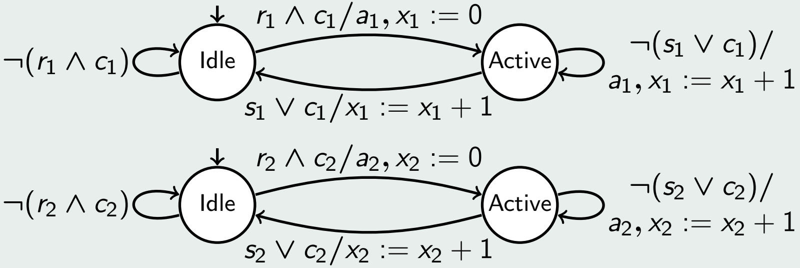
- Enforcing Mutual Exclusion Between Active States & Bounds on x_i's

$$\Phi(\langle t_1, t_2, x_1, x_2 \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle}) \wedge (x_1 \leqslant 10 \wedge x_2 \leqslant 10)$$

Example Logico-numerical DCS Problem

Example Logico-numerical Program & Invariant Properties



- $X = \langle t_1, t_2, x_1, x_2, a_1, a_2 \rangle$ $\quad\mathcal{D}_X = \{\text{Idle}, \text{Active}\}^2 \times \mathbb{Z}^2 \times \mathbb{B}^2$
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- Enforcing Mutual Exclusion Between Active States & Bounds on x_i's & Relational Constraints on x_i's

$$\Phi(\langle t_1, t_2, x_1, x_2 \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle}) \wedge (x_1 \leqslant 10 \wedge x_2 \leqslant 10) \wedge (x_1 \leqslant x_2)$$

Example Logico-numerical DCS Problem

Example Logico-numerical Program & Invariant Properties

```
typedef State = enum {Idle, Active};
state t1, t2: State;                               (* State Variables *)
      x1, x2: int;
input r1, r2, s1, s2: bool;              (* Non-controllable Inputs *)
controllable c1, c2: bool;                   (* Controllable Inputs *)
transition
  t1' = if t1 = Idle   and  r1 and c1  then Active else
        if t1 = Active and (s1 or  c1) then Idle   else t1;
  x1' = if t1 = Idle   and  r1 and c1  then 0      else
        if t1 = Active                 then x1 + 1 else x1;
  t2' = if t2 = Idle   and  r2 and c2  then Active else
        if t2 = Active and (s2 or  c2) then Idle   else t2;
  x2' = if t2 = Idle   and  r2 and c2  then 0      else
        if t2 = Active                 then x2 + 1 else x2;
initial x1 = 0 and t1 = Idle and x2 = 0 and t2 = Idle;
invariant t1 <> t2 or t1 = Idle;              (* To be Enforced (Φ) *)
```

Example Logico-numerical DCS Problem

Example Logico-numerical Program & Invariant Properties

```
typedef State = enum {Idle, Active};
state t1, t2: State;                          (* State Variables *)
      x1, x2: int;
input r1, r2, s1, s2: bool;       (* Non-controllable Inputs *)
controllable c1, c2: bool;            (* Controllable Inputs *)
transition
  t1' = if t1 = Idle   and  r1 and c1  then Active else
        if t1 = Active and (s1 or  c1) then Idle   else t1;
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        if t1 = Active                 then x1 + 1 else x1;
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        if t2 = Active and (s2 or  c2) then Idle   else t2;
  x2' = if t2 = Idle   and  r2 and c2  then 0      else
        if t2 = Active                 then x2 + 1 else x2;
initial x1 = 0 and t1 = Idle and x2 = 0 and t2 = Idle;
invariant t1 <> t2 or t1 = Idle      (* To be Enforced (Φ) *)
      and x1 <= 10 and x2 <= 10;
```

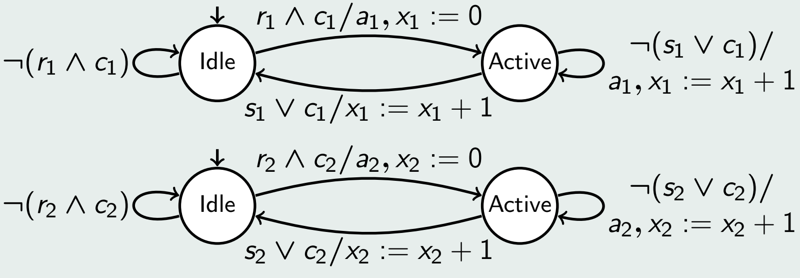
Example Logico-numerical DCS Problem

Example Logico-numerical Program & Invariant Properties

```
typedef State = enum {Idle, Active};
state t1, t2: State;                        (* State Variables *)
      x1, x2: int;
input r1, r2, s1, s2: bool;       (* Non-controllable Inputs *)
controllable c1, c2: bool;            (* Controllable Inputs *)
transition
  t1' = if t1 = Idle   and  r1 and c1  then Active else
        if t1 = Active and (s1 or  c1) then Idle   else t1;
  x1' = if t1 = Idle   and  r1 and c1  then 0      else
        if t1 = Active                 then x1 + 1 else x1;
  t2' = if t2 = Idle   and  r2 and c2  then Active else
        if t2 = Active and (s2 or  c2) then Idle   else t2;
  x2' = if t2 = Idle   and  r2 and c2  then 0      else
        if t2 = Active                 then x2 + 1 else x2;
initial x1 = 0 and t1 = Idle and x2 = 0 and t2 = Idle;
invariant t1 <> t2 or t1 = Idle        (* To be Enforced (Φ) *)
      and x1 <= 10 and x2 <= 10 and x1 <= x2;
```

Results of Over-approximating Synthesis

Program



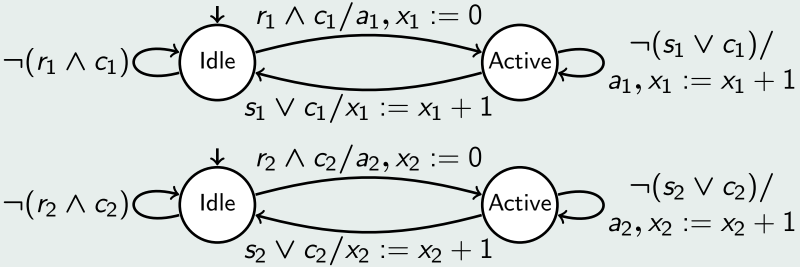
$\Phi(\langle t_1, t_2, x_1, x_2 \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle}) \wedge (x_1 \leqslant 10 \wedge x_2 \leqslant 10)$

- Power Domain, Intervals & Convex Polyhedra

$$I'^{c}_{Bad} = \left\{ \begin{array}{l} t_1 = \text{Idle} \wedge t_2 = \text{Idle} \wedge x_1 \leqslant 10 \wedge x_2 \leqslant 10 \vee \\ t_1 = \text{Idle} \wedge t_2 = \text{Active} \wedge x_1 \leqslant 10 \wedge x_2 \leqslant 9 \vee \\ t_1 = \text{Active} \wedge t_2 = \text{Idle} \wedge x_1 \leqslant 9 \wedge x_2 \leqslant 10 \end{array} \right\}$$

Results of Over-approximating Synthesis (cont'd)

Program



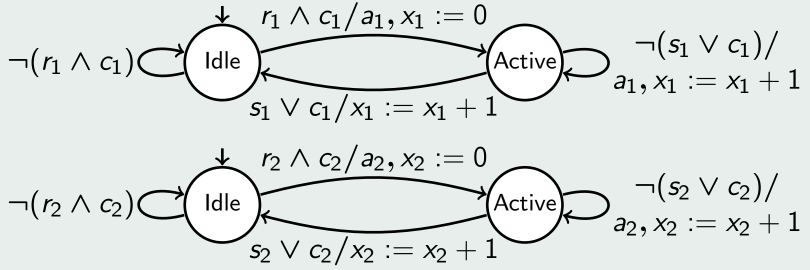
$\Phi(\langle t_1, t_2, x_1 \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle}) \wedge (x_1 \leqslant 10 \wedge x_2 \leqslant 10) \wedge (x_1 \leqslant x_2)$

- Power Domain, Intervals

$$
\begin{aligned}
I'_{Bad} &\supseteq \{t_1 = \text{Idle} \ \wedge \ t_2 = \text{Idle} \ \wedge \ x_1 < 11 \ \wedge \ x_2 < 10\} \\
&\supseteq \{t_1 = \text{Idle} \ \wedge \ t_2 = \text{Idle} \ \wedge \ x_1 = \ 0 \ \wedge \ x_2 = \ 0\} = \mathcal{X}_0
\end{aligned}
$$

Results of Over-approximating Synthesis (cont'd)

Program



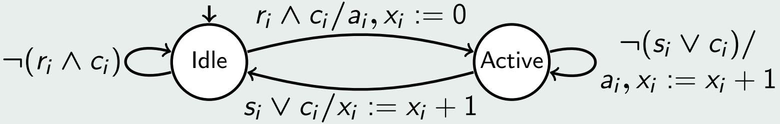
$\Phi(\langle t_1, t_2, x_1 \ldots \rangle) = (t_1 \neq t_2 \vee t_1 = \text{Idle}) \wedge (x_1 \leqslant 10 \wedge x_2 \leqslant 10) \wedge (x_1 \leqslant x_2)$

- Power Domain, Convex Polyhedra

$$I'^{c}_{Bad} = \left\{ \begin{array}{l} t_1 = \text{Idle} \;\wedge\; t_2 = \text{Idle} \quad \wedge x_1 \leqslant x_2 \leqslant 10 \vee \\ t_1 = \text{Idle} \;\wedge\; t_2 = \text{Active} \wedge x_1 \leqslant x_2 \leqslant \;\; 9 \vee \\ t_1 = \text{Active} \;\wedge\; t_2 = \text{Idle} \wedge x_1 < x_2 \leqslant 10 \end{array} \right\}$$

Performance Evaluation

Example Logico-numerical Programs & Invariant Properties



- $X = \langle t_1 \dots t_n, x_1 \dots x_n, a_1 \dots a_n \rangle$ $\mathcal{D}_X = \{\text{Idle}, \text{Active}\}^n \times \mathbb{Z}^n \times \mathbb{B}^n$
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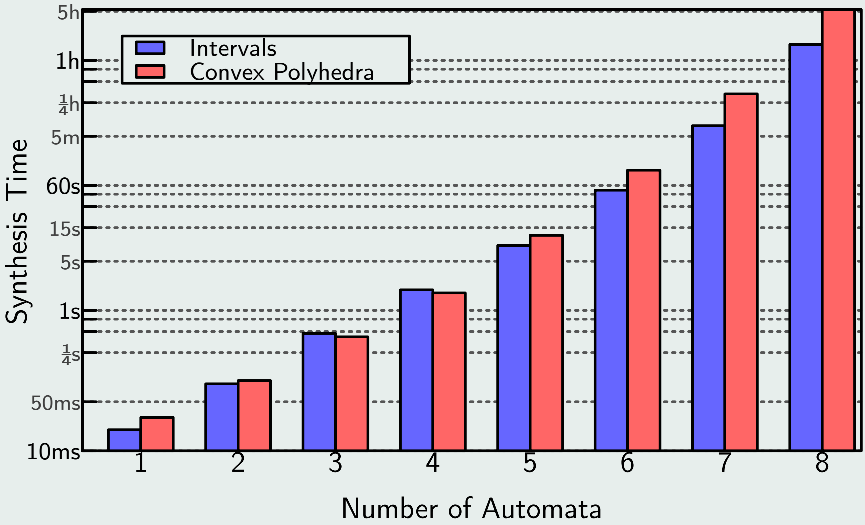
- Enforcing Mutual Exclusion Between Active States & Bounds on x_i's

$$\Phi(\langle t_1 \dots t_n, x_1 \dots x_n, \dots \rangle) =$$

$$\bigoplus_{i \in [1,n]} (t_i = \text{Active}) \wedge \bigwedge_{i \in [1,n]} (x_i \leqslant 10)$$

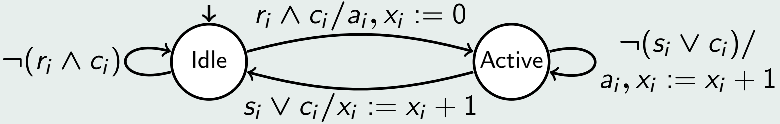
Performance of Over-approximating Synthesis

Performance Results: Mutual Exclusion & Bounds



Performance Evaluation

Example Logico-numerical Programs & Invariant Properties



- $X = \langle t_1 \ldots t_n, x_1 \ldots x_n, a_1 \ldots a_n \rangle$ $\qquad \mathcal{D}_X = \{\text{Idle}, \text{Active}\}^n \times \mathbb{Z}^n \times \mathbb{B}^n$
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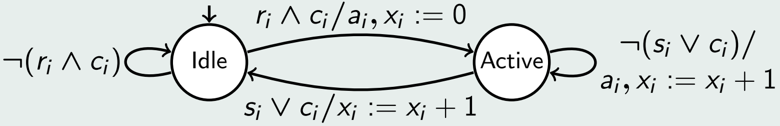
Performance Evaluation

Example Logico-numerical Programs & Invariant Properties



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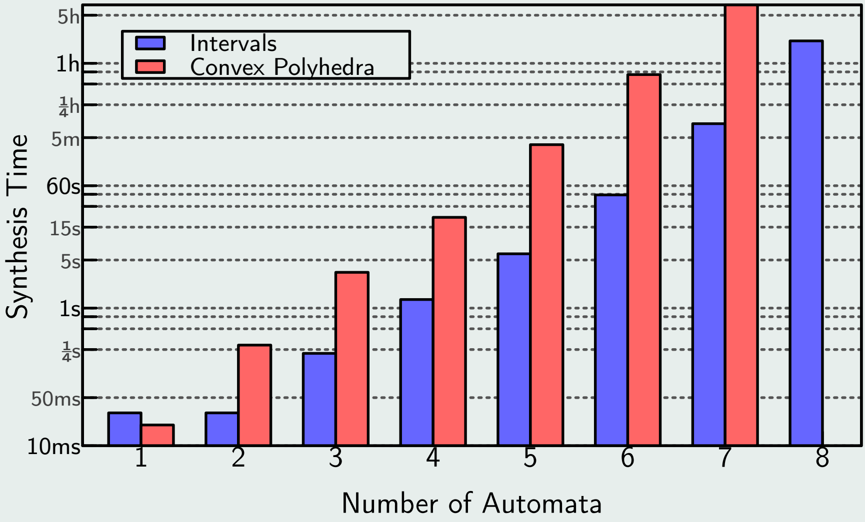
$$\bigoplus_{i \in [1,n]} (t_i = \text{Active}) \wedge \bigwedge_{i \in [1,n]} (x_i \leqslant 10) \wedge \bigwedge_{i \in [1,n[} (x_i \leqslant x_{i+1})$$

Performance of Over-approximating Synthesis (cont'd)

Performance Results: Mutual Exclusion & Bounds & Relational Constraints



Outline

- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
- Discrete Controller Synthesis Principles
- ReaX: Technical Choices, Implementation & Evaluations
- Conclusions

Conclusion & Further Works

ReaX

- Discrete Controller Synthesis for Logico-numerical Synchronous/Reative Programs
- Efficient Exact Synthesis
- Over-approximating Synthesis with Abstract Interpretation Techniques
- Heptagon/BZR Backend

$\rightsquigarrow$ Favorably Replaces Sigali

Conclusion & Further Works

ReaX

- Discrete Controller Synthesis for Logico-numerical Synchronous/Reative Programs
- Efficient Exact Synthesis
- Over-approximating Synthesis with Abstract Interpretation Techniques
- Heptagon/BZR Backend

$\leadsto$ Favorably Replaces Sigali

Forthcoming Challenges

- Enforcement of Quantitative Properties
- Improving Precision
- Synthesis Failure Diagnosis
- Avoiding Deadlocks

Thanks!

Outline

- Discrete Controller Synthesis for Logico-numerical Programs
- State of the Art
- Discrete Controller Synthesis Principles
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Outline

- Backup
  - On the Necessary Convexity of Bad
  - Producing an Executable Controller
  - Triangularization of the Controller
  - Performance Comparison with Sigali for a Realistic Model

On the Necessary Convexity of *Bad*

Example Problem

With Usual Numerical Abstract Domains:

$$\alpha(\{x \in \mathbb{Z} \mid x \leqslant 0 \wedge 10 \leqslant x\}) = \top$$

A Solution

- Split *Bad* into a Disjunction of n Convex Clauses Bad_i
- Compute Every I'_{Bad_i}
- Compute the Controller using $\bigcup_{i \in [1,n]} I'_{Bad_i}$

$\rightsquigarrow$ Deadlocks!

Producing an *Executable* Controller

- New Constraint $\mathcal{A}_\Phi$ $(\subseteq \mathcal{X} \times \mathcal{U} \times \mathcal{C})$
  - Relating States with *Allowed Inputs*

$$\mathcal{A}_\Phi = \mathcal{T}_S^{-1}(I_{Bad}^c) \cap \mathcal{A}_\Phi$$

- Controller $(\mathcal{K}_\Phi : \mathcal{X} \times \mathcal{U} \to \wp(\mathcal{C}))$

$$\mathcal{K}_\Phi = \lambda(x, \upsilon).\, \{\iota \,|\, (x, \upsilon, \iota) \in \mathcal{A}_\Phi\}$$

Resulting System

$$[S'] = \langle \mathcal{X}, \mathcal{U}, \lambda(\sigma, \upsilon).\mathcal{T}_S(\sigma, \upsilon, \text{choose} \circ \mathcal{K}_\Phi(\sigma, \upsilon)), \mathcal{A}_\Phi, \mathcal{X}_0 \rangle$$

- $\text{choose} : \wp(\mathcal{C}) \to \mathcal{C}$: Non-Deterministic Choice!

Triangularization of the Controller

- Code Generation for $\lambda(\sigma, \upsilon).\mathsf{choose}(\mathcal{K}_{\Phi}(\sigma, \upsilon))$
- Using Oracles ("Phantom Variables")
  - "Preferred Value" for the Controllable Inputs

Performance Comparison with Sigali for a Realistic Model

What About A Realistic Model?

	Sigali	ReaX
2-tiers	6s	3.2s
4-tiers	105s	12s