Dynamique de formation de groupes dans les réseaux sociaux

J-C BERMOND
avec A. Chaintreau, D. Mazauric et G. Ducoffe
STINT 05/12/2017

Simple model (J. Kleinberg, K. Ligett)

Information-sharing in social networks

Games and Economic Behavior 2013

- Two persons (vertices, players) are either friends or enemies

Friendship graph or Conflict Graph G^-

- Partition of the persons into n groups X_i (some empty)

(a person is in exactly one group)

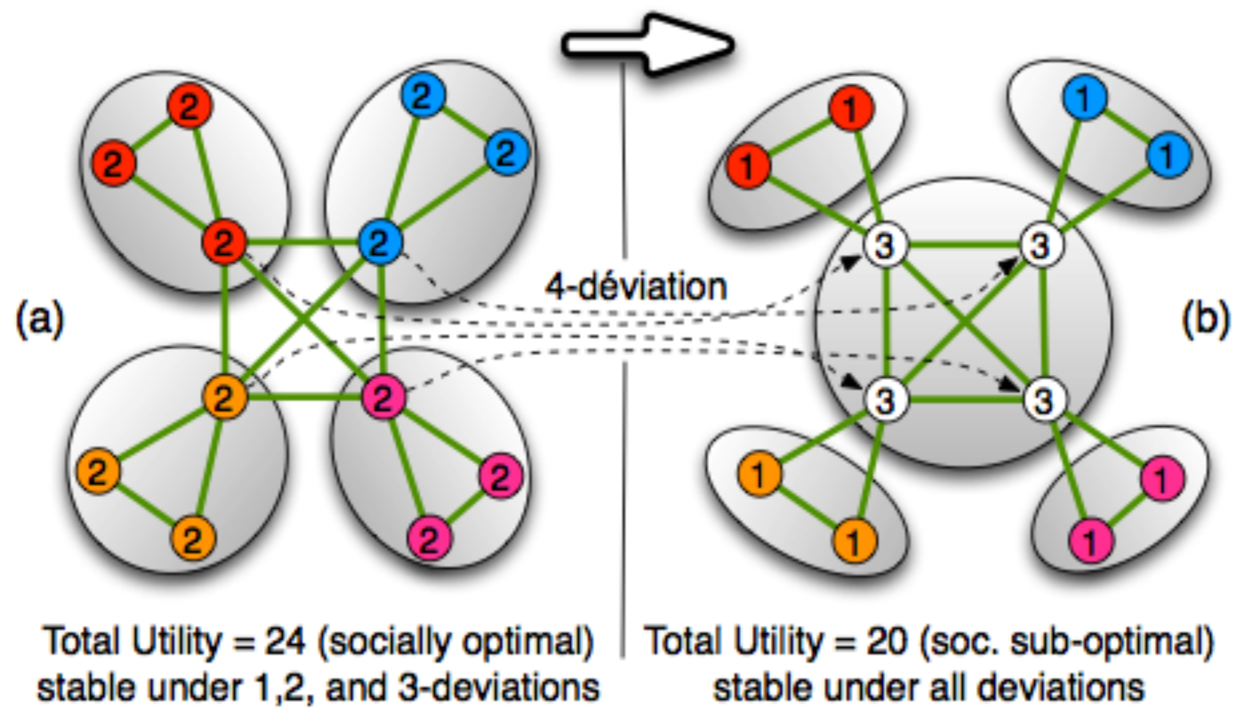
Emphasis on the fact

two "enemies" are not in the same group.

- Utility for a person = number of friends in her group

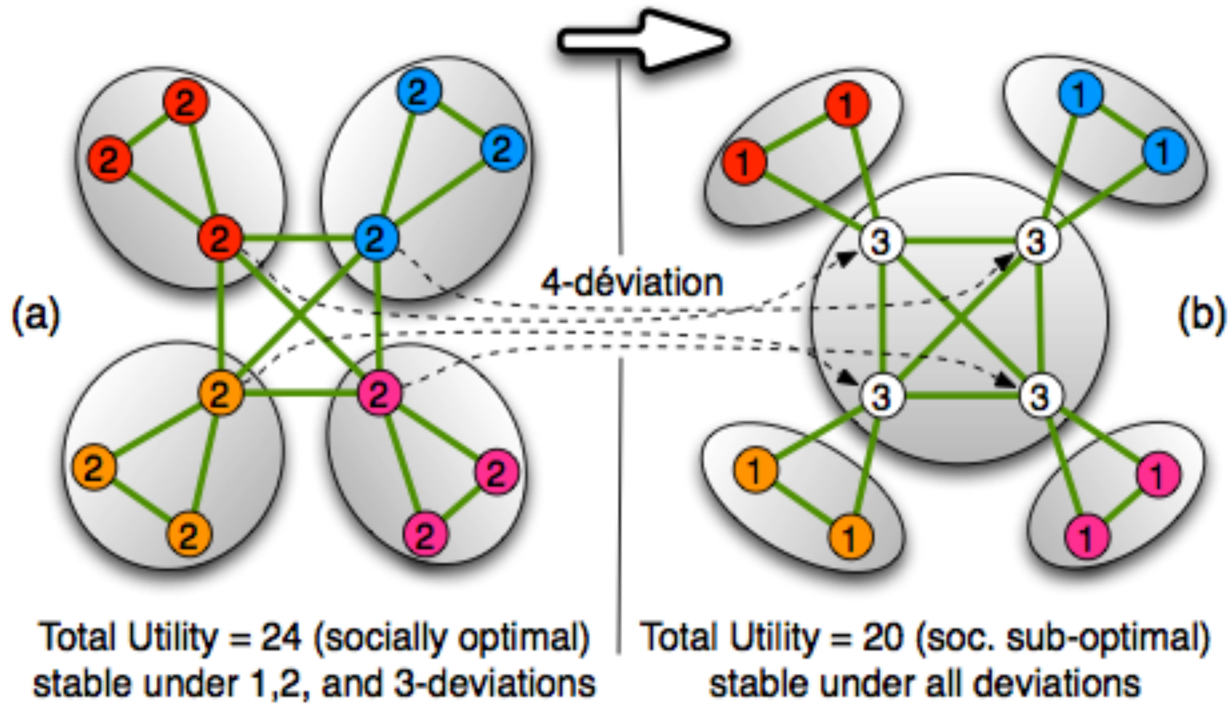
= size of the group to which she belongs – 1

(if there is an enemy utility = $-\infty$)


4-déviation
(a)
(b)
Total Utility = 24 (socially optimal)
stable under 1,2, and 3-deviations
Total Utility = 20 (soc. sub-optimal)
stable under all deviations

Dynamics of the model (k-deviations)

- Selfish behavior (bounded cooperation between users)
- 1-deviation : a person leaves the group to which she belongs to join another group but only if it increases her utility (in particular she does not join a group with an enemy)
- k-deviation : a set of at most k persons leaves the group to which they belong to join another group but only if each person increases her utility (if the group they join is empty it can be viewed as creating a new group)
- Partition k-stable if there does not exist a k-deviation


4-déviation
(a)
(b)
Total Utility = 24 (socially optimal)
stable under 1,2, and 3-deviations
Total Utility = 20 (soc. sub-optimal)
stable under all deviations

Coloring games

P. Panagolopoulou and P.Spirakis 2008

- « In this paper, we propose an efficient vertex coloring algorithm that is based on local search: Starting with an arbitrary proper vertex coloring (e.g. the trivial proper coloring where each vertex is assigned a unique color), we do local changes, by allowing each vertex (one at a time) to move to another color class of higher cardinality, until no further local moves are possible. «
- Players = persons = vertices
- Colors = groups
- Existence of a Nash equilibrium in 0(nα(G$^-$) giving known bounds in such an equilibrium

$$k \leq \min\left\{\Delta_2(G)+1,\ \frac{n+\omega(G)}{2},\ n-\alpha(G)+1,\ \frac{1+\sqrt{1+8m}}{2}\right\}.$$

Existence of k-stable partitions?

Main questions :

1- Does there always exist a k-stable partition ?

2- Is it is easy (polynomial time) to find one?

3- What about the convergence = number of k-deviations to reach a k-stable partition in the worst case ?

- Answers to 1 and 2 easy if the conflict graph is empty (partition formed by the unique group of n persons)

Existence of 1-stable partition (Potential function)

Lemma : In a 1-deviation the global utility increases by at least 2

Proof :

A vertex leaves a group of size p to join a group of size q ≥ p

For this vertex increase by : q-p+1

For the q vertices of the groups of size q increase by 1

For the p-1 vertices of the group of size p decrease by 1

Altogether increase ≥ (q-p+1)+q –(p-1) =2q -2p + 2 ≥ 2

Total utility at most n(α(G^-)-1)

so the number of 1-deviations is at most 0(nα(G^-)

Existence and Convergence using potential functions

Known results (Kleinberg-Ligett ; Escoffier Gourvès-Monnot 2012))

There always exist a k-stable partition

which can be found in :

- k =1,2 : $O(n^2)$
- k =3 : $O(n^3)$
- k ≥ 4 : $O(2^n)$

Existence of k-stable partitions

Better response dynamics Algorithm

1- Start from the partition composed with n singleton groups

2- While there exists a k-deviation for the current partition P_i do it and compute the partition P_{i+1} obtained

3- If there is no k-deviation the partition obtained is stable

This Algorithm converges to a k-stable partition in finite time

Proof :

- Partition vector associated to P: $\overrightarrow{\Lambda}(P) = (\lambda_n(P), \ldots, \lambda_1(P))$, where $\lambda_i(P)$ is the number of groups of size i.

Set S of vertices joins a group of size j => new group of size |S|+j

The groups to which the vertices belonged are of size less so

$\overrightarrow{\Lambda}(P_i) <_L \overrightarrow{\Lambda}(P_{i+1})$ where $<_L$ is the lexicographical ordering.

Convergence

- $L(k,G^-)$ = size of a longest sequence of k-deviations with a conflict graph G^-
- $L(k,n)$ = maximum over all the conflict graph with n vertices
- Observation : L(k,n) is always attained in the empty conflict graph

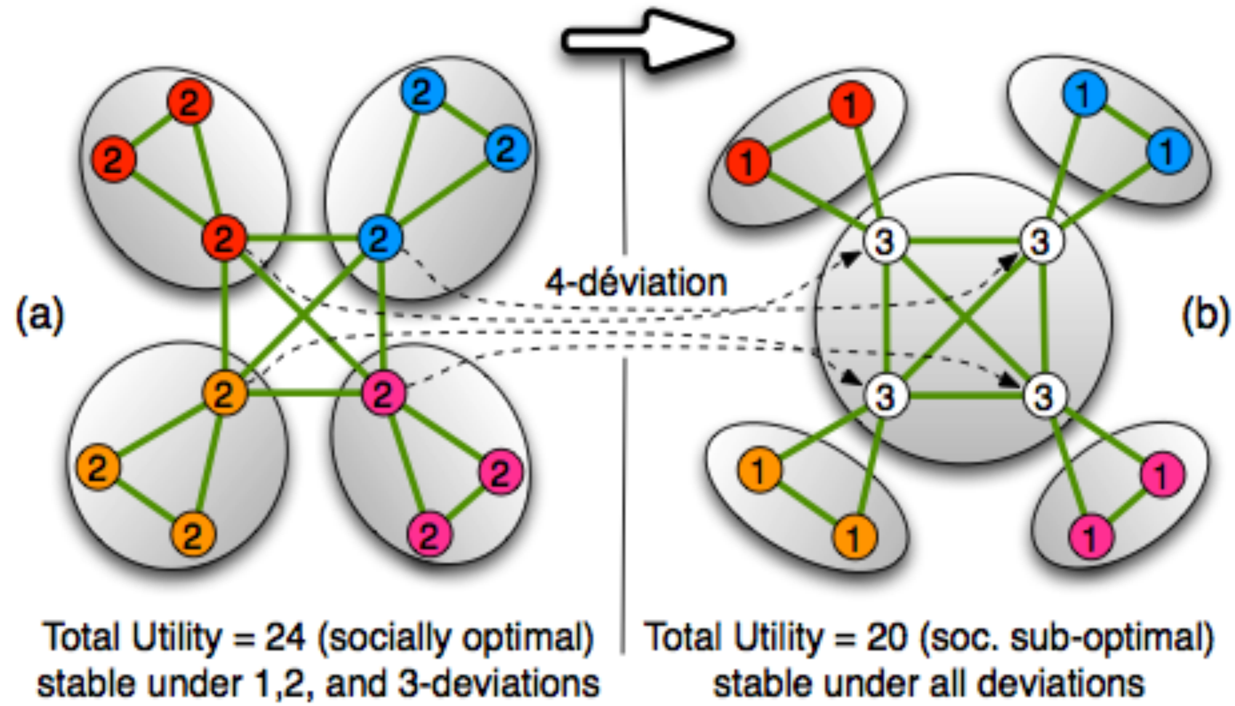
Definition 2 ([Bry73]). *An integer partition of $n \geq 1$, is a non-increasing sequence of integers $Q = q_1 \geq q_2 \geq \ldots \geq q_n \geq 0$ such that $\sum_{i=1}^{n} q_i = n$.*

- To a partition P into groups X_i with size q_i we associate the integer partition Q(P) with values q_i
- Converse true if the graph is empty
- Number of integer partitions $$p_n = \Theta((e^{\pi\sqrt{\frac{2n}{3}}})/n)$$

Results

Table 1: Previous Bounds and results we obtained on $L(k,n)$.

k	Prior to our work	**Our results**	
1	$O(n^2)$ [KL13]	exact analysis, which implies $L(1,n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 7
2	$O(n^2)$ [KL13]	exact analysis, which implies $L(2,n) \sim \frac{(2n)^{3/2}}{3}$	Theorem 10
1-2	$O(n\alpha(G^-))$ [PS08]	$L(k,G^-) = \Omega(n.\alpha(G^-)$ for some G^- and $\alpha(G^-) = 0(\sqrt{n})$	Theorem 13
3	$O(n^3)$ [EGM12, KL13]	$\Omega(n^2)$	Theorem 14
≥ 4	$O(2^n)$ [KL13]	$\Omega(n^{\Theta(\ln(n))})$, $O(\exp(\pi\sqrt{2n/3})/n)$	Theorem 15



4-déviation
(a)
(b)
Total Utility = 24 (socially optimal)
stable under 1,2, and 3-deviations
Total Utility = 20 (soc. sub-optimal)
stable under all deviations

Examples

- First partition P into 4 groups of size 3

Partition vector (0,0,…,0,4,0,0)

Integer Partition Q(P) = 3,3,3,3,0,0,0,0,0,0,0,0

- Second partition Q’: 1 group of 4, 4 groups of 2

Partition Vector (0,…0,1,0,4,0)

Integer partition Q’ = Q(P’) = 4,2,2,2,2, 0,0,0,0,0,0,0

Case k =1 : Dominance ordering Dominance lattice

- $Q' = (q'_1 \geq \ldots \geq q'_n)$ dominates $Q = (q_1 \geq \ldots \geq q_n)$

if $\sum_{j=1}^{i} q'_j \geq \sum_{j=1}^{i} q_j$, *for all* $1 \leq i \leq n$.

Equivalence between chains in the dominance lattice and sequences of 1-deviations (Conflict graph empty)

Lemma : Let P' be the partition obtained from P after a 1-deviation ; then Q'= Q(P') dominates Q= Q(P)

Lemma : Conflict graph empty: If Q' dominates Q Then the partition P' is obtained from P via a sequence of 1-deviations.

Exact value of L(1,n)

- Using the result of Greene and Kleitman (1986)

Theorem: Let n = m(m+1)/2 +r, 0 ≤ r ≤m ,

Then $L(1, n) = 2\binom{m+1}{3} + mr.$

Exact value of L(1,n)

- Using the result of Greene and Kleitman (1986)

Theorem: Let n = m(m+1)/2 +r, $0 \leq r \leq m$,

then $$L(1,n) = 2\binom{m+1}{3} + mr.$$

Algorithm : Choose among the possible deviations the one which leads to the smallest

Increase of the global utility

Exact value of L(2,n)

- Using the result of Greene and Kleitman (1986)

Theorem: Let $n = m(m+1)/2 + r$, $0 \le r \le m$,

Then $L(1,n) = 2\binom{m+1}{3} + mr.$

Conflict graph empty : Any 2-deviation can be replaced by 1 or 2 1-deviations.

So $L(2,n) = L(1,n)$

k= 1,2 General Conflict graph

- Upper bound (potential function) $0(n\alpha(G^-))$
- Empty G^- $\alpha(G^-)$: n but O (n $^{3/2}$)
- Conjecture : $0(n.\sqrt{\alpha(G^-)})$.

Theorem 11. *For* $n = \binom{m+1}{2}$ *there exists a conflict graph* G *with* $\alpha(G^-) = m = \sqrt{n}$ *and a sequence of* $\binom{m+1}{3}$ *valid* 1*-deviations that is a sequence of* $\Omega(n^{\frac{3}{2}}) = \Omega(n.\alpha(G^-))$ *1-deviations.*

Open problems

- For k =3 Conjecture : L(3,n) of order n^2
- For k ≥ 4 polynomial algorithm to find a k-stable partition
- Parallel deviations
- Case of neutral relations and in general of relations with various weights of friendship or incompatibility (OK for 1-deviations but not for 2)
- Digraphs with symmetric or symmetric relations
- Case a person can be in multiple groups

Unstable case k =2

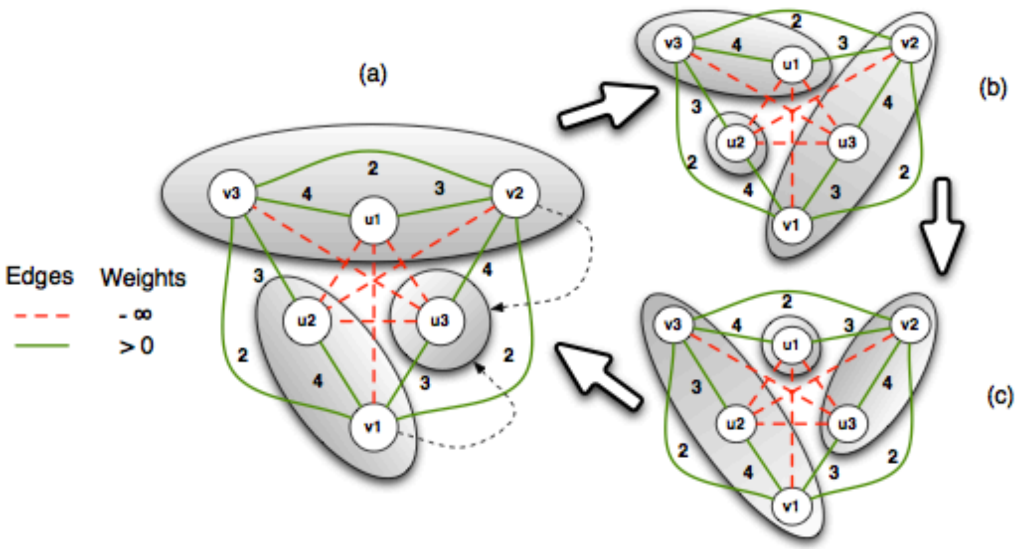


Figure 1: A network with set of weights $\mathcal{W} = \{-\infty, 2, 3, 4\}$ that does not admit a 2-stable partition. **(a)** 1-stable partition that is not 2-stable and that can be obtained after a 2-deviation in partition depicted in (c). **(b)** 1-stable partition that is not 2-stable and that can be obtained after a 2-deviation in partition depicted in (a). **(c)** 1-stable partition that is not 2-stable and that can be obtained after a 2-deviation in partition depicted in (b).

THANK YOU