The Rödl Nibble

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Beyond design

Design = perfect case **BUT** do not always exists.

No decomposition of K_5 into copies of K_3.

($|E(K_5)| = 10$ and $|E(K_3)| = 3$.)

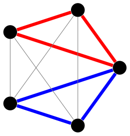
No decomposition of $K_6^{(3)}$ into copies of $K_4^{(3)}$.

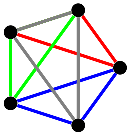
(degree in $K_6^{(3)} = 10$ and degree in $K_4^{(3)} = 3$)

Approximate design

packing of F in G : set of edge-disjoint copies of F in G.

covering of G by F : set of copies of F such that every edge of G is in one of the copies.





Approximate design

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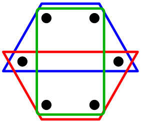
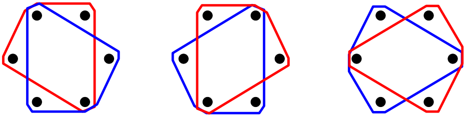




Approximate design

packing of F in G : set of edge-disjoint copies of F in G.

covering of G by F : set of copies of F such that every edge of G is in one of the copies.

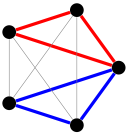
$m(n,k,t)$: maximum size of a packing of $K_k^{(t)}$ in $K_n^{(t)}$
$M(n,k,t)$: minimum size of a covering of $K_n^{(t)}$ by $K_k^{(t)}$.

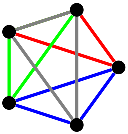
Trivial inequalities

$$m(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}} \leq M(n,k,t)$$

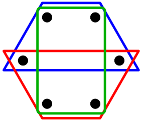
with equalities iff there is a t-$(n,k,1)$-design.

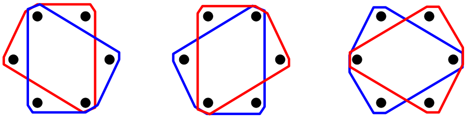
Examples





$$m(5,3,2) = 2 < \frac{\binom{5}{2}}{\binom{3}{2}} = \frac{10}{3} \leq M(5,3,2) = 4$$





$$m(6,4,3) = 3 < \frac{\binom{6}{3}}{\binom{4}{3}} = 5 < M(6,4,3) = 6$$

Erdős–Hanani Conjecture

$$m(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}} \leq M(n,k,t)$$

Conjecture (Erdős–Hanani, 1963)

$$\lim_{n\to+\infty} \frac{m(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1 \qquad \textit{and} \qquad \lim_{n\to+\infty} \frac{M(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1.$$

Erdős–Hanani: True for $t = 2$ (graphs).

Rödl Theorem

$$m(n,k,t) \leq \frac{\binom{n}{t}}{\binom{k}{t}} \leq M(n,k,t)$$

Theorem (Rödl, 1985)

$$\lim_{n\to+\infty} \frac{m(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1 \qquad \textit{and} \qquad \lim_{n\to+\infty} \frac{M(n,k,t)}{\binom{n}{t}/\binom{k}{t}} = 1.$$

The auxiliary hypergraph $\mathcal{H}$

Hypergraph $\mathcal{H}$: vertices = t-subsets of $[n]$
hyperedges = sets of $\binom{k}{t}$ t-subsets of a k-subset of $[n]$.

packing of $K_k^{(t)}$ in $K_n^{(t)}$ $\leftrightarrow$ matching in $\mathcal{H}$
matching : set of pairwise vertex-disjoint hyperedges.

covering of $K_n^{(t)}$ by $K_k^{(t)}$ $\leftrightarrow$ cover of $\mathcal{H}$
cover : set of hyperedges s.t. every vertex is in one of them.

Theorem (Rödl, 1985)

$\mathcal{H}$ has a matching of size $(1-o(1))\ N/r$ $\qquad N = \binom{n}{t}$
and a cover of size $(1-o(1))\ N/r$ $\qquad r = \binom{k}{t}$.

A more general result

- $\mathcal{H}$ has N vertices with $N = \binom{n}{t}$.
- $\mathcal{H}$ is r-uniform with $r = \binom{k}{t}$.
- $\mathcal{H}$ is D-regular with $D = \binom{n-t}{k-t}$.

Observation : The existence of the desired matching and cover also holds for large class of r-uniform hypergraphs D-regular graphs.

Frank and Rödl, Pippenger and Spencer 1989, Kahn 1996.

Equivalence of the two statements

In a D-regular r-uniform hypergraph of order N

matching of size $(1 - o(1))\ N/r \Leftrightarrow$ cover of size $(1 + o(1))\ N/r$

$(\Rightarrow)$ matching of size $\frac{N(1-\epsilon)}{r}$, $\Rightarrow$ at most ϵN non-covered vertices
$\Rightarrow$ cover of size at most $\frac{N(1-\epsilon)}{r} + \epsilon N$.

$(\Leftarrow)$ cover of size $\frac{N(1+\epsilon)}{r}$. Each vertex x covered $c(x)$ times.
For each x choose $c(x) - 1$ hyperedges and remove them.
At most $\sum_{v \in V}(c(x) - 1) = \frac{(1+\epsilon)N}{r} \times r - N = \epsilon N$ are removed.
$\Rightarrow$ matching of size at least $\frac{N(1+\epsilon)}{r} - \epsilon N$.

General idea: nibbling

Fix ϵ.

Take a random set of $\epsilon N/r$ edges. W.h.p. only $O(\epsilon^2 N)$ vertices covered more than once. So at least $\epsilon N - O(\epsilon^2 N)$ covered vertices.

Remove the covered vertices.

Choose again a random set of edges covering roughly an ϵ-fraction of the vertices wiht almost no overlap.

And so on until at most ϵN vertices remain.

Cover each of the remaining vertices with a dedicated hyperedge.

The Theorem

$r > 2$ fixed.
For $\kappa \geq 1$ and $a > 0$, there exists $\gamma > 0$ and d_0 s.t. the following holds for all $N \geq D \geq d_0$.

Every r-uniform hypergraph $H = (V, \mathcal{E})$ on N vertices such that

(0) $d(x) > 0$ for all $x \in V$.

(1) All vertices x except at most γN satisfy $d(x) = (1 \pm \gamma)D$.

(2) For all $x \in V$, $d(x) < \kappa D$.

(3) For each pair (x, y) of distincts vertices, $d(x, y) < \gamma D$.

has a **cover of size** $(1 + a)\frac{N}{r}$.

$d(x, y)$: **codegree** = number of edges containing both x and y.

In $\mathcal{H}$, we have $d(x, y) = \binom{n-t-1}{k-t-1} = o(D)$.

The Nibble Lemma

$r > 2$ fixed.
For $K \geq 1$, $\epsilon > 0$ and $\delta' > 0$, there exists $\delta > 0$ and D^* s.t. the following holds for all $N \geq D \geq D^*$.

Every r-uniform hypergraph $H = (V, \mathcal{E})$ on N vertices such that

(1) All vertices x except at most δN satisfy $d(x) = (1 \pm \delta)D$.

(2) For all $x \in V$, $d(x) < KD$.

(3) For each pair (x, y) of distincts vertices, $d(x, y) < \delta D$.

contains a set $\mathcal{E}'$ of hyperedges s.t.

(iv) $|\mathcal{E}'| = \frac{N}{r}\epsilon(1 \pm \delta')$.

(v) $V' = V \setminus \bigcup_{S \in \mathcal{E}'} S$ has size $N\mathrm{e}^{-\epsilon}(1 \pm \delta')$.

(vi) All vertices x of V' except at most $\delta'|V'|$ the degree $d'(x)$ of x in $H[V']$ satisfies $d'(x) = D\mathrm{e}^{-\epsilon(r-1)}(1 \pm \delta')$.

Proving the Theorem with the Nibble Lemma

$a > 0$. Take δ, ϵ very small, and p s.t. $\mathrm{e}^{-\epsilon p} < \epsilon$.
$K_i = \kappa \mathrm{e}^{i(r-1)}$, $D_i = D e^{-\epsilon i(r-1)}$ with $D \geq d_0$.
Choose $\delta = \delta_p > \delta_{p-1} > \cdots > \delta_0$ s. t. we can apply the Nibble Lemma each time and $\delta_{i-1} \leq \delta_i \mathrm{e}^{-\epsilon(r-1)}$.

Step i: $H_{i-1} \rightsquigarrow H_i$.

(1) All x but at most $\delta_{i-1} N_{i-1}$ satisfy $d(x) = (1 \pm \delta_{i-1}) D_{i-1}$.

(2) For all $x \in V_{i-1}$, $d(x) < K_{i-1} D_{i-1}$.

(3) For each pair (x, y) of distincts vertices, $d(x, y) < \delta_{i-1} D_{i-1}$.

A set $\mathcal{E}'_i$ of hyperedges s.t.

(iv) $|\mathcal{E}'_i| = \frac{|V_{i-1}|}{r} \epsilon (1 \pm \delta_i)$.

(v) $V_i = V_{i-1} \setminus \bigcup_{S \in \mathcal{E}_i} S$ has size $|V_{i-1}| \mathrm{e}^{-\epsilon} (1 \pm \delta_i)$.

(vi) All vertices x of V_i except at most $\delta_i |V_i|$ the degree of x in H_i satisfies $d(x) = D_{i-1} \mathrm{e}^{-\epsilon(r-1)} (1 \pm \delta_i) = (1 \pm \delta_i) D_i$.

Proving the Theorem with the Nibble Lemma

$a > 0$. Take δ, ϵ very small, and p s.t. $\mathrm{e}^{-\epsilon p} < \epsilon$.
$K_i = \kappa \mathrm{e}^{i(r-1)}$, $D_i = De^{-i(r-1)}$ with $D \geq d_0$.

Choose $\delta = \delta_p > \delta_{p-1} > \cdots > \delta_0$ s. t. we can apply the Nibble Lemma each time and $\delta_{i-1} \leq \delta_i \mathrm{e}^{-\epsilon(r-1)}$.

$P = \prod_{i=0}^{p}(1+\delta_i) \leq \prod_{i=0}^{p}(1+\delta \mathrm{e}^{-i\epsilon(r-1)}) \leq \frac{1+4\delta}{1+2\delta}$.

$|V_i| \leq N\mathrm{e}^{-i\epsilon}P \leq N\mathrm{e}^{-i\epsilon}(1+2\delta)$.

So $|\mathcal{E}'_i| = (\frac{\epsilon}{r}|V_{i-1}|)(1 \pm \delta_i) \leq \frac{\epsilon}{r}N\mathrm{e}^{-(i-1)\epsilon}(1+2\delta)P$
$\leq \frac{\epsilon}{r}N\mathrm{e}^{-(i-1)\epsilon}(1+4\delta)$.

Proving the Theorem with the Nibble Lemma

$a > 0$. Take δ, ϵ very small, and p s.t. $\mathrm{e}^{-\epsilon p} < \epsilon$.

$|V_i| \leq N\mathrm{e}^{-i\epsilon}(1+2\delta)$ and $|\mathcal{E}'_i| \leq \frac{\epsilon}{r} N\mathrm{e}^{-(i-1)\epsilon}(1+4\delta)$.

Cover of size $\sum_{i=1}^{p} |\mathcal{E}'_i| + |V_p|$.

$$\begin{aligned}
(1+4\delta)\frac{\epsilon}{r}N\sum_{i=0}^{p-1}\mathrm{e}^{-i\epsilon} + |V_p| &\leq (1+4\delta)\frac{\epsilon N}{r}\frac{1}{1-\mathrm{e}^{-\epsilon}} + (1+2\delta)N\mathrm{e}^{-\epsilon p} \\
&\leq \frac{N}{r}(1+4\delta)\left(\frac{1}{1-\mathrm{e}^{-\epsilon}} + r\epsilon\right) \\
&< (1+a)\frac{N}{r}.
\end{aligned}$$

Proving the Nibble Lemma

$r > 2$ fixed.
For $K \geq 1$, $\epsilon > 0$ and $\delta' > 0$, there exists $\delta > 0$ and D^* s.t. the following holds for all $N \geq D \geq D^*$.

Every r-uniform hypergraph $H = (V, \mathcal{E})$ on N vertices such that

(1) All vertices x except at most δN satisfy $d(x) = (1 \pm \delta)D$.

(2) For all $x \in V$, $d(x) < KD$.

(3) For each pair (x, y) of distincts vertices, $d(x, y) < \delta D$.

contains a set $\mathcal{E}'$ of hyperedges s.t.

(iv) $|\mathcal{E}'| = \frac{N}{r}\epsilon(1 \pm \delta')$.

(v) $V' = V \setminus \bigcup_{e \in \mathcal{E}'} e$ has size $N\mathrm{e}^{-\epsilon}(1 \pm \delta')$.

(vi) All vertices x of V' except at most $\delta'|V'|$ the degree $d'(x)$ of x in $H[V']$ satisfies $d'(x) = D\mathrm{e}^{-\epsilon(r-1)}(1 \pm \delta')$.

Proving the Nibble Lemma

$\mathcal{E}'$ a random subset of $\mathcal{E}$: every $e \in \mathcal{E}$ is picked randomly, independently with probability $p = \epsilon/D$.

- With very high probability (say ≥ 0.9) (iv) holds.
- With very high probability (say ≥ 0.9) (v) holds.
- With very high probability (say ≥ 0.9) (vi) holds.

$\Rightarrow$ with positive probability (≥ 0.7), (iv), (v) and (vi) hold.

With very high probability, (iv) holds

(iv) $|\mathcal{E}'| = \frac{N}{r}\epsilon(1 \pm \delta')$.

(1): All vertices x except at most δN satisfy $d(x) = (1 \pm \delta)D$.

$\Longrightarrow$

$$\frac{1}{r}(1-\delta)^2 DN \leq |\mathcal{E}| \leq \frac{1}{r}(1+\delta)DN + \delta KDN)$$

$$|\mathcal{E}| = (1 \pm \delta_1)\frac{DN}{r}.$$

$$\mathbf{E}(|\mathcal{E}'|) = |\mathcal{E}|/p = (1 \pm \delta_1)\frac{\epsilon N}{r}$$

Need to prove that $|\mathcal{E}'|$ **is concentrated** around its expected value.

2nd Moment Method

Variance : $\mathbf{Var}(X) = \mathbf{E}\left(\ (X - E(X))^2\ \right)$.

Chebyshev Inequality : For any λ,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| \geq \lambda\sqrt{\mathbf{Var}(X)}\ \right) \leq \frac{1}{\lambda^2}\ .$$

$$\mathbf{Pr}\left(X = \mathbf{E}(X) \pm \lambda\sqrt{\mathbf{Var}(X)}\ \right) \geq 1 - \frac{1}{\lambda^2}\ .$$

Variance and covariance

Assume $X = X_1 + \cdots + X_m$.

$$\mathbf{Var}(X) = \sum_{i=1}^{m} \mathbf{Var}(X_i) + \sum_{i \neq j} \mathbf{Cov}(X_i, X_j)$$

Covariance : $\mathbf{Cov}(Y, Z) = \mathbf{E}(YZ) - \mathbf{E}(Y)\,\mathbf{E}(Z)$.

If Y and Z are independent, then $\mathbf{Cov}(Y, Z) = 0$.

If $X_i = 1$ with probability p_i and 0 otherwise, then

$$\mathbf{Var}(X_i) = p_i(1 - p_i) \leq p_i = \mathbf{E}(X_i)$$

$$\mathbf{Var}(X) \leq \mathbf{E}(X) + \sum_{i \neq j} \mathbf{Cov}(X_i, X_j)$$

With very high probability, (iv) holds

(iv) $|\mathcal{E}'| = \frac{N}{r}\epsilon(1 \pm \delta')$.

(1): All vertices x except at most δN satisfy $d(x) = (1 \pm \delta)D$.

$$\Longrightarrow \qquad \frac{1}{r}(1-\delta)^2 DN \leq |\mathcal{E}| \leq \frac{1}{r}(1+\delta)DN + \delta KDN)$$

$$|\mathcal{E}| = (1 \pm \delta_1)\frac{DN}{r}.$$

$$\mathbf{E}(|\mathcal{E}'|) = |\mathcal{E}|/p = (1 \pm \delta_1)\frac{\epsilon N}{r}$$

$$\mathbf{Var}(|\mathcal{E}'|) = |\mathcal{E}|p(1-p) \leq (1 \pm \delta_1)\frac{\epsilon N}{r}.$$

By Chebyshev Inequality,

$$\mathbf{Pr}\left(|\mathcal{E}'| = (1 \pm \delta_2)\frac{\epsilon N}{r}\right) > 0,9.$$

With very high probability, (v) holds

(v) $V' = V \setminus \bigcup_{e \in \mathcal{E}'} e$ has size $N\mathrm{e}^{-\epsilon}(1 \pm \delta')$.

$I_x = 1$ if $x \notin \bigcup_{e \in \mathcal{E}'} e$ and $I_x = 0$ otherwise.

$|V'| = \sum_{x \in V} I_x$.

x good if $d(x) = (1 \pm \delta D)$, bad otherwise.

$$x \text{ good} : \mathbf{E}(I_x) = \mathbf{Pr}(I_x = 1) = (1-p)^{d(x)} = \left(1 - \frac{\epsilon}{D}\right)^{(1 \pm \delta D)} = \mathrm{e}^{-\epsilon}(1 \pm \delta_3).$$

x bad: $0 \leq \mathbf{E}(I_x) \leq 1$, but at most δN bad vertices.

Linearity of the Expected Value : $\mathbf{E}(|V'|) \leq N\mathrm{e}^{-\epsilon}(1 \pm \delta_4)$.

With very high probability, (v) holds

$$\begin{aligned}\mathbf{Var}(|V'|) &= \sum_{x\in V}\mathbf{Var}(I_x) + \sum_{x,y\in V, x\neq y}\mathbf{Cov}(I_x, I_y)\\ &\leq \mathbf{E}(|V'|) + \sum_{x,y\in V, x\neq y}\mathbf{Cov}(I_x, I_y)\end{aligned}$$

$$\begin{aligned}\mathbf{Cov}(I_x, I_y) &= \mathbf{E}(I_x I_y) - \mathbf{E}(I_x)\,\mathbf{E}(I_y)\\ &= (1-p)^{d(x)+d(y)-d(x,y)} - (1-p)^{d(x)+d(y)}\\ &\leq (1-p)^{-d(x,y)} - 1 \leq \left(1-\frac{\epsilon}{D}\right)^{-\delta(D)} - 1 \leq \delta_5.\end{aligned}$$

$$\mathbf{Var}(|V'|) \leq \mathbf{E}(|V'|) + \delta_5 N^2 \leq \delta_6\left(\mathbf{E}(|V'|)\right)^2 .$$

Chebyshev : With proba $\geq 0,9$,

$$|V'| = (1\pm\delta_7)\,\mathbf{E}(|V'|) = (1\pm\delta_8)N\mathrm{e}^{-\epsilon} .$$