Open problem session – STINT Meeting

4-6 December 2017

Partition of in graphs with large average degree

Communiqué par P. Ossona de Mendez.

Stiebitz proved that if $\delta(G) \geq s + t + 1$, then there is a partition (V_1, V_2) of $V(G)$ such that $\delta(G_1) \geq s$ and $\delta(G_2) \geq t$.

Conjecture 1. If $|E(G)| \geq (s + t + 1)|V(G)|$, then there is a partition (V_1, V_2) of $V(G)$ such that $|E(G_1)| \geq s|V(G_1)|$ and $|E(G_2)| \geq t|V(G_2)|$.

Odd girth of 4-chromatic graphs

Communiqué par M . Stehlik.

Conjecture 2 (Esperet, Stehlik)**.** If $\chi(G) = 4$, then its odd girth is at most $\frac{1}{2}\lceil 1 + \sqrt{8n - 7}\rceil$.

Esperet and Stehlik proved it is true if odd cycles pairwise intersect, and tight in this case. Nilli [1] proved that the odd girth of a 4-chromatic graph is at most $8\sqrt{n}$ and Jiang [2] lowered this bound to $2\sqrt{n} + 3$.

Characterizing 4-colourable graphs via orientations

Communiqué par S. Thomassé .

A graph is 3-colourable if and only if it has an orientation such that all oriented cycles have algebraic length $0 \mod 3$

Problem 3. Can you find an analogous statement for 4-colourable graphs ?

Non approximability of some cluster index

Communiqué par S. Pérennes

Let $\sigma = (V_1, \ldots, Vp)$ be a partition of a graph G. The number of traversal edges (edges with endvertices in different part of the partition) is denoted by $\mathrm{tr}(\sigma)$. Its *cluster index* is $\mathrm{ci}(\sigma) = \max\{|V_i| \mid 1 \leq i \leq p\} + \mathrm{tr}(\sigma)$. The *cluster index* of a graph G, denoted by $\mathrm{ci}(G)$, is the minimum cluster index over all its partitions. Finding the cluster index of a graph G is NP-hard. We conjecture that it is not approximable.

Conjecture 4. There is no constant-approximation polynomial-time algorithm for cluster index.

Sampling directed cuts

Communiqué par S. Pérennes

Problem 5. Find a random generator of a minimum (s,t)-cut in a digraph D.

We do not think that it is done in the undirected case.

Problem 6. Find a random generator of a minimum (s,t)-cut in a graph D.

Generalizing Hall's Theorem

Communiqué par S. Thomassé sur recommandation de N. Le

Let G_1 and G_2 be two bipartite graphs with the same bipartition (A, B). We say that $G_1 \preceq G_2$ if for every $X \subseteq A$ then $|N_{G_1}(X)| \leq |N_{G_2}(X)|$.

Conjecture 7 (Volec)**.** If $G_1 \preceq G_2$, then the number of matchings saturating A in G_1 is no larger than the number of matchings saturating A in G_2.

This conjecture generalizes the celebrated Hall's Theorem.

Induced bipartite subgraph of minimum degree 3

Communiqué par F. Maffray

Problem 8 (Esperet and Kang)**.** What is the complexity of deciding whether a graph contains an induced subgraph which is bipartite and of minimum degree 3 ?

Problem 9 (Esperet and Kang)**.** Does every graph with sufficiently large girth and minimum degree contain an induced subgraph which is bipartite and of minimum degree 3 ?

Cooperative colouring

Communiqué par F. Havet

Given a set $\mathcal{G} = (G_1, \ldots, G_k)$ of graphs on the same vertex set V, a *cooperative colouring* for $\mathcal{G}$ is a choice of sets $S_1, \ldots, S_k$ such that I_j is a stable set of G_j and $\bigcup_{j \leq k} S_j = V$. If all G_i are the same graph G then a cooperative coloring is just a k-colouring of G.

Let $f(d)$ be the minimum k such that every set of k trees with maximum degree d have a cooperative colouring.

Aharoni et al. proved that $f(2) = 3$ using topological argument. Using probabilistic arguments, Aharoni et al. proved that $\log_2(\log_2(d))(1 + o(\log \log d)) \leq f(d) \leq 4 \log_2(d)$.

Problem 10. Improve the aforementioned bounds for f.

Problem 11. Prove that $f(3) \leq 5$.

References

[1] A. Nilli. Short odd cycles in 4-chromatic graphs. *J. Graph Theory*, 31(2):145–147, 1999.

[2] T. Jiang. Small odd cycles in 4-chromatic graphs. *J. Graph Theory*, 37(2):115–117, 2001.

[3] R. Aharoni, E. Berger, M. Chudnovsky, F. Havet. Cooperative colorings of trees and of bipartite graphs. *manuscript*.

[4] R. Aharoni, R. Holzman, D. Howard, and P. Sprüssel. Cooperative Colorings and Independent Systems of Representatives *Electronic Journal of Combinatorics*, 22 (2), P2.27, 2015.