A tight Erdős-Posa function for wheel minors

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Definition

Let G and H be two graphs.

• $\nu_H(G) =$ maximum number of pairwise disjoint H minors in G.

• $\tau_H(G) =$ minimum size of a subset $X \subseteq V(G)$ such that $G - X$ is H-minor free.

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Theorem (Robertson, Seymour, 1986)

H has the Erdős-Posa property if and only if H is planar.

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For every planar graph H, the EP property holds for H with function $k \log^c k$.

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Our main contribution:

Theorem

The conjecture holds for wheels.

Minimal counter-example to the EP property

Let H be a graph and let $f : \mathbb{N} \to \mathbb{R}$ be a function.

A graph G is a **minimal counterexample** to the EP property for H if the following properties hold:

- $\tau_H(G) > \nu_H(G) log \nu_H(G)$,
- subject to the above constraint $\nu_H(G)$ is minimum,
- subject to the above constraints, G is minor minimal.

General tools

Theorem (Fomin et al. 2012)

For every planar graph H, for every $k \in \mathbb{N}$, every graph G with $\tau_H(G) = k$ satisfies:

$$|V(G)| \leq poly(k)$$

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Proof.

By Fomin et al, $|V(G)| \leq poly(\tau_H(G))$.

Set $k = \nu_H(G)$. By Chekuri and Chuzoy, $\tau_H(G) = O(klog^c k)$.

Hence, $|V(G)| \leq poly(klog^c k) \leq poly(k)$. □

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Lemma

Let H be a planar graph and let $f : \mathbb{N} \to \mathbb{R}$ be a function.
A minimal counterexample G to the EP property for H does not contain a H-minor of size $O(\log n)$.

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Lemma

Let H be a planar graph and let $f : \mathbb{N} \to \mathbb{R}$ be a function.
A minimal counterexample G to the EP property for H does not contain a H-minor of size $O(\log n)$.

Proof.

Toward a contradiction, let M be an H-minor on $O(\log n)$ vertices.
Set $k = \nu_H(G)$.
Since $n = poly(k)$, $\log n = O(k)$.
Moreover, since $\nu_H(G - M) \leq k - 1$, by minimality of G we have:

$$\begin{aligned}
\tau_H(G) &\leq |V(M)| + \tau(G - V(M)) \\
&\leq O(\log n) + O((k-1)\log(k-1)) \\
&\leq O(k) + O((k-1)\log(k-1)) \\
&\leq O(k \log k)
\end{aligned}$$

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If an n-vertex graph has average degree at least $O(t\sqrt{\log t})$, then it contains a K_t minor on $O(\log n)$ vertices.

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Lemma

For every planar graph H, there is a function $g : \mathbb{N} \to \mathbb{N}$ such that, if G is a minimal counterexample, then every H-minor free induced subgraph J of G that has exactly p vertices with a neighbor in $V(G - J)$ satisfies $|J| < g(p)$.

Back to the wheels

Let W_t be the wheel on t vertices and let G be a minimal counterexample for W_t.

Fix some constants: $t\sqrt{\log t} << c_1 << p << c_2$.

- Let C be a maximum collection of vertex disjoint cycles in G whose lengths are in $[c_1, c_2]$,
- let P be a maximum collection of vertex disjoint paths in $G - C|$ of length p,
- let R be tthe collection of components in $G - (C \cup P)$

Observe that every piece of $C \cup P \cup R$ is H-minor free.

Thank you for your attention