3-colorability of (claw, H)-free graphs

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k-coloring of graph G - a mapping $f : V(G) \to \{1, \ldots, k\}$ s.t.

$$\forall uv \in E(G) : f(u) \neq f(v)$$

$\chi(G)$ - chromatic number of G

- coloring problems:

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- the problem COLORING is NP-complete problem
(even 3-COLORING is NP-hard)

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$\Rightarrow$ complexity of 3-COLORING when H is a forest?

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- computational complexity of 3-COLORING in other subclasses of claw-free graphs?

Kráľ, Kratochvíl, Tuza, Woeginger:

- 3-COLORING is NP-complete for (claw, C_r)-free graphs whenever $r \geq 4$
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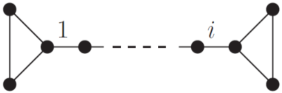
Malyshev:

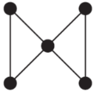
- 3-COLORING is poly-time solvable for (claw, H)-free graphs for $H = P_5, C_3^*, C_3^{++}$

Theorem (Lozin, Purcell)

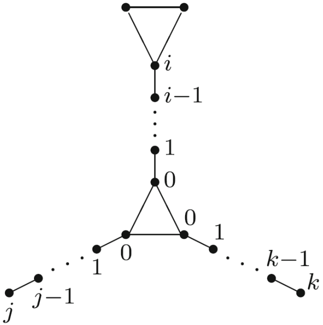
The 3-COLORING *problem can be solved in polynomial time in the class of (claw, H)-free graphs only if every connected component of H is either a* Φ_i *with an odd i or a* $T^{\Delta}_{i,j,k}$ *with an even i or an induced subgraph of one of these two graphs.*



$\Phi_i,\ i > 0$



Φ_0



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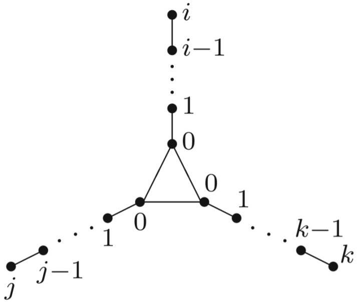
- for 1 triangle:
  if H is a graph every connected component of the form $T^1_{i,j,k}$, then the clique-width of (claw, H)-free graphs of bounded vertex degree is bounded by a constant (Lozin, Rautenbach)



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  $H = \Phi_0$: Randerath, Schiermeyer, Tewes (polynomial-time algorithm), Kamiński, Lozin (linear-time algorithm)
  $H = T^{\Delta}_{0,0,k}$: Kamiński, Lozin
  $H \in \{\Phi_1, \Phi_3\}$: Lozin, Purcell
  $H = \Phi_2, \Phi_4$: Maceková, Maffray

- every graph on 5 vertices contains either a C_3, or a $\overline{C_3}$, or a C_5 $\Rightarrow$ as K_4 and W_5 are not 3-colorable, every claw-free graph, which is 3-colorable, has $\Delta(G) \leq 4$

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- $\delta(G) \geq 3$
- G is 2-connected

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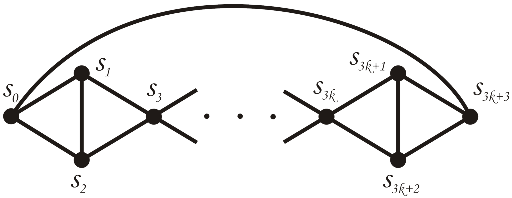
Definition

Any claw-free graph that is 2-connected, K_4-free, and where every vertex has degree either 3 or 4 is called a standard claw-free graph.

$F_{3k+4}, k \geq 1$:



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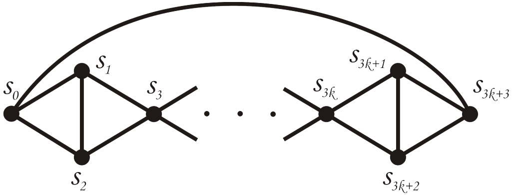
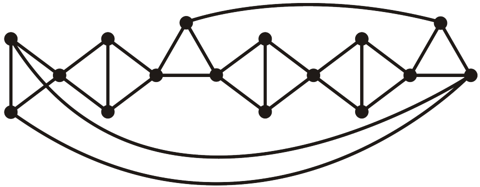


F'_{16}:



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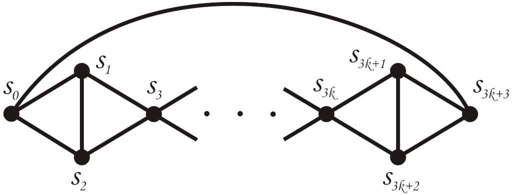

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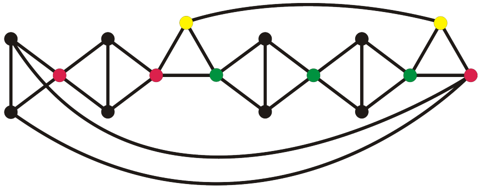

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In a graph G, we say that a non-empty set $R \subset V(G)$ is removable if any 3-coloring of $G \setminus R$ extends to a 3-coloring of G.

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Lemma

In a graph G, let $R \subset V(G)$, and let $x, y \in R$ be such that every vertex in $R \setminus \{x, y\}$ has no neighbor in $V(G) \setminus R$, and each of x, y has at most two neighbors in $V(G) \setminus R$. Assume that either:

(a) *$G[R]$ admits a 3-coloring where x and y have the same color and a 3-coloring where x and y have distinct colors; or*

(b) *$G[R]$ admits a 3-coloring, and each of x, y has at most one neighbor in $V(G) \setminus R$; or*

(c) *$G[R]$ admits a 3-coloring in which x and y have different colors, and one of x, y has at most one neighbor in $V(G) \setminus R$.*

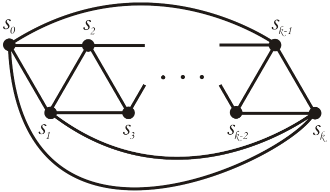
Then R is removable.

- given diamond $D \rightarrow$ vertices of degree 2 = peripheral, vertices of degree 3 = central
- types of diamonds in G:
  - pure diamond $\rightarrow$ both central vertices of diamond have degree 3 in G
  - perfect diamond $\rightarrow$ pure diamond in which both peripheral vertices have degree at most 3 in G
- operations with diamonds:
  - pure diamond contraction
  - perfect diamond deletion

Lemma

Let G be a connected claw-free graph with maximum degree at most 4. Assume that G contains a diamond and no K_4. Then one of the following holds:

- *G is either a tyre, or a pseudo-tyre, or $K_{2,2,1}$, or*



$K_{2,2,2}$, or $K_{2,2,2} \setminus e$, or

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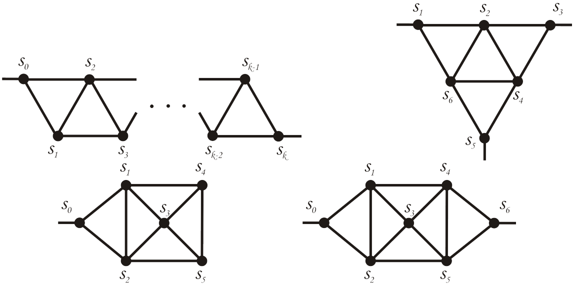
- *G is either a tyre, or a pseudo-tyre, or $K_{2,2,1}$, or $K_{2,2,2}$, or $K_{2,2,2} \setminus e$, or*
- *G contains a strip.*



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- *G contains a strip.*

- if G is a tyre or a pseudo-tyre, then it is 3-colorable only if $|V(G)| \equiv 0 \pmod 3$
- if G is isomorphic to $K_{2,2,1}$, $K_{2,2,2}$, or $K_{2,2,2} \setminus e$, then it is 3-colorable
- if G contains a strip and is Φ_2-free, then we can reduce it

When G is a claw-free graph that contains a strip, we define a reduced graph G' as follows:

- if G contains a linear strip S, then G' is obtained by removing the vertices $s_1, \ldots, s_{k-1}$ and identifying the vertices s_0 and s_t (if $k = 0 \bmod 3$), or adding the edge $s_0 s_k$ (if $t \neq 0 \bmod 3$)
- if G contains a square strip S, then G' is obtained by removing the vertices $s_1, \ldots, s_5$ and identifying the vertices s_0 and s_6
- if G contains a semi-square strip S, then G' is obtained by removing the vertices $s_1, \ldots, s_5$
- if G contains a triple strip, then G' is obtained by removing the vertices s_2, s_4, s_6 and adding the three edges $s_1 s_3, s_1 s_5, s_3 s_5$

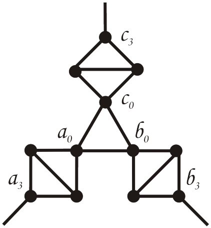
Lemma

Let G be a claw-free graph that contains a strip S and no K_4, and let G' be the reduced graph obtained from G by strip reduction. Then:

(i) *G' is claw-free.*

(ii) *G is 3-colorable if and only if G' is 3-colorable.*

(iii) *If G is Φ_2-free, and S is not a diamond, then G' is Φ_2-free.*



Lemma

Let G be a (claw, Φ_2)-free graph. Let $T \subset V(G)$ be a set that induces a $(1,1,1)$-tripod. Let G' be the graph obtained from G by removing the vertices of $T \setminus \{a_3, b_3, c_3\}$ and adding the three edges a_3b_3, a_3c_3, b_3c_3. Then:

- *G' is (claw, Φ_2)-free,*
- *G is 3-colorable if and only if G' is 3-colorable.*

Theorem

Let G be a standard (claw, Φ_2)-free graph. Then either:

- *G is a tyre, a pseudo-tyre, a $K_{2,2,1}$, or a $K_{2,2,2}$ or a $K_{2,2,2} \setminus e$, or*
- *G contains F_7 as an induced subgraph, or*
- *G is diamond-free, or*
- *G has a set whose reduction yields a Φ_2-free graph, or*
- *G has a removable set.*

Definition

The chordality of a graph G is the length of the longest chordless cycle in G.

Theorem (Lozin, Purcell)

For every fixed p, the 3-colorability problem is polynomial-time solvable in the class of claw-free graphs of chordality at most p.

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Lemma

Let G be a standard (claw, Φ_2)-free graph that contains no diamond. Assume that G contains a chordless cycle C of length at least 10. Then every vertex of G not in C which has a neighbor in C is adjacent to exactly two consecutive vertices of the cycle.

Theorem

One can decide 3-COLORING *problem in polynomial time in the class of* $(claw, \Phi_2)$*-free graphs.*

Sketch of the proof.

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- G contains a diamond - if yes, then 3-COLORING of $G \leftrightarrow$ 3-COLORING on a smaller (claw, Φ_2)-free graph; otherwise G is diamond-free
- G contains a chordless cycle of length at least 10 - if no, G has bounded chordality; otherwise G has specifical structure and either it contains a removable set, or we can easily color the vertices of G with three colors

Lemma

Let G be a standard (claw, Φ_k)-free graph, $k \geq 4$. Assume that G contains a strip S which is not a diamond. Then either we can find in polynomial time a removable set, or $|V(G)|$ is bounded by a function that depends only on k.

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Definition

Let a Φ_0 be pure if none of its two triangles extends to a diamond.

Lemma

Let G be a standard (claw, Φ_4)-free graph. Assume that every strip in G is a diamond. If G contains a pure Φ_0, then either $|V(G)| \leq 127$ or we can find a removable set.

Theorem

Let G be a standard (claw, Φ_4)-free graph in which every strip is a diamond. Assume that G contains a diamond, and let G' be the graph obtained from G by reducing a diamond. Then one of the following holds:

- *G' is (claw, Φ_4)-free, and G is 3-colorable if and only if G' is 3-colorable;*
- *G contains F_7, F_{10} or F'_{16} (and so G is not 3-colorable);*
- *G contains a pure Φ_0;*
- *G contains a removable set;*
- *G contains a (1,1,1)-tripod.*

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Corollary

One can decide 3-COLORING *in polynomial time in the class of (claw, Φ_4)-free graphs.*

Thank you for your attention!