On the minimum size of an identifying code over all orientations of a graph

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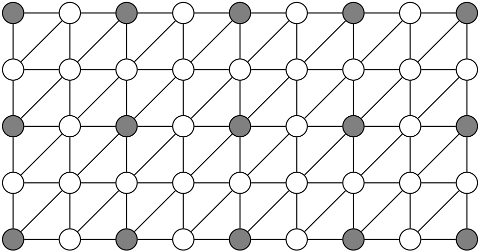
Definitions

neighbourhood of v : $N(v) = \{u \mid uv \in E(G)\}$.
closed neighbourhood of v : $N[v] = N(v) \cup \{v\}$.

$C \subseteq V(G)$
identifier of v : $I(v) = N[v] \cap C$.

C is an identifying code if

- $I(v) \neq \emptyset$ for all $v \in V(G)$;
- $I(v) \neq I(u)$ for any $v \neq u$.



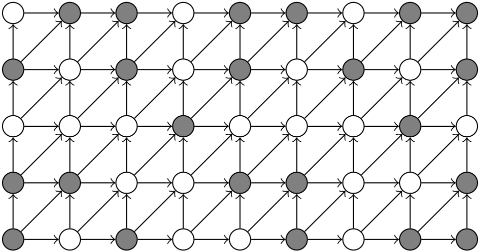
Definitions

out-neighbourhood of v : $N^+(v) = \{u \mid uv \in A(D)\}$.
closed out-neighbourhood of v : $N^+[v] = N^+(v) \cup \{v\}$.

$C \subseteq V(D)$
identifier of v : $I(v) = N^+[v] \cap C$.

C is an identifying code if

- $I(v) \neq \emptyset$ for all $v \in V(D)$;
- $I(v) \neq I(u)$ for any $v \neq u$.



Existence theorem

u and v are twins if $N[u] = N[v]$. resp. $N^+[u] = N^+[v]$.

For all C, two twins have the same identifier.

Theorem: G admits an identifying code iff G has no twins.

Proof: If two twins, no identifying code.
If no twins, then $V(G)$ is an identifying code. □

Problem 1: Let G be a finite (di)graph with no twins.
What is the **minimum size** id(G) of an identifying code ?

Identifying codes of orientations

Theorem: D admits an identifying code iff D has no twins.

Corollary: If D is an orientation of G, then D has an identifying code.

Problem 2: Let G be a graph.
What is the **minimum size** idor(G) of an **identifying code** of an **orientation** D of G?

First bounds on idor

$$\log_2(n+1) \leq \text{idor}(G) \leq n$$

empty graphs : $\text{idor}(E_n) = n$.
complete graphs : $\text{idor}(K_n) = \lceil \log_2(n+1) \rceil$.

Proposition: $\text{idor}(G) \leq \text{idor}(G \setminus e)$.

Slightly better upper bounds for idor

Lemma: (V_1, V_2) partition of G

$$\text{idor}(G) \leq \text{idor}(G\langle V_1\rangle) + \text{idor}(G\langle V_2\rangle)$$

[[Orient all arcs from V_1 to V_2.]]

$\text{idor}(G) \leq |V(G)| - \omega(G) + \lceil \log_2(\omega(G)+1) \rceil.$

$\text{idor}(G) \leq |V(G)| - \delta(G)/2 + 1.$

We cannot expect better than $|V(G)| - g(\delta(G))$
for some g s.t. $\frac{k}{2} - 1 \leq g(k) \leq 2^k - 1.$

$\text{idor}(K_{k,n-k}) \geq n - 2^k + 1.$

Lower bounds for idor

Lemma: $\mathrm{idor}(G) \geq \frac{2}{\Delta(G)+2}|V(G)|$.

Discharging Method

Initial charge : $w_0(v) = 1$ for all $v \in V(G)$. Total charge $= |V(G)|$.

Discharging rule : every vertex sends $\frac{1}{|I(v)|}$ to every vertex of $I(v)$.

Final charge: if $v \notin C$, then $w(v) = 0$;
if $v \in C$, then $w(v) \leq 1 + d^-(v)/2 \leq \frac{2+\Delta(G)}{2}$;

$$|V(G)| = \sum_{v \in C} w(v) \leq |C| \frac{\Delta(G)+2}{2}.$$

Lower bounds for idor

Lemma: $\mathrm{idor}(G) \geq \dfrac{2}{\Delta(G)+2}|V(G)|.$

This bound is tight.
Proposition: If G is the incidence graph of a Δ-regular graph,
then $\mathrm{idor}(G) = \frac{2}{\Delta+2}|V(G)|$.

[[G incidence graph of H.
Orient all arcs from $E(H)$ to $V(H)$.]]

Trees

Theorem: If T is a tree of order $n \geq 2$, then $\text{idor}(T) \geq \lceil (n+1)/2 \rceil$.

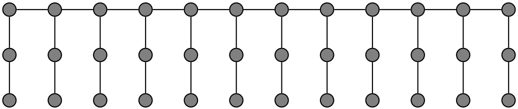
Paths: $\text{idor}(P_n) = \lceil (n+1)/2 \rceil$.

Stars: $\text{idor}(S_n) = n - 1$.

Theorem: If T is a tree, then $\text{idor}(T) \leq \left\lfloor \frac{|V(T)|+\text{leav}(T)}{2} \right\rfloor$.

Corollary: If T is a tree, then $\text{idor}(T) \leq \frac{3\alpha(T)}{2}$.

Theorem: If T is a tree, $T \neq P_2, P_4$, then $\text{idor}(T) \leq \frac{4}{3}\alpha(T)$.



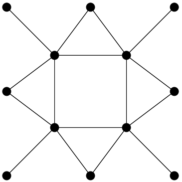
idor(G) versus id(G)

$\log_2(n+1) \leq \mathrm{id}(G) \leq n$ for G twin-free
$\log_2(n+1) \leq \mathrm{idor}(G) \leq n$.

$\mathrm{idor}(G) \leq 2^{\mathrm{id}(G)}$ and $\mathrm{id}(G) \leq 2^{\mathrm{idor}(G)}$ (G twin-free)

$\mathrm{idor}(K_{k,2^k-k-1}) = k$ and $\mathrm{id}(K_{k,2^k-k-1}) = 2^k - 3$.

Theorem: $\mathrm{idor}(G) \leq \frac{3}{2}\,\mathrm{id}(G)$ for all graph G.



Complexity

IDOR
Input: A graph G and an integer k.
Parameter: k.
Question: $\mathrm{idor}(G) \leq k$?

IDOR is NP-complete even for bipartite cubic graphs or bipartite planar graphs of maximum degree 3.

It is FPT. (If $\mathrm{idor}(G) \leq k$, then $|G| \leq 2^k - 1$.)

Problem: Does IDOR admit a polynomial kernel ?

Large Idor

LARGE-IDOR
Input: A graph G and an non-negative integer k.
Parameter: k.
Question: $\text{idor}(G) \geq |V(G)| - k$?

It is in XP.

k-atom G : $\text{idor}(G) = |V(G)| - k$ and $\text{idor}(H) > |V(H)| - k$ for all proper induced subgraphs H of G.

Lemma: Every k-atom has order at most $\binom{k}{2} + 2k + 1$.

Problem: Is LARGE-IDOR FPT ?

Is Code

IsCode
Input: A graph G and a set $C \subseteq V(G)$.
Question: Is there an orientation D of G for which C is an identifying code ?

IsCode is **NP-hard**.

Many polynomial subcases.

- If $C = V(G)$, then the answer is trivially 'yes', and if $2^{|C|} - 1 < V(G)$, then the answer is trivially 'no'.
- If $G\langle C\rangle$ has a **bounded number of edges**, then IsCode can be solved in polynomial time using matchings.

  Given G, $C \subseteq V(G)$, and D_C orientation of $G\langle C\rangle$, one can check in polynomial time whether there exists an orientation D of G such that $D\langle C\rangle = D_C$ and C is an identifying code of D.