JCALM 2017
Introduction to Designs and Steiner Triple Systems

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May 3, 2017

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Decomposition of Complete Graphs

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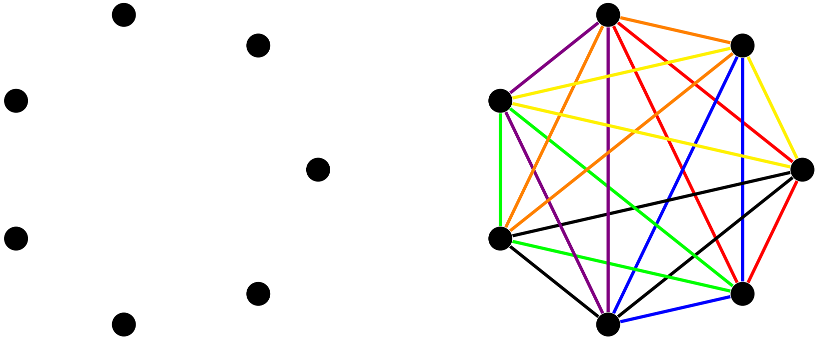
(7,3,1) design: Decompose K_7 into K_3



Decomposition: Partition of edges.

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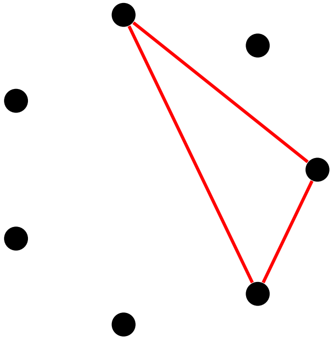

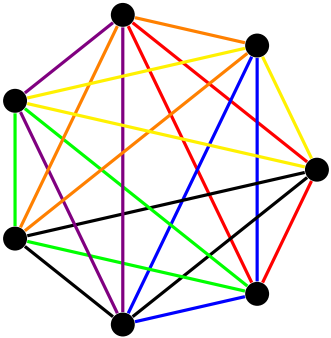

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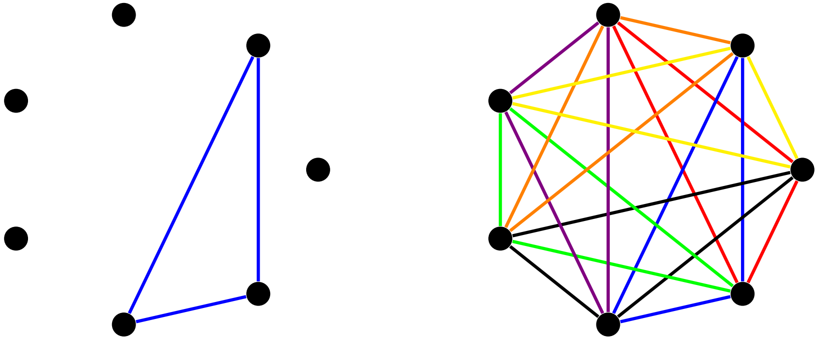

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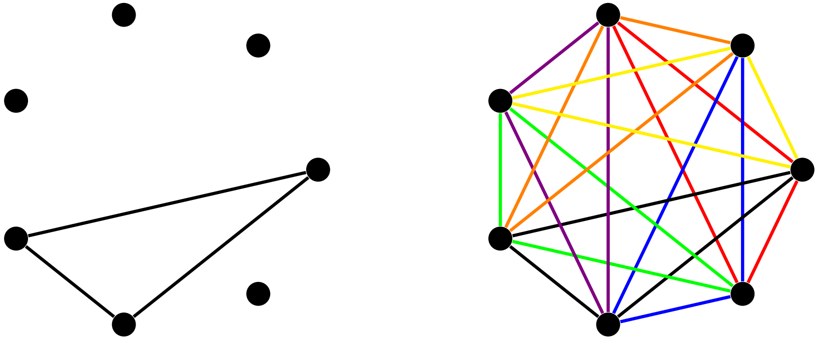

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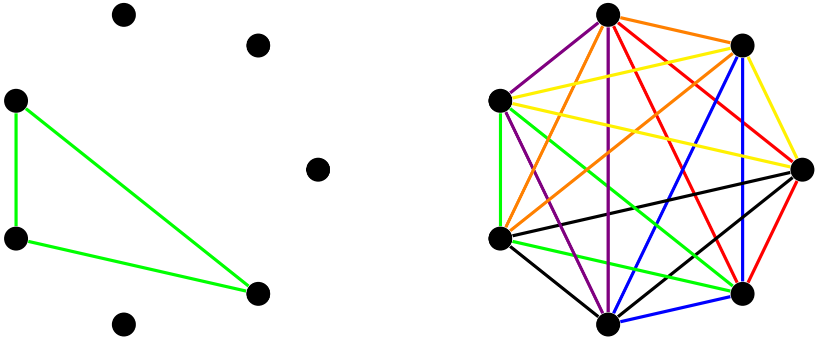

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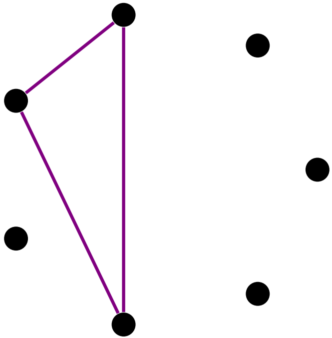

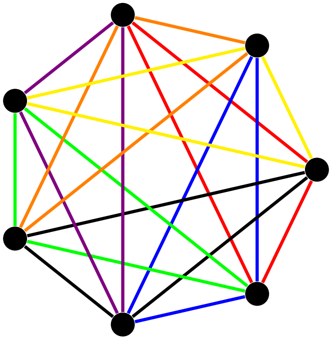

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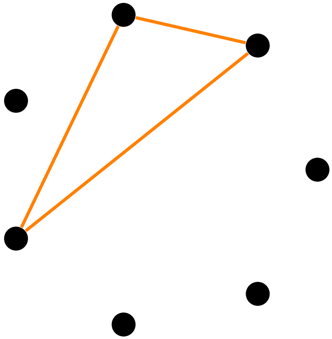

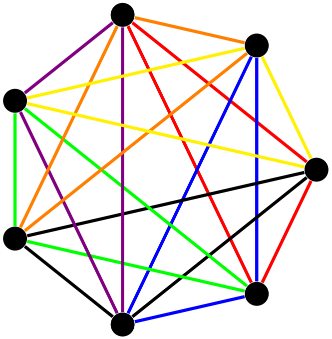

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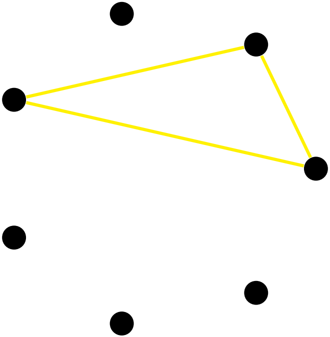

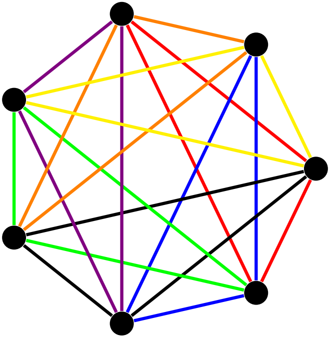

Decomposition: Partition of edges.

K_4 and K_5 don't decompose into K_3

- K_4 doesn't decompose into K_3:
  for all $v \in V(K_4)$ and $u \in V(K_3)$, $deg(v) = 3$ and $deg(u) = 2$ but $2 \nmid 3$.

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  $|E(K_5)| = 10$ and $|E(K_3)| = 3$ but $3 \nmid 10$.

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- K_5 doesn't decompose into K_3:
  $|E(K_5)| = 10$ and $|E(K_3)| = 3$ but $3 \nmid 10$.

- Main question: when do designs exist?

t-designs

Definition

A $t-(n, k, \lambda)$ **design** is a set of subsets of size k (**blocks**) of a set V of n vertices such that all subsets of size t belong to exactly λ blocks.

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- Abbreviated to (n, k, λ) design when $t = 2$.

2-designs ($t = 2$) with $\lambda = 1$: Decompose K_n into K_k

Definition

An $(n, k, 1)$ **design** is a set of subsets of size k (**blocks**) of a set V of n vertices such that any pair of elements belongs to exactly 1 block.

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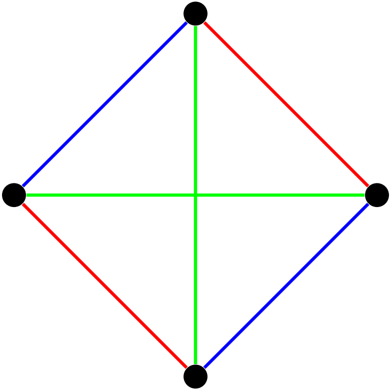
An $(n, k, 1)$ **design** is a decomposition of K_n (edge partition) into K_k.

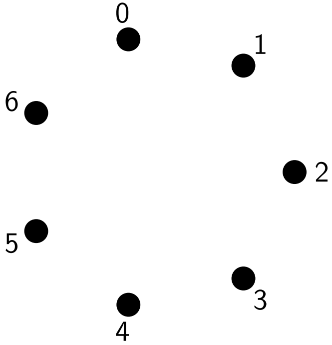
- Each block of size k: K_k.
- Any pair ($t = 2$) of elements in exactly one ($\lambda = 1$) block: each edge of K_n in exactly one K_k.

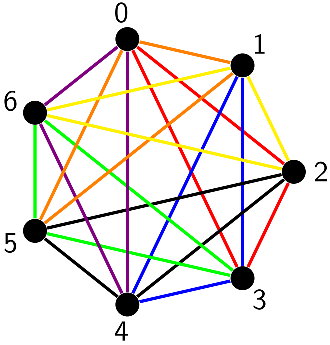
(4,2,1) design: Decompose K_4 into K_2



$(n, 2, 1)$ designs $(k = 2)$ trivially exist (decomposing K_n into K_2).

(7,3,1) design: Decompose K_7 into K_3





Decomposition obtained by rotation of K_3.

$B_i = \{\{i, i+2, i+3\} \mid i \in [7]\}$ (indices $\pmod 7$).

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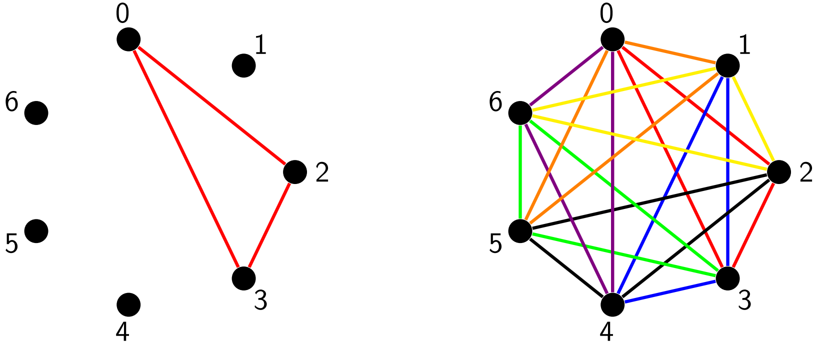

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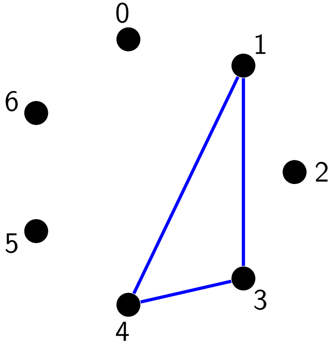

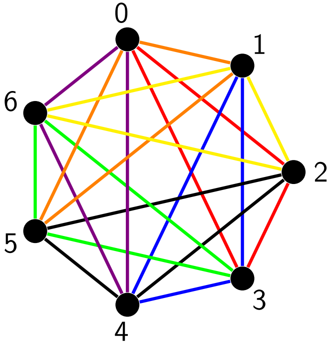

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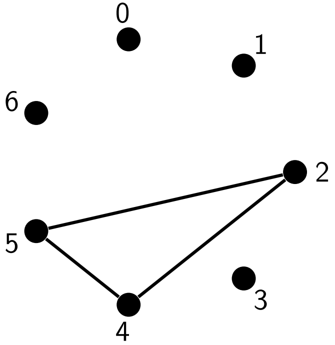

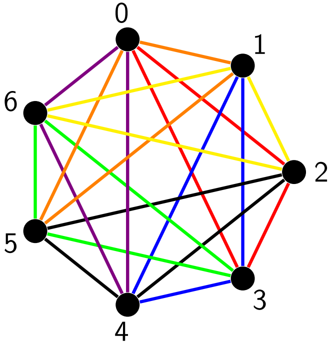

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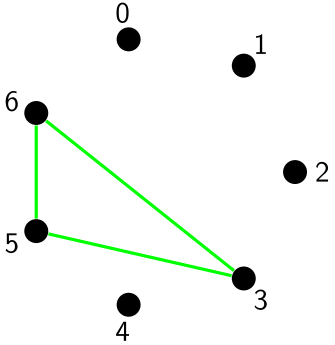

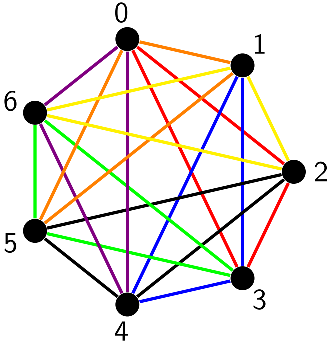

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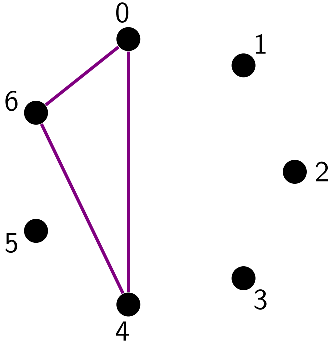

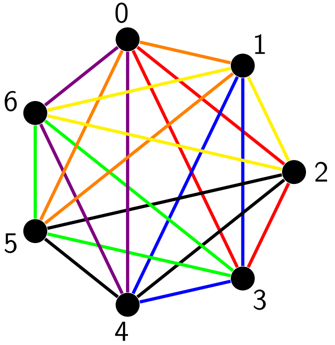

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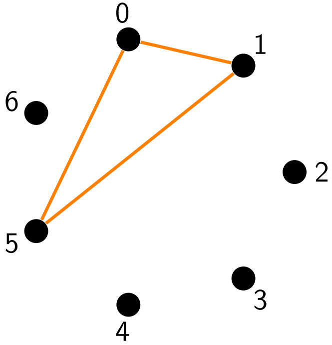

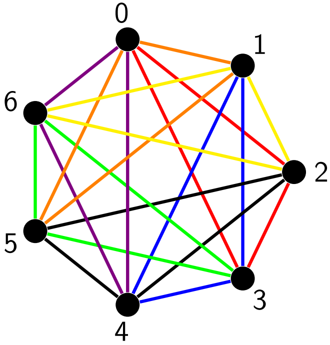

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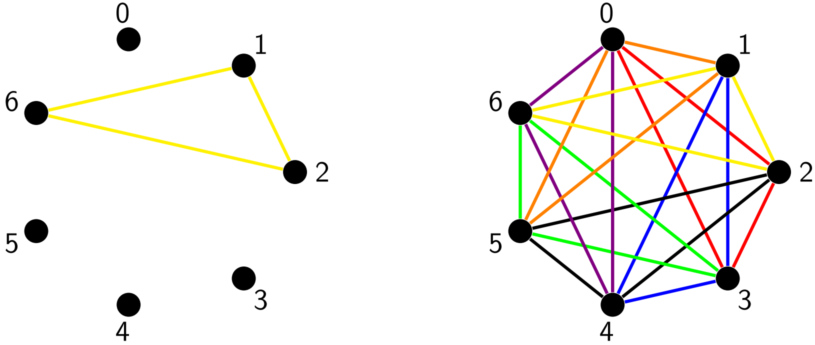
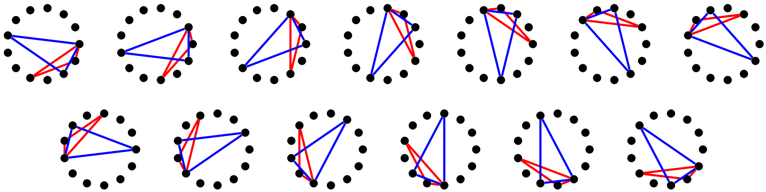


Decomposition obtained by rotation of K_3.

$B_i = \{\{i, i+2, i+3\} \mid i \in [7]\}$ (indices (mod 7)).

(13,3,1) design: Decompose K_{13} into K_3

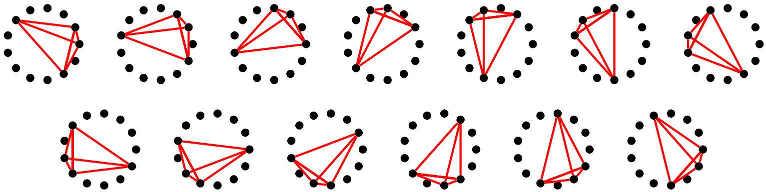


Decomposition obtained by rotation of two K_3.

$B_i = \{\{i, i+2, i+7\} \mid i \in [13]\}$ (indices (mod 13)).

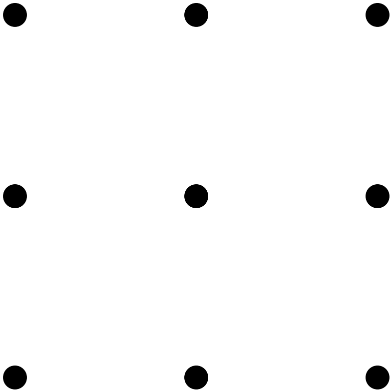
$B_j = \{\{j, j+1, j+4\} \mid j \in [13]\}$ (indices (mod 13)).

(13,4,1) design: Decompose K_{13} into K_4



Decomposition obtained by rotation of K_4.
$B_i = \{\{i, i+4, i+5, i+7\} \mid i \in [13]\}$ (indices (mod 13)).

(9,3,1) design: Decompose K_9 into K_3



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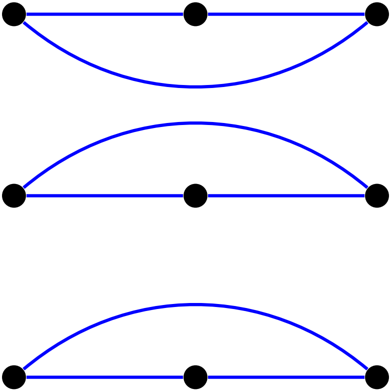
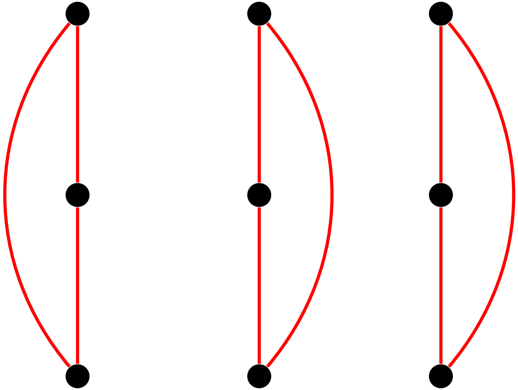


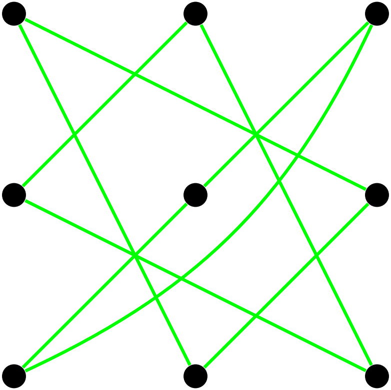
Blue blocks partition vertices.

(9,3,1) design: Decompose K_9 into K_3



Red blocks partition vertices.

(9,3,1) design: Decompose K_9 into K_3



Green blocks partition vertices.

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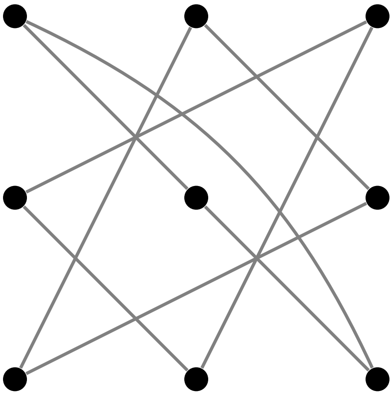

Gray blocks partition vertices.

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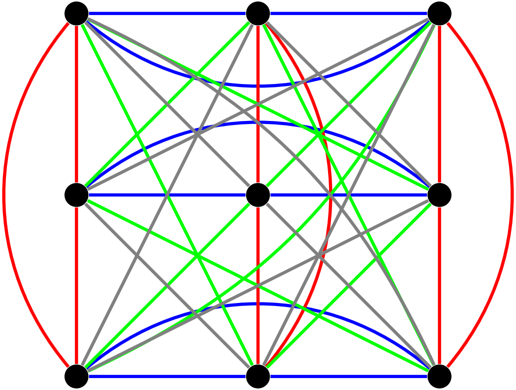

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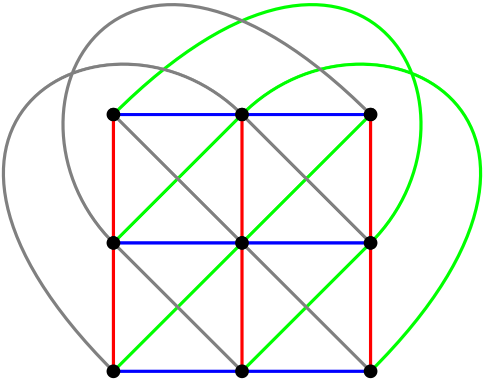
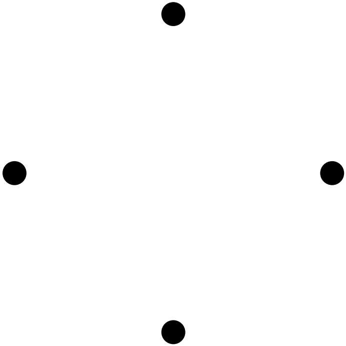


Resolvable designs

Definition

A design is **resolvable** if the blocks ($\mathbf{K_k}$) can be grouped into sets that partition the vertices.

Resolvable (4,2,1) design: Resolvable decomposition of K_4 into K_2



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For n even, K_n decomposes into $n-1$ **perfect matchings**.

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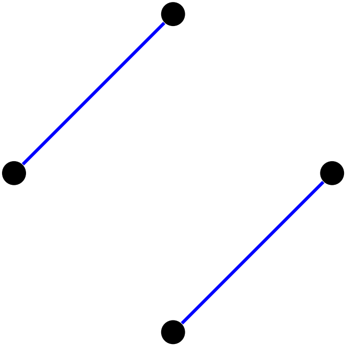

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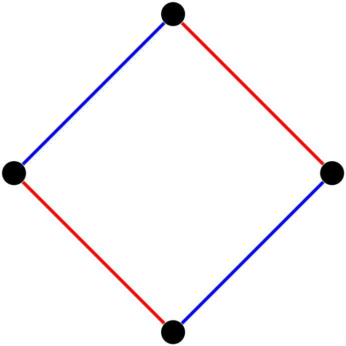

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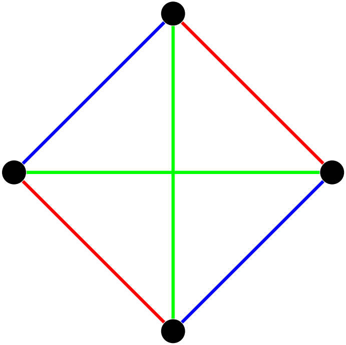

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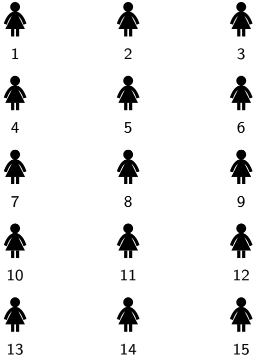
Kirkman's Schoolgirl Problem: Resolvable decomposition of K_{15} into K_3

Kirkman's Schoolgirl Problem (Reverend Kirkman, 1850)

15 young ladies in a school walk out 3 abreast for 7 days in succession. Is it possible to arrange them daily, so that no 2 walk twice abreast?



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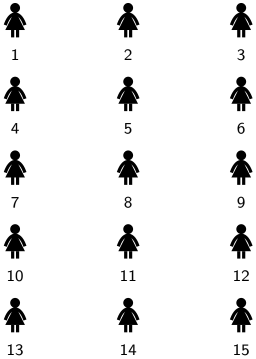
- Seeked actual solution for resolvable (15,3,1) design.



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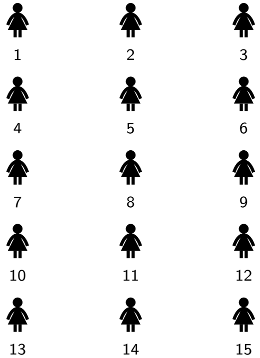
- Seeked actual solution for resolvable (15,3,1) design.
- Same can be applied to Coati team.



Coati Schoolgirl Problem - Day 1

Michelle	Jeanne-Claudia	Christine
Nicole N.	Frédérica V.	Patricia
Fiona	Andrea	Stéphanie
Juliana	Nicole H.	Joanna
Christelle	Frédérica G.	Valentina

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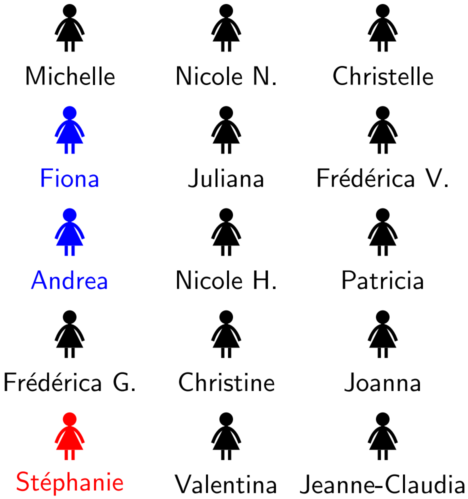
Coati Schoolgirl Problem - Day 2



Coati Schoolgirl Problem - Day 2

Michelle	Nicole N.	Christelle
Fiona	Juliana	Frédérica V.
Andrea	Nicole H.	Patricia
Frédérica G.	Christine	Joanna
Stéphanie	Valentina	Jeanne-Claudia

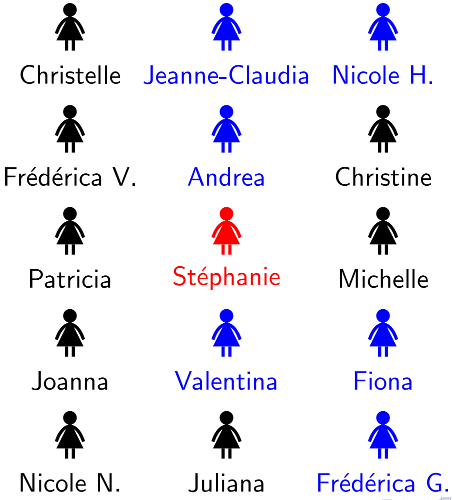
Coati Schoolgirl Problem - Day 3

Nicole N.	Fiona	Jeanne-Claudia
Juliana	Christelle	Andrea
Nicole H.	Frédérica G.	Stéphanie
Christine	Patricia	Valentina
Joanna	Michelle	Frédérica V.

Coati Schoolgirl Problem - Day 3

Nicole N.	Fiona	Jeanne-Claudia
Juliana	Christelle	Andrea
Nicole H.	Frédérica G.	Stéphanie
Christine	Patricia	Valentina
Joanna	Michelle	Frédérica V.

Coati Schoolgirl Problem - Day 4



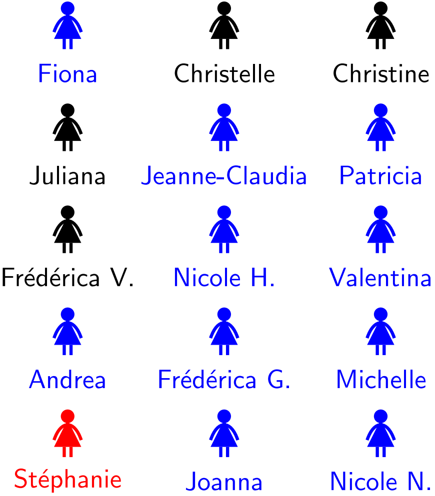
Coati Schoolgirl Problem - Day 4

Christelle	Jeanne-Claudia	Nicole H.
Frédérica V.	Andrea	Christine
Patricia	Stéphanie	Michelle
Joanna	Valentina	Fiona
Nicole N.	Juliana	Frédérica G.

Coati Schoolgirl Problem - Day 5

Fiona	Christelle	Christine
Juliana	Jeanne-Claudia	Patricia
Frédérica V.	Nicole H.	Valentina
Andrea	Frédérica G.	Michelle
Stéphanie	Joanna	Nicole N.

Coati Schoolgirl Problem - Day 5



Coati Schoolgirl Problem - Day 6

Christelle	Frédérica V.	Stéphanie
Jeanne-Claudia	Andrea	Joanna
Nicole H.	Christine	Nicole N.
Frédérica G.	Patricia	Fiona
Valentina	Michelle	Juliana

Coati Schoolgirl Problem - Day 6

Christelle	Frédérica V.	Stéphanie
Jeanne-Claudia	Andrea	Joanna
Nicole H.	Christine	Nicole N.
Frédérica G.	Patricia	Fiona
Valentina	Michelle	Juliana

Coati Schoolgirl Problem - Day 7

Christine	Stéphanie	Juliana
Patricia	Joanna	Christelle
Valentina	Nicole N.	Andrea
Michelle	Fiona	Nicole H.
Jeanne-Claudia	Frédérica V.	Frédérica G.

Coati Schoolgirl Problem - Day 7

Christine	Stéphanie	Juliana
Patricia	Joanna	Christelle
Valentina	Nicole N.	Andrea
Michelle	Fiona	Nicole H.
Jeanne-Claudia	Frédérica V.	Frédérica G.

Necessary condition (1) for an $(n, k, 1)$ design: Decomposition of K_n into K_k

Lemma

If an $(n, k, 1)$ design exists, then $n(n-1) \equiv 0 \pmod{k(k-1)}$.

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$|E(K_n)| = \frac{n(n-1)}{2}$ must be a multiple of $|E(K_k)| = \frac{k(k-1)}{2}$. □

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K_5 doesn't decompose into K_3.

Necessary condition (2) for an $(n, k, 1)$ design: Decomposition of K_n into K_k

Lemma

If an $(n, k, 1)$ *design exists, then* $n - 1 \equiv 0 \pmod{k-1}$.

Necessary condition (2) for an $(n, k, 1)$ design: Decomposition of K_n into K_k

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If an $(n, k, 1)$ design exists, then $n - 1 \equiv 0 \pmod{k-1}$.

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For all $v \in V(K_n)$ and $u \in V(K_k)$, $deg(v) = n - 1$ must be a multiple of $deg(u) = k - 1$. □

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Necessary Conditions are Sufficient in General

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Necessary Conditions are Sufficient in General

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In general, for n large enough, these 2 conditions are sufficient.

Steiner Triple Systems: Decomposition of K_n into K_3

Definition

A **Steiner Triple System** is an $(n, 3, 1)$ design.

Steiner Triple Systems: Decomposition of K_n into K_3

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A **Steiner Triple System** is an $(n, 3, 1)$ design.

- Steiner asked the question if they exist in 1853, without knowing Kirkman had already proved their existence in 1847.

Necessary conditions for Steiner Triple Systems: Decomposition of K_n into K_3

Lemma

If an $(n, 3, 1)$ *design exists, then* $n \equiv 1$ *or* $3 \pmod 6$.

Necessary conditions for Steiner Triple Systems: Decomposition of K_n into K_3

Lemma

If an $(n, 3, 1)$ design exists, then $n \equiv 1$ or $3 \pmod 6$.

Proof.

- **NC(2)** $\Rightarrow n - 1 \equiv 0 \pmod 2 \Rightarrow n$ odd $\Rightarrow n \equiv 1, 3$ or $5 \pmod 6$.

$\square$

Necessary conditions for Steiner Triple Systems: Decomposition of K_n into K_3

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If an $(n, 3, 1)$ design exists, then $n \equiv 1$ or $3 \pmod 6$.

Proof.

- **NC(2)** $\Rightarrow n - 1 \equiv 0 \pmod 2 \Rightarrow n$ odd $\Rightarrow n \equiv 1, 3$ or $5 \pmod 6$.
- **NC(1)** $\Rightarrow n(n-1) \equiv 0 \pmod 6 \Rightarrow n \equiv 1$ or $3 \pmod 6$ since if $n = 6k + 5$, then $\frac{n(n-1)}{6} = \frac{6k+5(6k+4)}{6} = \frac{18k^2+27k+10}{3} \notin \mathbb{Z}$.

$\square$

Necessary Conditions are Sufficient for Steiner Triple Systems: Decomposition of K_n into K_3

Theorem (Kirkman, 1847)

An $(n, 3, 1)$ design exists $\Leftrightarrow n \equiv 1$ or $3 \pmod 6$.

i.e. K_n can be decomposed into $K_3 \Leftrightarrow n \equiv 1$ or $3 \pmod 6$.

Lemma 1: K_{2m+1} dec. $K_3 \Leftrightarrow K_{m*2}$ dec. K_3

Lemma 1

K_{2m+1} can be decomposed into $K_3 \Leftrightarrow$
$K_{2,\cdots,2} = K_{m*2}$ can be decomposed into K_3.

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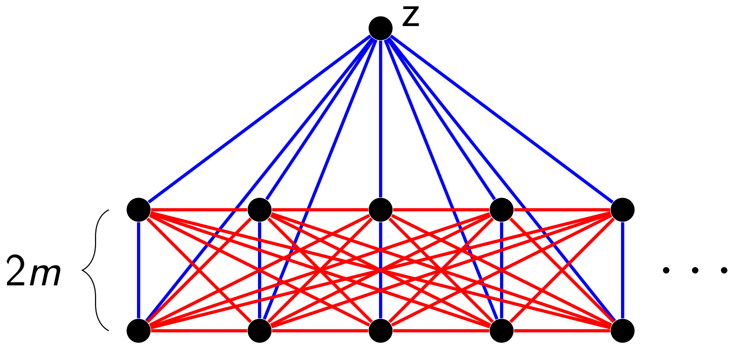

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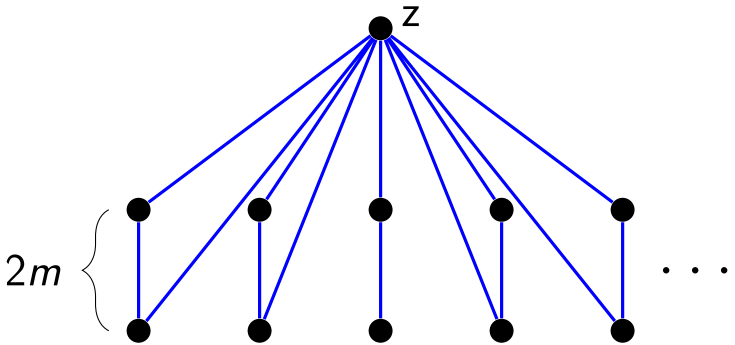

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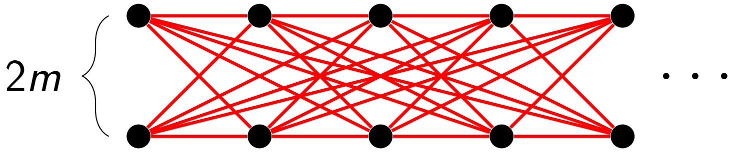

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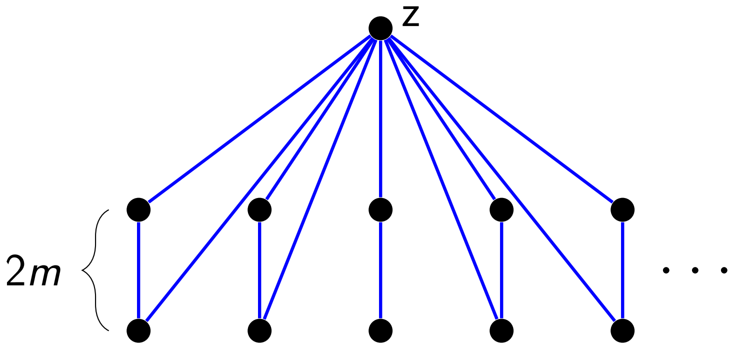

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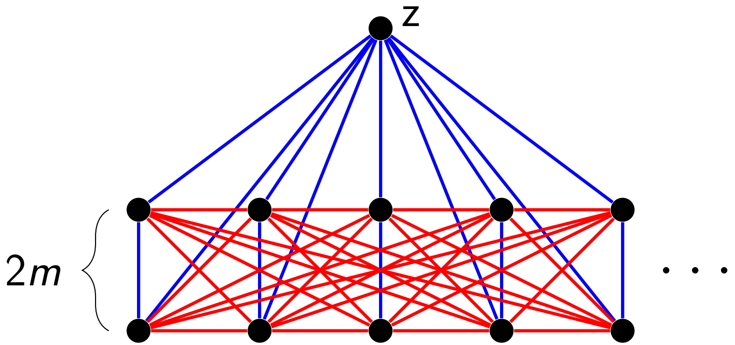

Proof of Lemma 1: K_{2m+1} dec. $K_3 \Leftrightarrow K_{m*2}$ dec. K_3

Proof.

- Let $V(K_{2m+1}) = \{v_1, \cdots, v_{2m+1}\}$, D be a decomposition of K_{2m+1} into K_3, and A be the set of all blocks containing v_{2m+1}.

$\square$

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- $|A| = m$ so w.l.o.g. suppose $A = \{v_i, v_{i+m}, v_{2m+1} \mid i \in [m]\}$. Then, $D \setminus A$ is a decomposition of H into K_3 where H is the graph K_{2m} with the edges $v_i v_{i+m}, i \in [m]$ removed. H is isomorphic to K_{m*2}.

□

Proof of Lemma 1: K_{2m+1} dec. $K_3 \Leftrightarrow K_{m*2}$ dec. K_3

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- In the other direction, if D' is a decomposition of H into K_3, then $D' \cup A$ is a decomposition of K_{2m+1} into K_3.

$\square$

Theorem 1: K_{3*q} dec. K_3

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$K_{q,q,q} = K_{3*q}$ can be decomposed into K_3.

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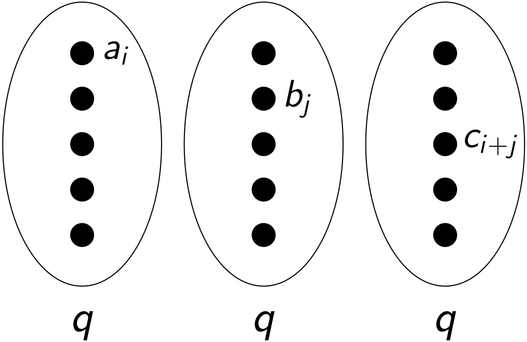

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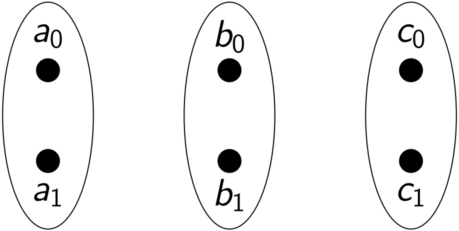

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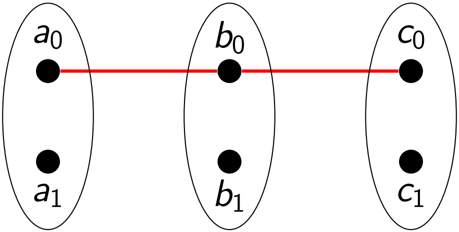

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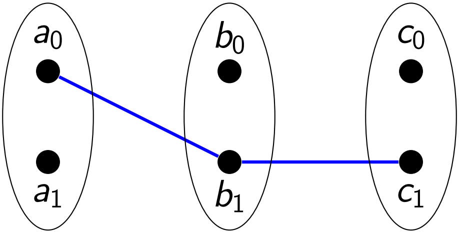

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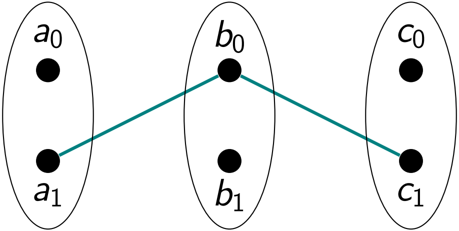

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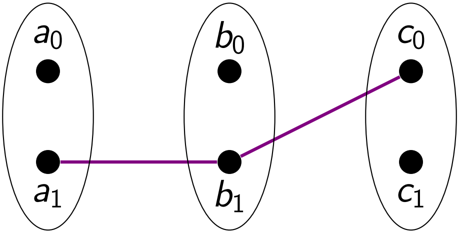

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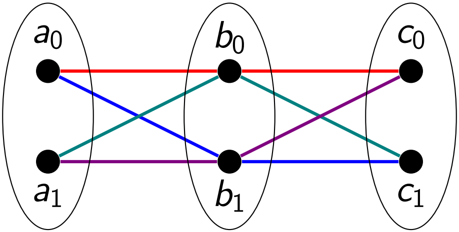

Proof of Theorem 1: K_{3*q} dec. K_3

Proof.

Let (A, B, C) be the tripartition of K_{3*q} with $A = \{a_i \mid i \in [q]\}$ and B and C defined analogously. $\{\{a_i, b_j, c_{i+j}\} \mid i \in [q], j \in [q]\}$ (indices $\pmod q$) is a decomposition of K_{3*q} into K_3. □

Theorem 2: $\forall q \geq 3$, $q \neq 6$, K_{4*q} dec. K_4

Theorem 2

For all $q \geq 3$, $q \neq 6$, $K_{q,q,q,q} = K_{4*q}$ can be decomposed into K_4.

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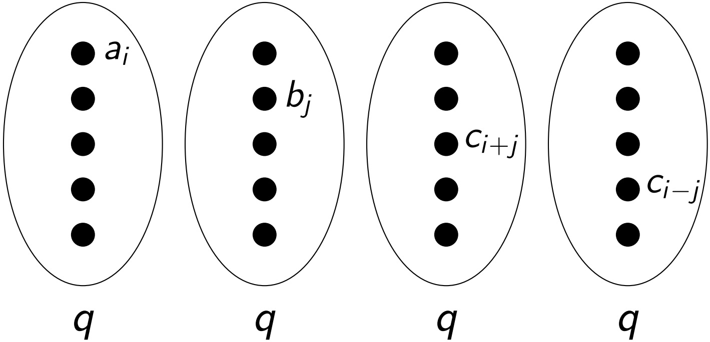
Proof for q odd.



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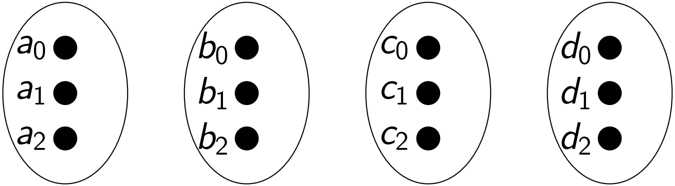

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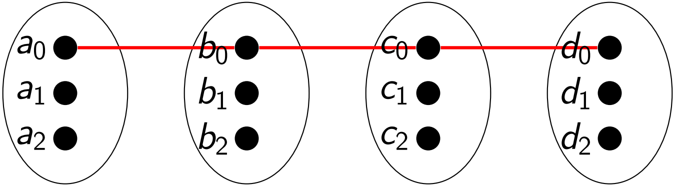

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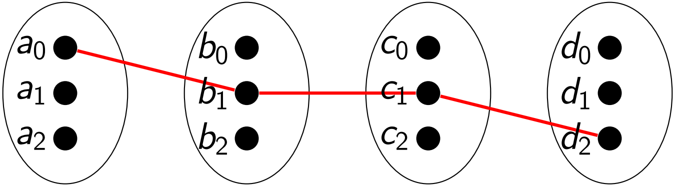

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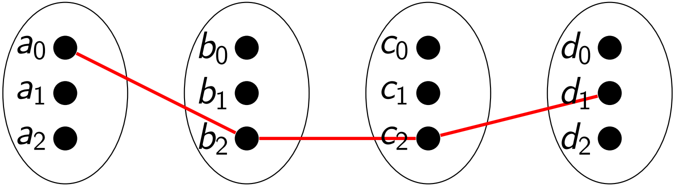

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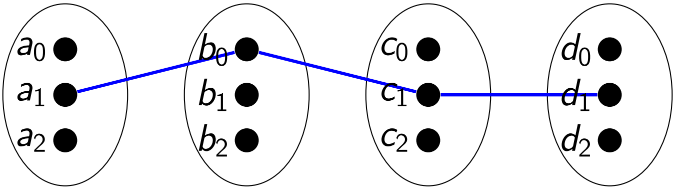

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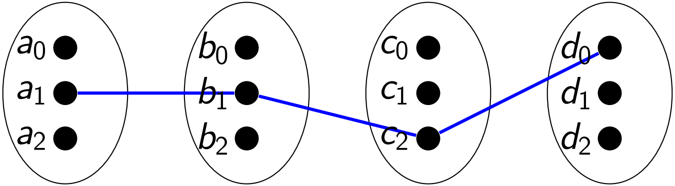

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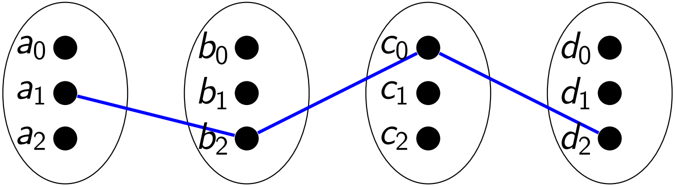

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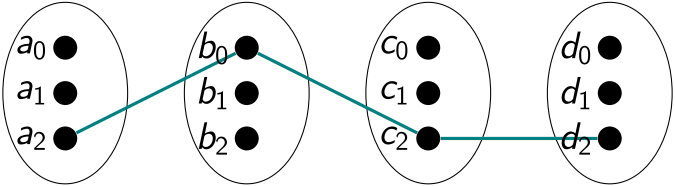

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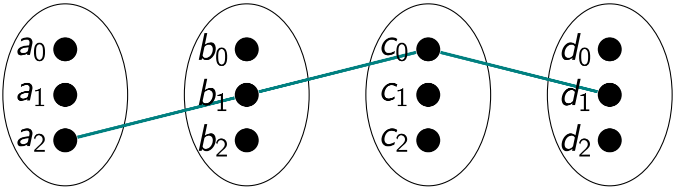

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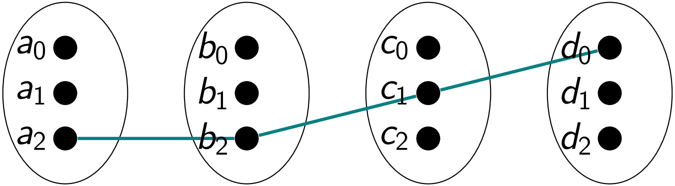

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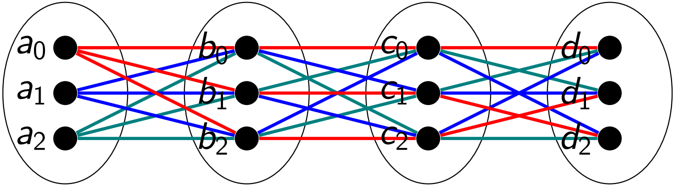
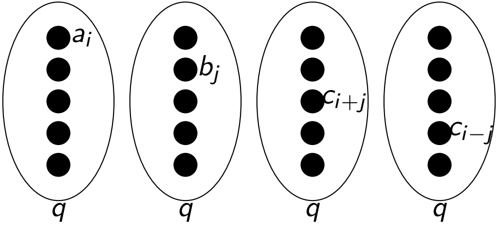


Proof of Theorem 2: $\forall q \geq 3$, $q \neq 6$, K_{4*q} dec. K_4



Proof. (q odd)

(i, j) and (i', j') don't exist such that $i + j = i' + j' \pmod{q}$ and $i - j = i' - j' \pmod{q}$. Such a pair exists $\Rightarrow 2i = 2i' \pmod{q}$ and $2j = 2j' \pmod{q} \Rightarrow i = i'$ and $j = j'$ since q is odd. $\square$

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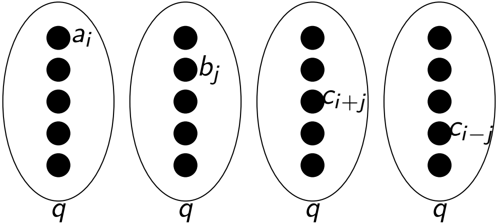

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Doesn't work when q is even: (i, j) and $(i + \frac{q}{2}, j + \frac{q}{2})$.
$(0, 0, 0, 0)$ and $(\frac{q}{2}, \frac{q}{2}, 0, 0)$.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

Theorem 3

If $r \equiv 0$ or $1 \pmod 3$, then K_r can be decomposed into a combination of K_3, K_4, and K_6.

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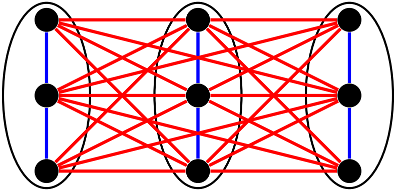
Proof.

We prove by induction. The result is trivially true for $r = 3, 4, 6$ and $r = 7$ we already saw. Suppose $r = 9m + j$ with $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

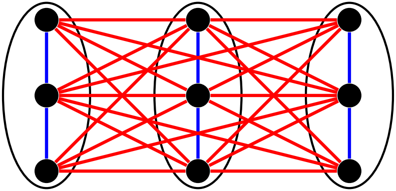
Example: $r = 9$.



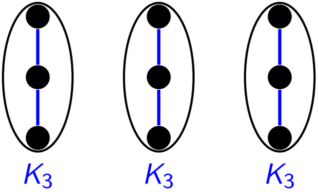
$K_9 = 3K_3 \cup K_{3,3,3}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

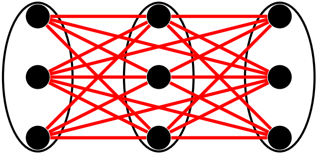
Example: $r = 9$.



$K_9 = 3K_3 \cup K_{3,3,3}$



Base Case: K_3 dec. K_3, K_4, K_6.

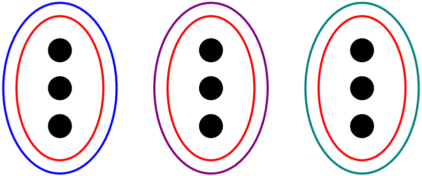


$K_{3,3,3}$

Thm 1: K_{3*q} dec. K_3.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

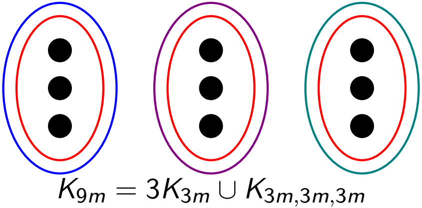
$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$.

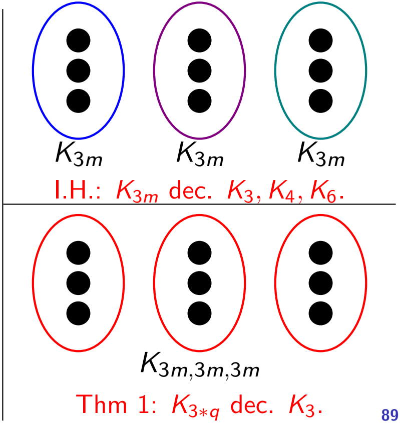


$K_{9m} = 3K_{3m} \cup K_{3m,3m,3m}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$.





Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

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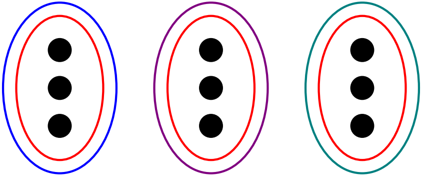
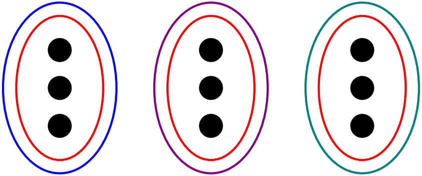


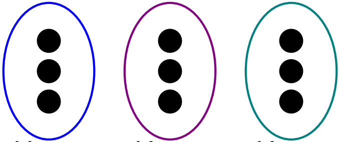
$K_{9m+3} = 3K_{3m+1} \cup K_{3m+1,3m+1,3m+1}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$.

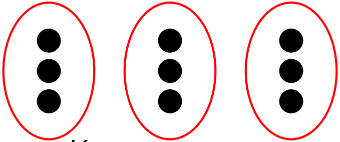


$K_{9m+3} = 3K_{3m+1} \cup K_{3m+1,3m+1,3m+1}$



K_{3m+1} K_{3m+1} K_{3m+1}

I.H.: K_{3m+1} dec. K_3, K_4, K_6.

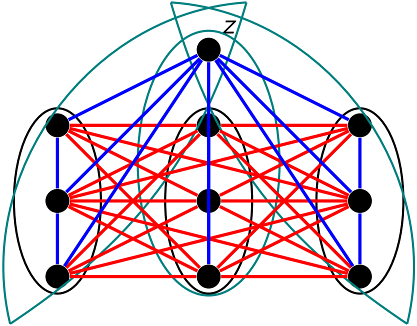


$K_{3m+1,3m+1,3m+1}$

Thm 1: K_{3*q} dec. K_3.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

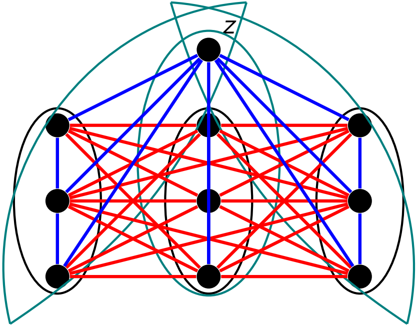
Example: $r = 10$.



$K_{10} = 3(K_3 \cup z) \cup K_{3,3,3}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

Example: $r = 10$.



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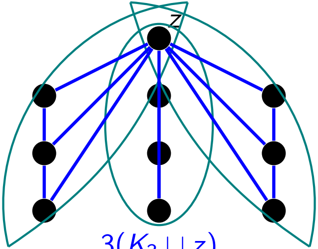
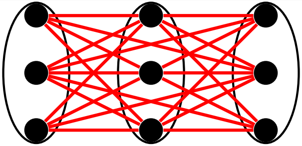


$3(K_3 \cup z)$

Base Case: K_4 dec. K_3, K_4, K_6.

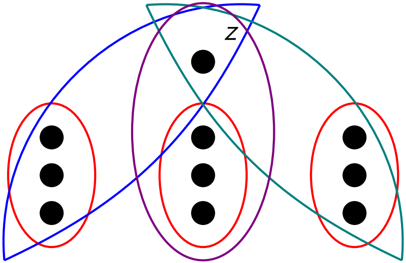


$K_{3,3,3}$

Thm 1: K_{3*q} dec. K_3.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$.



$K_{9m+1} = 3(K_{3m} \cup z) \cup K_{3m,3m,3m}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

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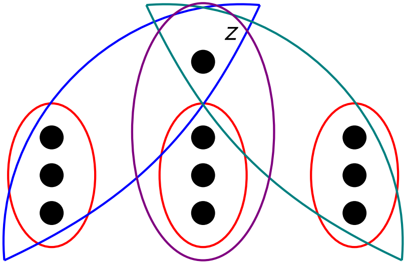
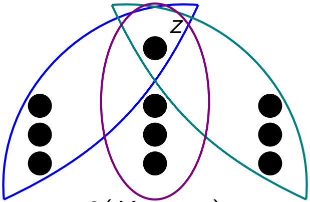
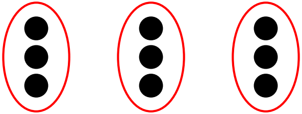


$K_{9m+1} = 3(K_{3m} \cup z) \cup K_{3m,3m,3m}$



$3(K_{3m} \cup z)$

I.H.: K_{3m+1} dec. K_3, K_4, K_6.



$K_{3m,3m,3m}$

Thm 1: K_{3*q} dec. K_3.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

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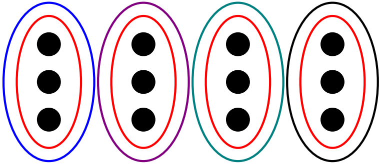
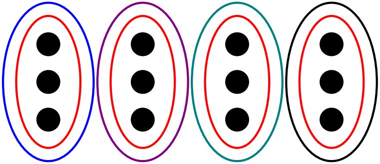


$K_{9m+3+h} =$
$3K_{3m+1} \cup K_h \cup K_{3m+1,3m+1,3m+1,h}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$. $h \in \{1, 3, 4\}$.



$K_{9m+3+h} =$
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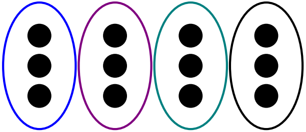
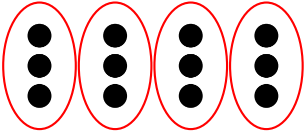


K_{3m+1} K_{3m+1} K_{3m+1} K_h

I.H.: K_{3m+1} dec. K_3, K_4, K_6.



$K_{3m+1,3m+1,3m+1,h}$

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

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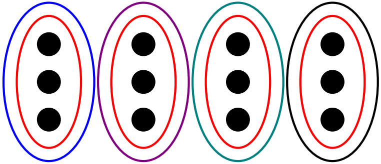

$K_{9m+3+h} =$
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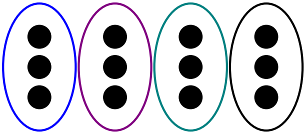
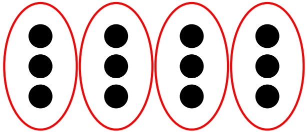


K_{3m+1} K_{3m+1} K_{3m+1} K_h

I.H.: K_{3m+1} dec. K_3, K_4, K_6.

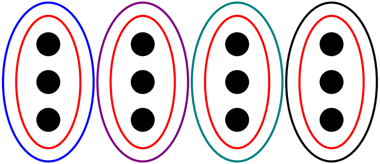


$K_{3m+1,3m+1,3m+1,3m+1}$

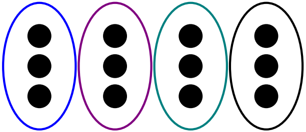
Thm 2: K_{4*q} dec. K_4.

Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, K_6

$r = 9m + j$ for $m \in \mathbb{Z}^+$ and $j \in \{0, 1, 3, 4, 6, 7\}$. $h \in \{1, 3, 4\}$.

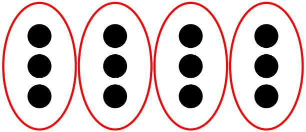


$K_{9m+3+h} =$
$3K_{3m+1} \cup K_h \cup K_{3m+1,3m+1,3m+1,h}$



K_{3m+1} K_{3m+1} K_{3m+1} K_h

I.H.: K_{3m+1} dec. K_3, K_4, K_6.



$K_{3m+1,3m+1,3m+1,h}$

$K_{3m+1,3m+1,3m+1,h}$ dec. K_3, K_4

(omit extra vertices of K_{4*q} dec. K_4).

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

- Suppose $r = 9m$. Partition $V(K_{9m})$ into three sets A_1, A_2, A_3 of size $3m$.

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

- Suppose $r = 9m$. Partition $V(K_{9m})$ into three sets A_1, A_2, A_3 of size $3m$.
- Decompose K_{9m} into 3 K_{3m} induced by A_1, A_2, A_3 and a K_{3*3m} with tripartition (A_1, A_2, A_3).

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- K_{3*3m} decomposes into K_3 by Thm 1.

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Proof.

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- By I.H., each K_{3m} decomposes into a combo of K_3, K_4, K_6.
- K_{3*3m} decomposes into K_3 by Thm 1.
- Suppose $r = 9m + 3$. Partition $V(K_{9m+3})$ into three sets of size $3m + 1$.

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

- Suppose $r = 9m + 1$. Partition $V(K_{9m+1})$ into three sets of size $3m$ and a vertex z.

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

- Suppose $r = 9m + 1$. Partition $V(K_{9m+1})$ into three sets of size $3m$ and a vertex z.
- Decompose K_{9m+1} into 3 K_{3m+1} induced by $A_1 \cup \{z\}, A_2 \cup \{z\}, A_3 \cup \{z\}$ and a K_{3*3m} with tripartition (A_1, A_2, A_3).

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

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- By I.H., each K_{3m+1} decomposes into a combo of K_3, K_4, K_6.

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- By I.H., each K_{3m+1} decomposes into a combo of K_3, K_4, K_6.
- K_{3*3m} decomposes into K_3 by Thm 1.

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

- Suppose $r = 9m + 4, 9m + 6$ or $9m + 7$. Partition $V(K_{9m+3+h})$ into three sets of size $3m + 1$ and a set of h vertices ($h \in \{1, 3, 4\}$).

Proof of Theorem 3: For $r \equiv 0$ or $1 \pmod 3$, K_r dec. K_3, K_4, and K_6

Proof.

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- Decompose K_{9m+3+h} into 3 K_{3m+1} induced by A_1, A_2, A_3, a $K_{3m+1,3m+1,3m+1,h}$, and a K_3 or K_4 if $h=3$ or $h=4$ resp.

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- Decompose K_{9m+3+h} into 3 K_{3m+1} induced by A_1, A_2, A_3, a $K_{3m+1,3m+1,3m+1,h}$, and a K_3 or K_4 if $h=3$ or $h=4$ resp.
- By I.H., each K_{3m+1} decomposes into a combo of K_3, K_4, K_6.

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- Decompose K_{9m+3+h} into 3 K_{3m+1} induced by A_1, A_2, A_3, a $K_{3m+1,3m+1,3m+1,h}$, and a K_3 or K_4 if $h=3$ or $h=4$ resp.
- By I.H., each K_{3m+1} decomposes into a combo of K_3, K_4, K_6.
- K_{4*3m+1} decomposes into K_4 by Thm 2. This decomp. transforms into a decomp. of $K_{3m+1,3m+1,3m+1,h}$ into a combo of K_3 and K_4 by omitting the extra vertices (some K_4 become K_3).

Necessary conditions are Sufficient for Steiner Triple Systems: Decomposition of K_n into K_3

Theorem (Kirkman, 1847)

An $(n, 3, 1)$ design exists $\Leftrightarrow n \equiv 1$ or $3 \pmod 6$.

i.e. K_n can be decomposed into $K_3 \Leftrightarrow n \equiv 1$ or $3 \pmod 6$.

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Let $r = \frac{n-1}{2}$. Then, $r \equiv 0$ or $1 \pmod 3$.

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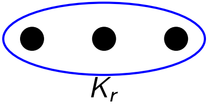

Thm 3: K_r dec. K_3, K_4, K_6.

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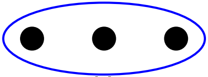
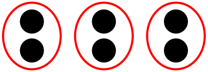


K_r

Thm 3: K_r dec. K_3, K_4, K_6.

$\longrightarrow$



K_{r*2}

K_{r*2} dec. $K_{3*2}, K_{4*2}, K_{6*2}$.

Necessary conditions are Sufficient for Steiner Triple Systems: Decomposition of K_n into K_3

Theorem (Kirkman, 1847)

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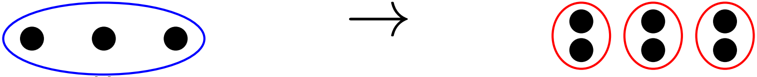

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Thm 3: K_r dec. K_3, K_4, K_6.

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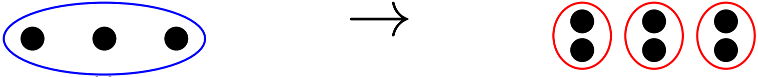

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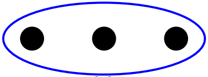
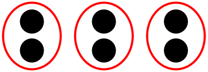


K_r

$\longrightarrow$



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$K_{2r+1} = K_n$ dec. K_3 by Lemma 1.

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- $K_{2r+1} = K_n$ can be decomposed into K_3 by Lemma 1.

Recall t-designs

Definition

A $t-(n, k, \lambda)$ **design** is a set of subsets of size k (**blocks**) of a set V of n vertices such that all subsets of size t belong to exactly λ blocks.

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- Abbreviated to (n,k,λ) design when $t=2$.

What else do we know?

Theorem (Kirkman, 1847 ($k = 3$) and Hanani, 1961 ($k = 4$), 1972 ($k = 5$))

For $\lambda = 1, t = 2$, and $k \in \{3, 4, 5\}$, an $(n, k, 1)$ design exists $\Leftrightarrow$ $n(n-1) \equiv 0 \pmod{k(k-1)}$ and $n - 1 \equiv 0 \pmod{k-1}$.

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36 officers problem (Euler, 1782): possible to arrange 6 regiments consisting of 6 officers each of different rank in a 6×6 square so that no rank or regiment will be repeated in any row or column.

All designs exist ... eventually

Theorem (Wilson, 1973 ($t = 2$) and Keevash, 2014 (all t))

For all k, λ, t, there exists n_0 such that for all $n \geq n_0$ satisfying the necessary conditions, there exists a $t - (n, k, \lambda)$ design.

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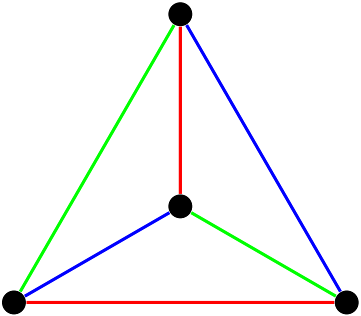
- If $n \in \{16, 21, 36, 46\}$ then there is no $(n, 6, 1)$ design.
- For 21 values of n we don't know if an $(n, 6, 1)$ design exists.

One of the few other cases fully solved

Theorem (Hanani, 1960)

A $3-(n,4,1)$ design exists $\Leftrightarrow$ $n \equiv 2$ or $4 \pmod 6$.

Affine plane ((4,2,1) design)



Definition

System of points and lines s.t.

- Two distinct points belong to exactly one line.
- Let p be a point and l a line not containing p. Exactly one line parallel to l intersects p.
- There exist at least 3 points that are not collinear.

Affine planes and $(q^2, q, 1)$ resolvable designs

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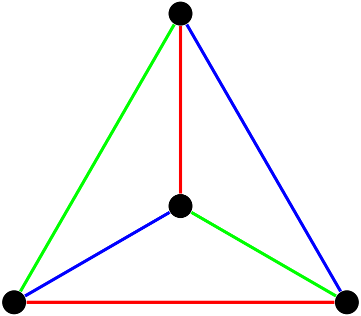
An affine plane is equivalent to a $(q^2, q, 1)$ resolvable design.

Affine plane to Projective plane ((4,2,1) design to (7,3,1) design)



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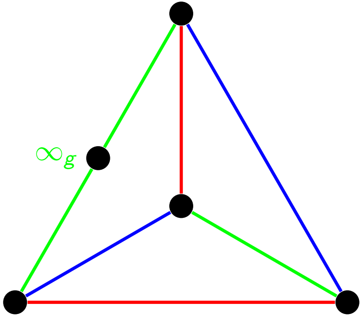

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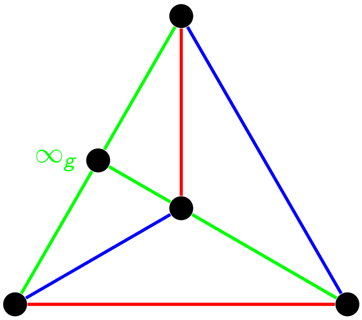

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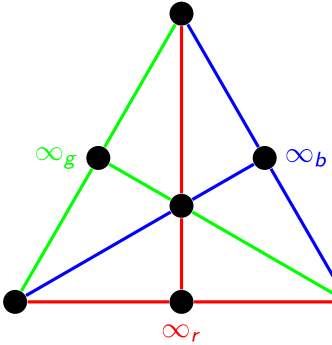

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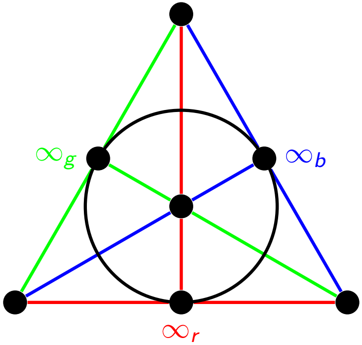
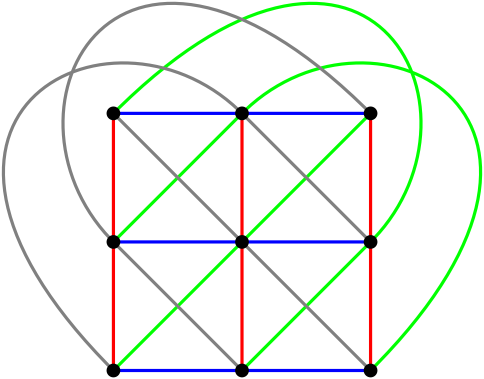


Fano Plane

Affine plane to Projective plane ((9,3,1) design to (13,4,1) design)



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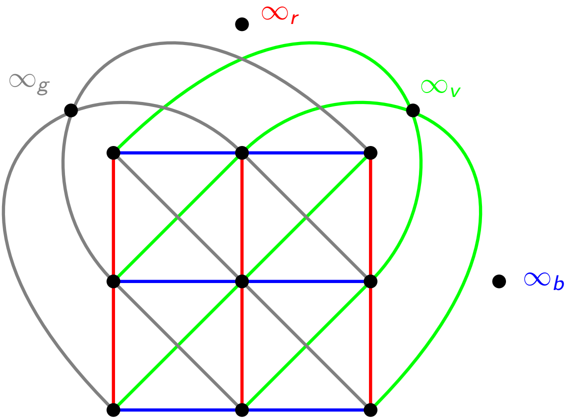

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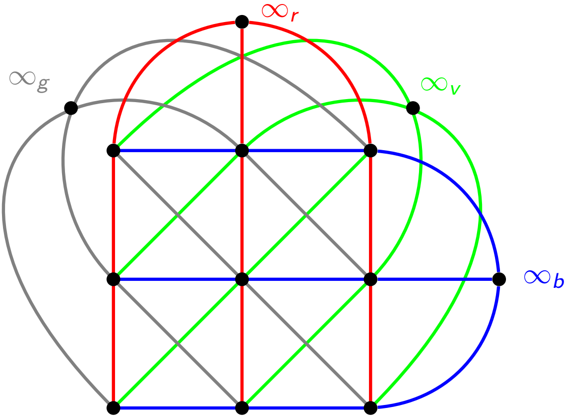

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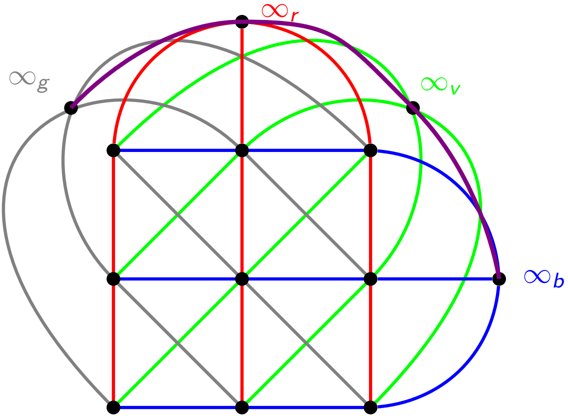

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A projective plane of order q is equivalent to a $(q^2+q+1, q+1, 1)$ design.

Properties of Graphs obtained from Projective planes

The bipartite graph $G(P)$ for a given projective plane P where one vertex class consists of the points of P and the other the lines of P has the following properties:

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- High girth: 6.

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- A $(q^2+q+1, q+1, 1)$ design exists?
- A projective plane of order q exists?

Non-existence of some designs

Theorem (Bruck-Ryser, 1949)

If q is not a prime power, q is not the sum of two squares, and $q \equiv 1$ or $2 \pmod 4$, then there does not exist a $(q^2, q, 1)$ design.

Existence of Projective Planes

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$q = 12$ is the first case where we don't know if it exists.

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- Still many values of n below the threshold of n_0 where we don't know if the designs exist.
- While we may know a lot of designs exist, we don't know how to construct many of them.

Thank you!