GENERALISED MYCIELSKI GRAPHS AND THE BORSUK–ULAM THEOREM

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Joint work with Tobias Müller

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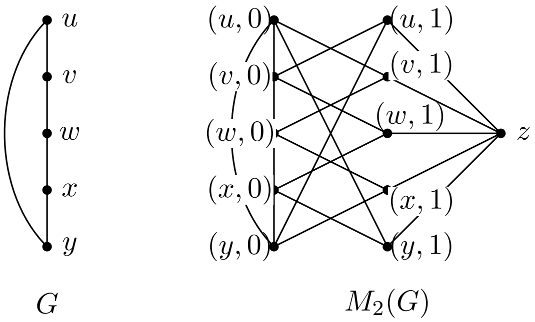
- There exist graphs of arbitrary girth and arbitrary chromatic number (Erdős 1959).

The Mycielski construction

Given a graph $G = (V, E)$, the graph $M_2(G)$ has:

- vertex set $V \times \{0, 1\} \cup \{z\}$;
- edges $\{(u, 0), (v, 0)\}$ and $\{(u, 0), (v, 1)\}$ iff $\{u, v\} \in E$;
- edges $\{(u, 1), z\}$ for all $u \in V$.

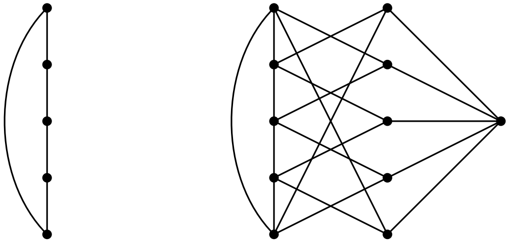


The chromatic number of Mycielski graphs

- Let c be a minimum colouring of $M_2(G)$.
- Define a colouring c' of G as

$$c'(u) = \begin{cases} c((u,0)) & \text{if } c((u,0)) \neq c(z) \\ c((u,1)) & \text{if } c((u,0)) = c(z). \end{cases}$$

- This shows $\chi(G) \leq \chi(M_2(G)) - 1$.



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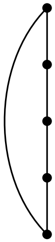

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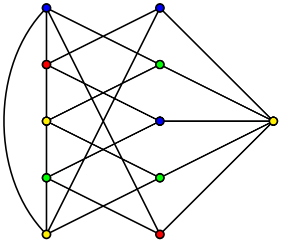

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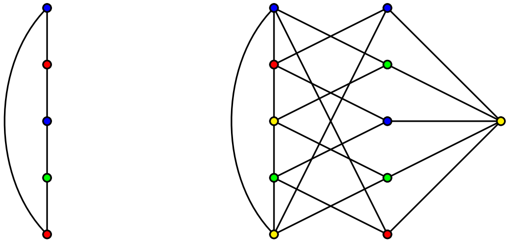


G $\qquad$ $M_2(G)$

Mycielski's theorem

Theorem (Mycielski 1955)

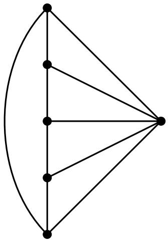
For any graph G, $\chi(M_2(G)) = \chi(G) + 1$. If G is triangle-free, then so is $M_2(G)$.

Corollary

There exist triangle-free graphs of arbitrary chromatic number.

The generalised Mycielski construction

- Construction generalised by Stiebitz (1985), and independently by Van Ngoc (1987).
- Given a graph $G = (V, E)$ and an integer $r \geq 1$, we define $M_r(G)$ as the graph with vertex set $V \times \{0, \ldots, r-1\} \cup \{z\}$, where there is an edge $\{(u,0),(v,0)\}$ and $\{(u,i),(v,i+1)\}$ whenever $\{u,v\} \in E$, and an edge $\{(u,r-1),z\}$ for all $u \in V$.
- Unlike Mycielski's original construction, the generalised Mycielski construction results in graphs of arbitrary odd girth.



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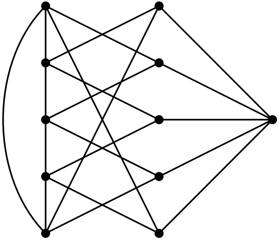

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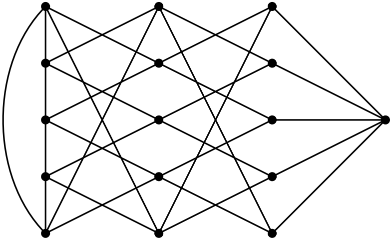

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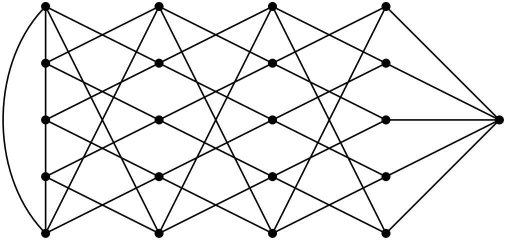
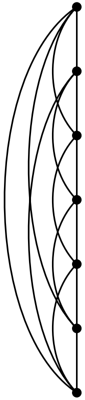


$M_4(C_5)$

Mycielski's theorem does not hold for the generalised case…



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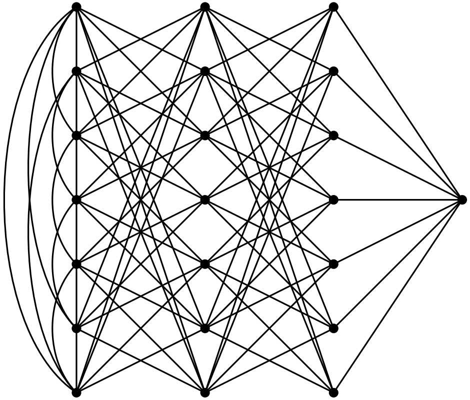

$M_3(G)$

Observation

$\chi(G) = \chi(M_3(G)) = 4$

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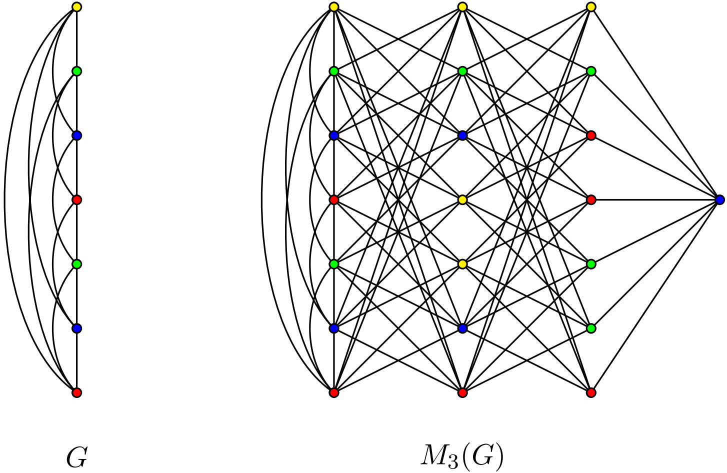

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…but it does if we start with an odd cycle!

Theorem (Stiebitz 1985)

The graph obtained from an odd cycle by n applications of $M_r(\cdot)$ has chromatic number $n+3$.

- Stiebitz's proof of the lower bound is topological.
- He shows that if the neighbourhood complex of G is k-connected, then the neighbourhood complex of $M_r(G)$ is $(k+1)$-connected.
- Since the neighbourhood complex of an odd cycle is 0-connected, the neighbourhood complex of a graph obtained from an odd cycle by n applications of $M_r(\cdot)$ is n-connected.
- He then applies the following bound of Lovász.

Theorem (Lovász 1978)

If the neighbourhood complex of G is k-connected, then $\chi(G) \geq k+3$.

Key result from algebraic topology

Borsuk–Ulam Theorem (Borsuk 1933)

For every continuous mapping $f : S^n \to \mathbb{R}^n$ there exists a point $x \in S^n$ with $f(x) = f(-x)$.

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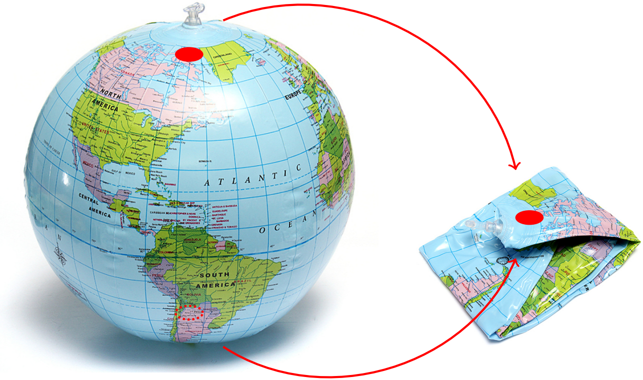

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- If “elementary combinatorial”=“discrete”, we show the answer is YES.

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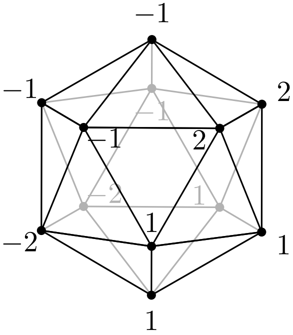
- Answer depends on the meaning of “elementary combinatorial”.
- If “elementary combinatorial”=“discrete”, we show the answer is YES.
- If “elementary combinatorial”=“avoiding topology”, we argue the answer is NO.

A discrete version of Borsuk–Ulam

Tucker's lemma (Tucker 1946)

- Let K be an antipodally symmetric triangulation of S^n.



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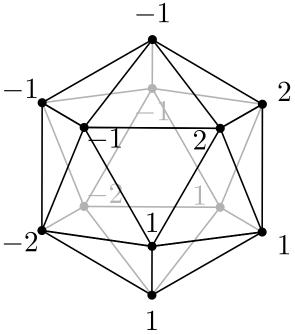

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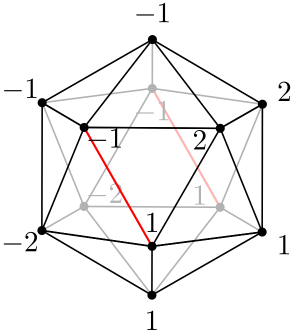
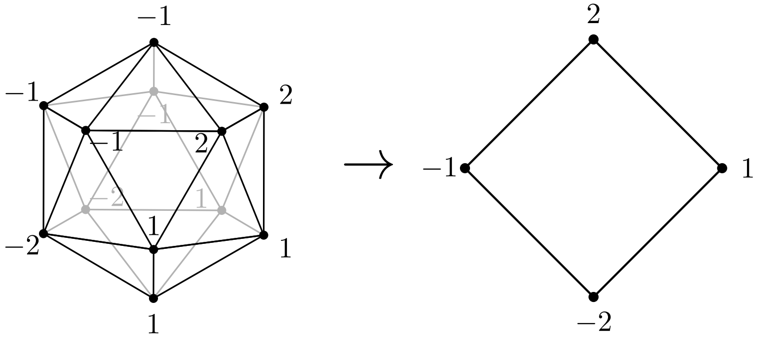


Equivalence of Tucker and Borsuk–Ulam

- Tucker follows from Borsuk–Ulam by considering λ as a simplicial map, and taking the affine extension $|\lambda|$.



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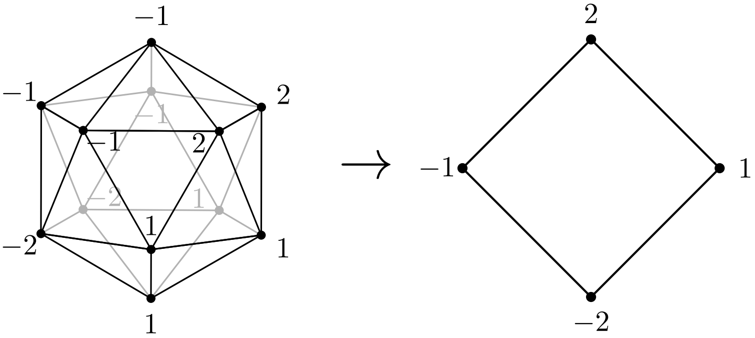
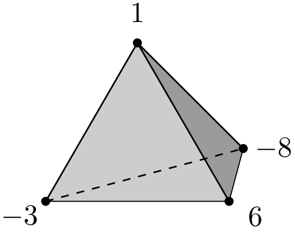


- Borsuk–Ulam follows from Tucker by taking sufficiently fine triangulations of S^n and using compactness.

Alternating and almost alternating simplices

- Let K be a simplicial complex.



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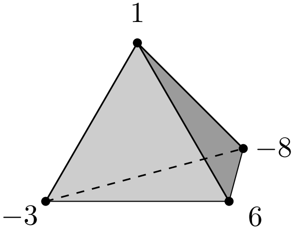

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- Let K be a simplicial complex.
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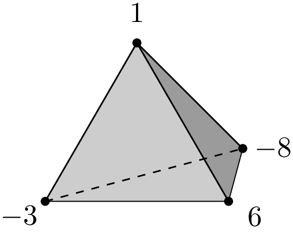
- A d-simplex $\sigma \in K$ is *positive alternating* if it has labels $\{+j_0, -j_1, +j_2, \ldots, (-1)^d j_d\}$, where $0 < j_0 < j_1 < \cdots < j_d$.



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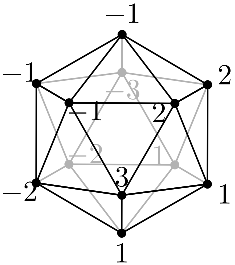

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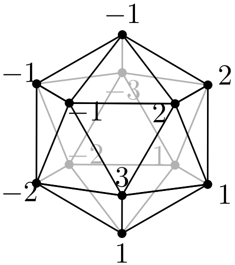

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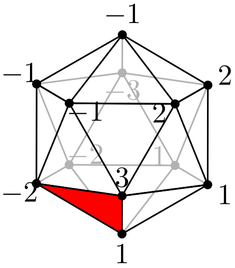
- Then there exists an odd number of positive alternating n-simplices.



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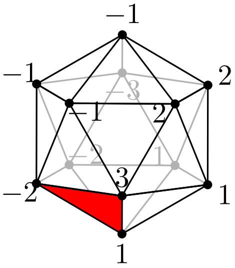
- In particular, $k \geq n + 1$.



Proof of Fan's lemma

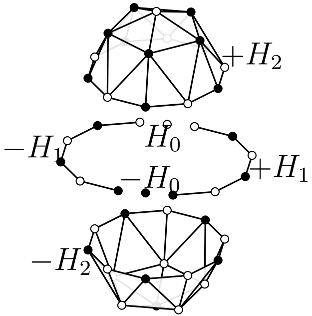
- All known combinatorial proofs of Fan's lemma impose some additional restrictions on the triangulation.
- As long as the class of triangulations contains a sequence of triangulations with simplex diameter tending to 0, we can use a compactness argument to deduce (a generalisation of) the Borsuk–Ulam theorem, and then deduce the general version of Fan's lemma.
- In our case, we are going to use triangulations *aligned with hemispheres*.

Flags of hemispheres

- A *flag of hemispheres* in S^n is a sequence $H_0 \subset \cdots \subset H_n$ where each H_d is homeomorphic to a d-ball, $\{H_0, -H_0\}$ are antipodal points, $H_n \cup -H_n = S^n$, and for $1 \leq d \leq n$,

$$\partial H_d = \partial(-H_d) = H_d \cap -H_d = H_{d-1} \cup -H_{d-1} \cong S^{d-1}.$$

- A symmetric triangulation K of S^n is *aligned with hemispheres* if there is a flag of hemispheres such that for every d, there is a subcomplex of the d-skeleton of K that triangulates H_d.



Fan's lemma for triangulations aligned with hemispheres

Fan's lemma (Prescott and Su 2005)

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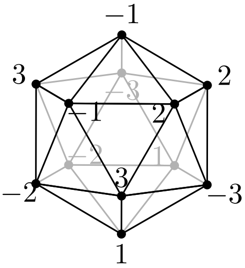
Remark

The proof of Prescott and Su is entirely discrete.

Another discrete version of Borsuk–Ulam

Corollary of Fan's lemma

- Let K be an antipodally symmetric triangulation of S^n (aligned with hemispheres).



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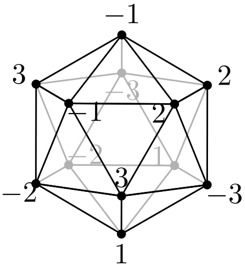
- Let $\lambda : V(K) \to \{\pm 1, \ldots, \pm(n+1)\}$ be a labelling such that $\lambda(-v) = -\lambda(v)$ for all $v \in V(K)$, and every n-simplex has vertices of both signs.



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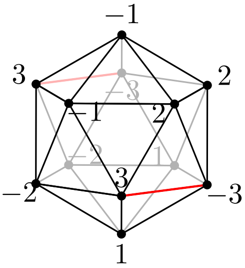

Proof

- Suppose $\lambda(u) + \lambda(v) \neq 0$ for every edge $\{u, v\} \in K$.
- Define a new labelling $\mu : V(K) \to \{\pm 1, \ldots, \pm(n+1)\}$ by $\mu(v) = (-1)^{|\lambda(v)|}\lambda(v)$.
- Observe that

$$\mu(-v) = (-1)^{|\lambda(-v)|}\lambda(-v) = -(-1)^{|\lambda(v)|}\lambda(v) = -\mu(v),$$

  and if $\mu(u) = -\mu(v)$, then $\lambda(u) = -\lambda(v)$.
- Therefore $\mu(u) + \mu(v) \neq 0$ for every edge $\{u, v\} \in K$.
- Hence μ satisfies the hypothesis of Fan's lemma.
- Therefore, there is an odd number of positive alternating n-simplices, i.e., simplices labelled $\{1, -2, \ldots, (-1)^n n, (-1)^{n+1}(n+1)\}$ by μ.
- Hence, there is an odd number of simplices labelled $\{1, 2, \ldots, n+1\}$ by λ.
- This contradicts the assumption that every n-simplex in K has vertices of both signs.

Graphs associated to 2-coloured triangulations

- Let K be a symmetric triangulation K of S^n with a proper antisymmetric 2-colouring κ of K
- We denote by $\tilde{G}(K, \kappa)$ the graph obtained from the 1-skeleton $K^{(1)}$ by deleting all monochromatic edges.
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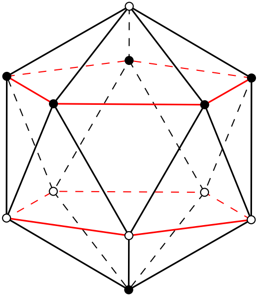

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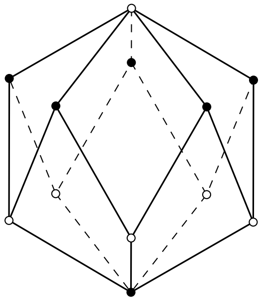

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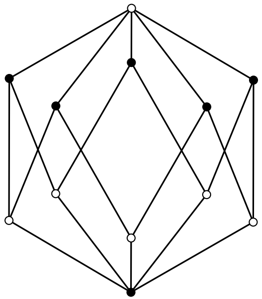

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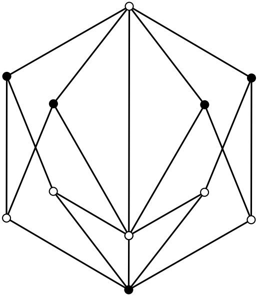

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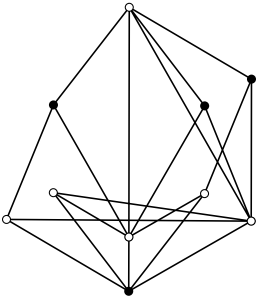

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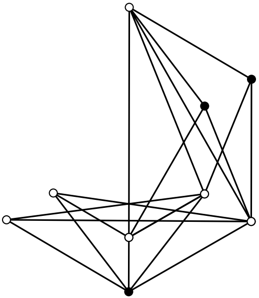

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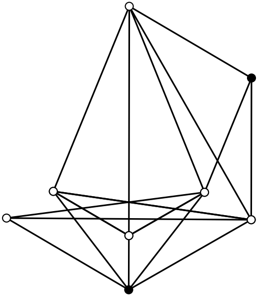

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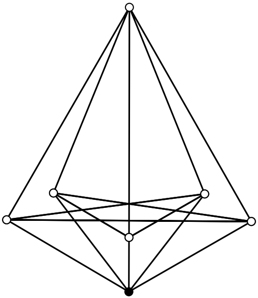

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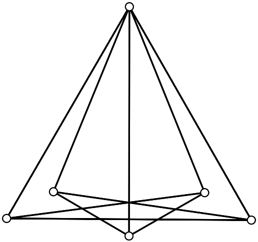
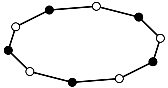
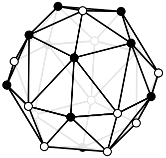


Theorem (Kaiser and MS 2015)

- Given $n \geq 1$, let K be a symmetric triangulation of S^n aligned with hemispheres, with a proper antisymmetric 2-colouring κ.
- For any $r \geq 1$, there exists a symmetric triangulation K' of S^{n+1} aligned with hemispheres, with a proper antisymmetric 2-colouring κ' such that $G(K', \kappa') \cong M_r(G(K, \kappa))$.

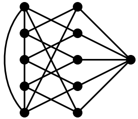


K



K'

$G(K, \kappa)$



$G(K', \kappa')$

The final step (1/2)

- Assume $k > 3$ and let $G \in \mathcal{M}_k$.
- The graph G is obtained from an odd cycle by $k-3$ iterations of $M_r(\cdot)$, where the value of r can vary from iteration to iteration.
- By the previous theorem, there exists a symmetric triangulation K of S^{k-2} aligned with hemispheres, and a proper antisymmetric 2-colouring κ such that $G \cong G(K, \kappa)$. Say the colours used in κ are black and white.
- Consider any (not necessarily proper) $(k-1)$-colouring $c : V(G) \to \{1, \ldots, k-1\}$.
- Set

$$\lambda(v) = \begin{cases} +c(p(v)) & \text{if } v \text{ is black} \\ -c(p(v)) & \text{if } v \text{ is white.} \end{cases}$$

The final step (2/2)

- $\lambda : V(K) \to \{\pm 1, \ldots, \pm(k-1)\}$ is an antisymmetric labelling such that every $(k-2)$-simplex has vertices of both signs.
- By the corollary to Fan's lemma, there exists an edge $\{u, v\} \in K$ such that $\lambda(u) + \lambda(v) = 0$.
- Hence, the edge $\{p(u), p(v)\} \in E(G)$ satisfies $c(p(u)) = |\lambda(u)| = |\lambda(v)| = c(p(v))$, i.e., c is not a proper colouring of G.
- This shows that $\chi(G) \geq k$.

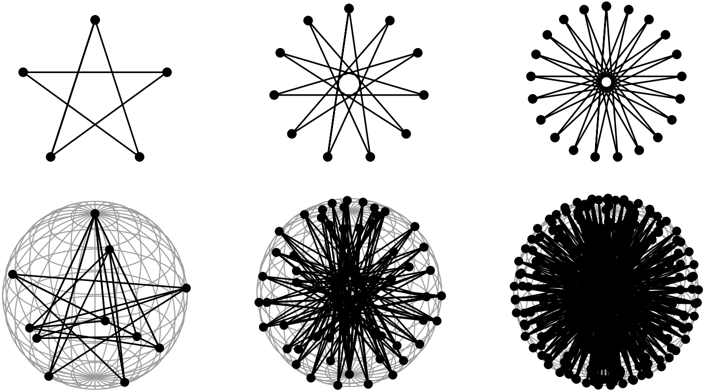
Borsuk graphs

- Connection between Borsuk–Ulam and chromatic number first noticed by Erdős and Hajnal (1967).
- They defined the *Borsuk graph* $BG(n, \alpha)$ as the (infinite) graph whose vertices are the points of $\mathbb{R}^{n+1}$ on S^n, and the edges connect points at Euclidean distance at least α where $0 < \alpha < 2$.
- As α tends to 2, the odd girth of $B(n, \alpha)$ tends to infinity.
- Borsuk–Ulam equivalent to $\chi(BG(n, \alpha)) \geq n + 2$.
- By using the standard $(n + 2)$-colouring of S^n based on the central projection of a regular $(n + 1)$-simplex, it can be shown that $\chi(BG(n, \alpha)) = n + 2$ for all α sufficiently large.

Generalised Mycielski graphs

Lemma (Müller and MS 2017)

For every $n \geq 0$ and every $\delta > 0$, there exists $G \in \mathcal{M}_{n+2}$ and a mapping $f : V(G) \to S^n$ such that $\|f(u) + f(v)\| < \delta$, for every edge $\{u, v\} \in G$. In particular, $G \subset BG(n, \sqrt{4 - \delta^2})$.



Stiebitz implies Borsuk–Ulam

- Suppose there exists a continuous antipodal map $f : S^n \to S^{n-1}$.
- For ε sufficiently small, $\chi(BG(n-1,\varepsilon)) = n+1$.
- Every continuous function on a compact set is uniformly continuous, so there exists $\delta > 0$ such that if $\|x - y\| < \delta$, then $\|f(x) - f(y)\| < 2\varepsilon$.
- By the previous lemma, there exists $G \in \mathcal{M}_{n+2}$ and a mapping $g : V(G) \to S^n$ such that $\|g(u) + g(v)\| < \delta$, for every edge $\{u, v\} \in E(G)$.
- The mapping $f \circ g : V(G) \to S^{n-1}$ satisfies $\|f(g(u)) + f(g(v))\| < 2\varepsilon$, for every edge $\{u, v\} \in E(G)$.
- The Euclidean distance between $f(g(u))$ and $f(g(v))$ is

$$\|f(g(u)) - f(g(v))\| > 2\sqrt{1 - \varepsilon^2},$$

- So $G \subset BG(n-1,\varepsilon)$, and $\chi(G) \leq \chi(BG(n-1,\varepsilon)) = n+1$.
- This contradicts Stiebitz's theorem.