Numeric Domains meet Algebraic Data Types

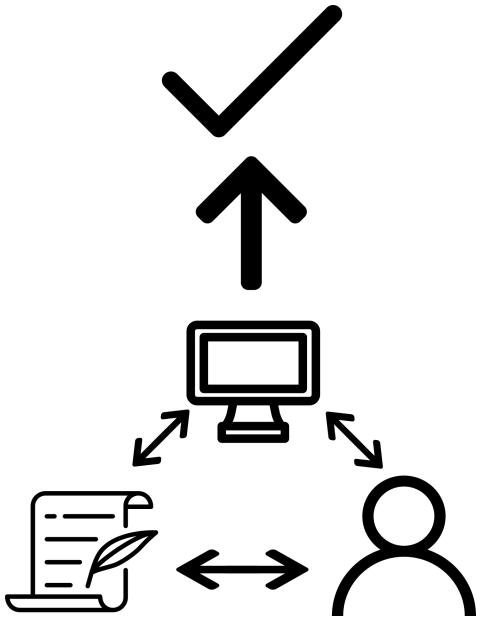
NSAD @ SPLASH 2020

Santiago Bautista Thomas Jensen Benoit Montagu

November 17, 2020

Motivation

Interactive Theorem Proving



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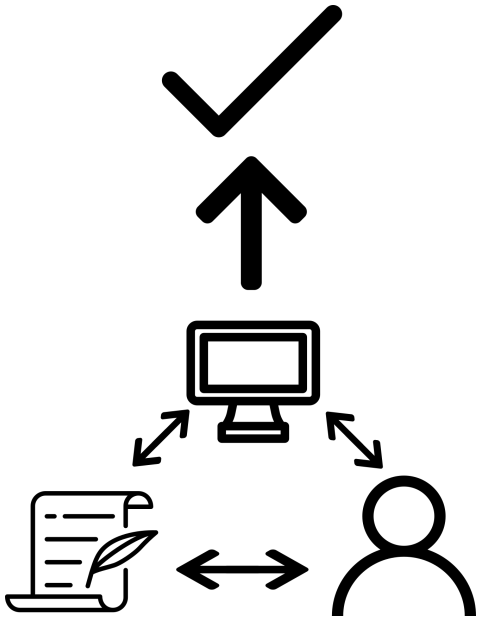

Requires manual effort

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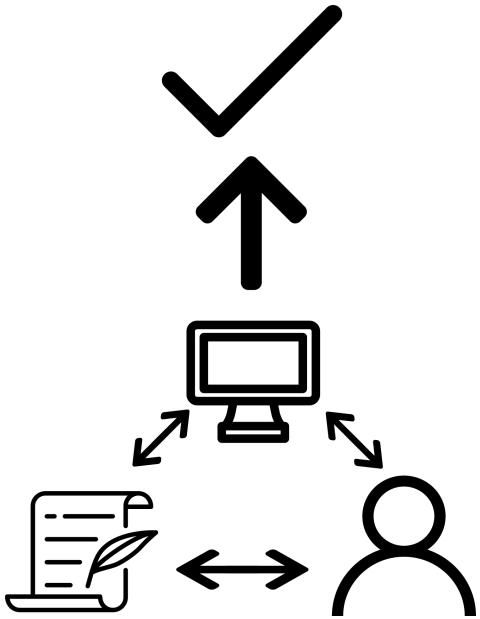


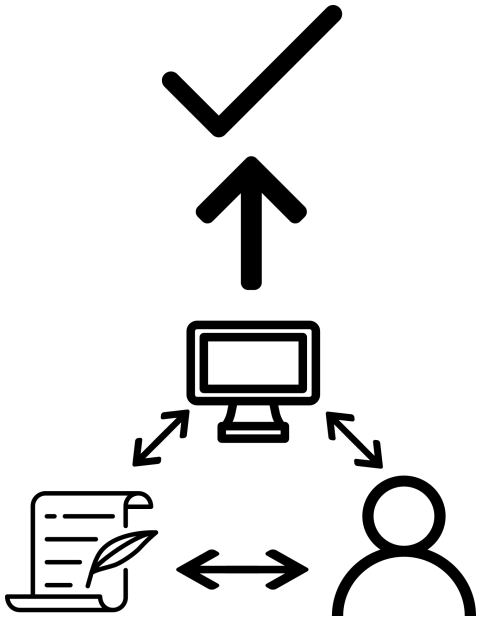
Requires manual effort

Static Analysis

Can alleviate manual effort

Motivation

Interactive Theorem Proving



Requires manual effort

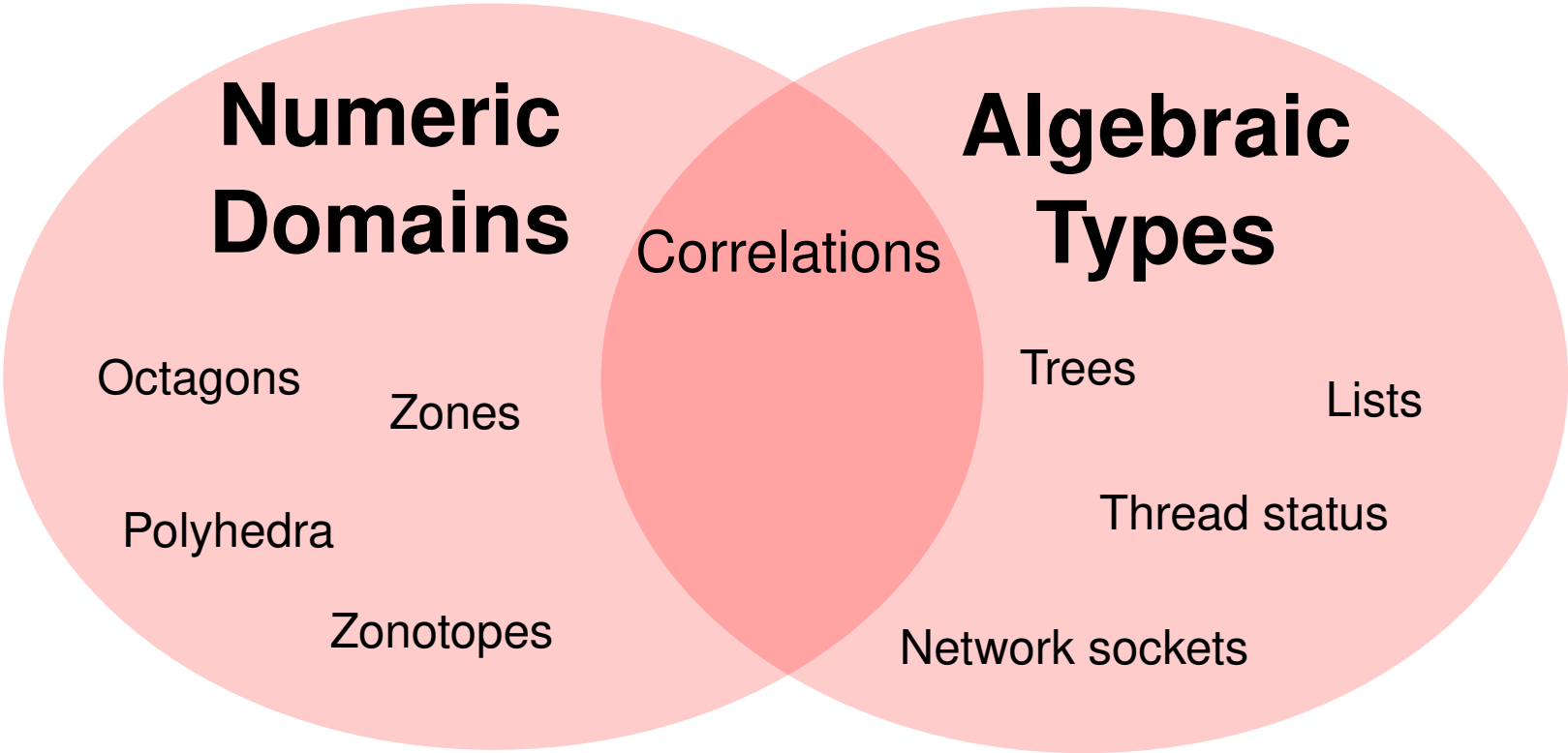
Static Analysis

Can alleviate manual effort

Example

ProvenCore & Correlations : $\frac{2}{3}$ proof obligations discharged

Overview


Numeric Domains
Octagons
Zones
Polyhedra
Zonotopes
Correlations
Algebraic Types
Trees
Lists
Thread status
Network sockets

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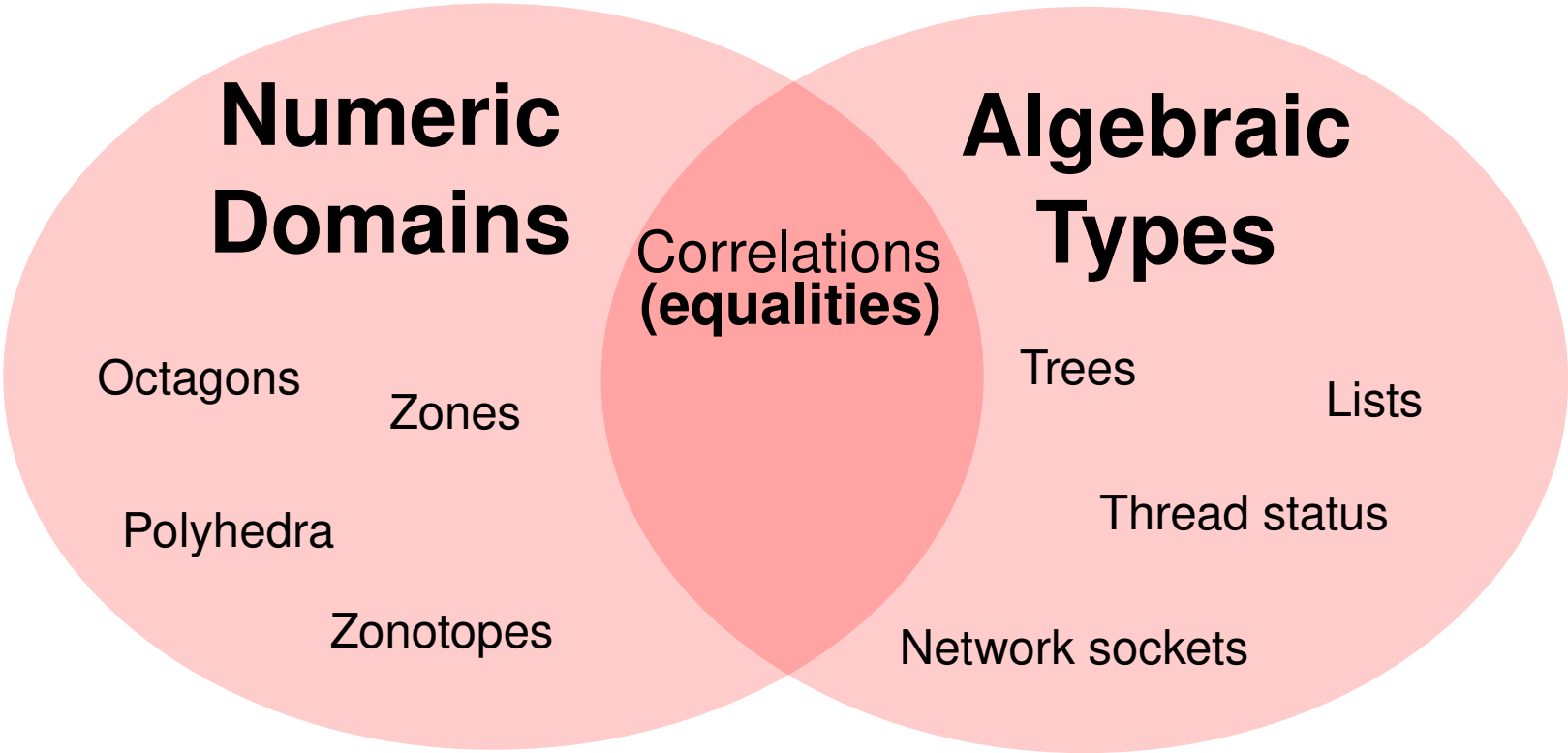
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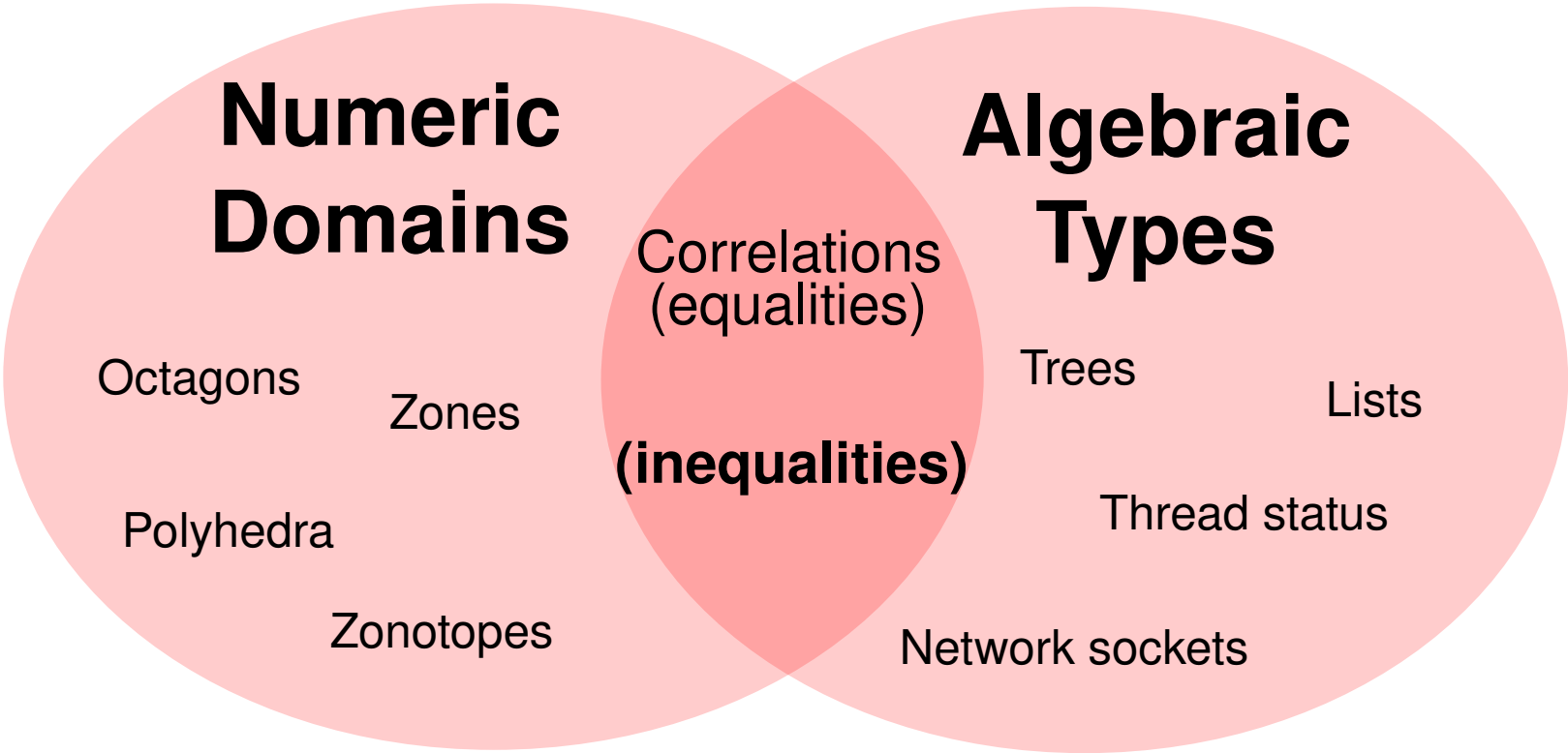
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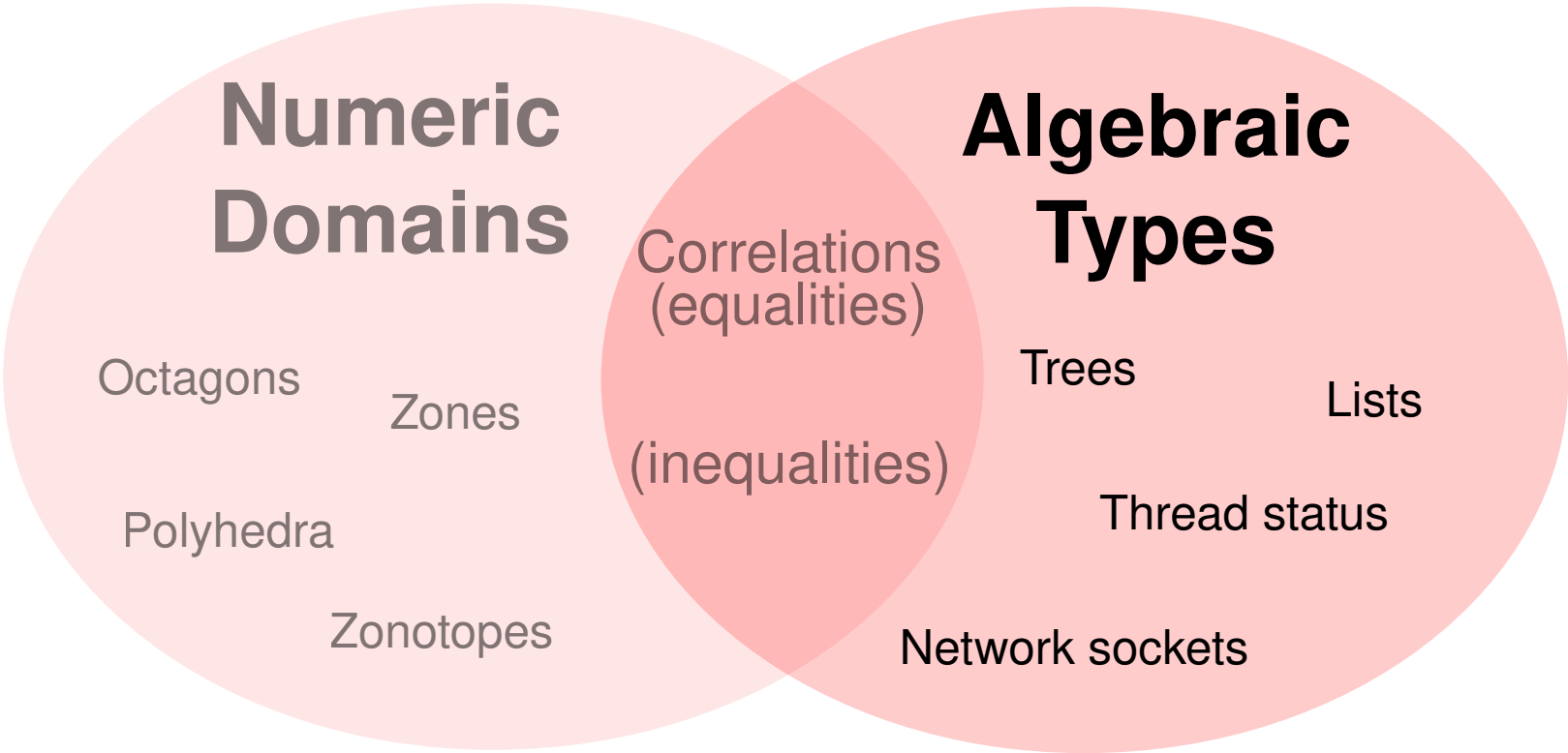
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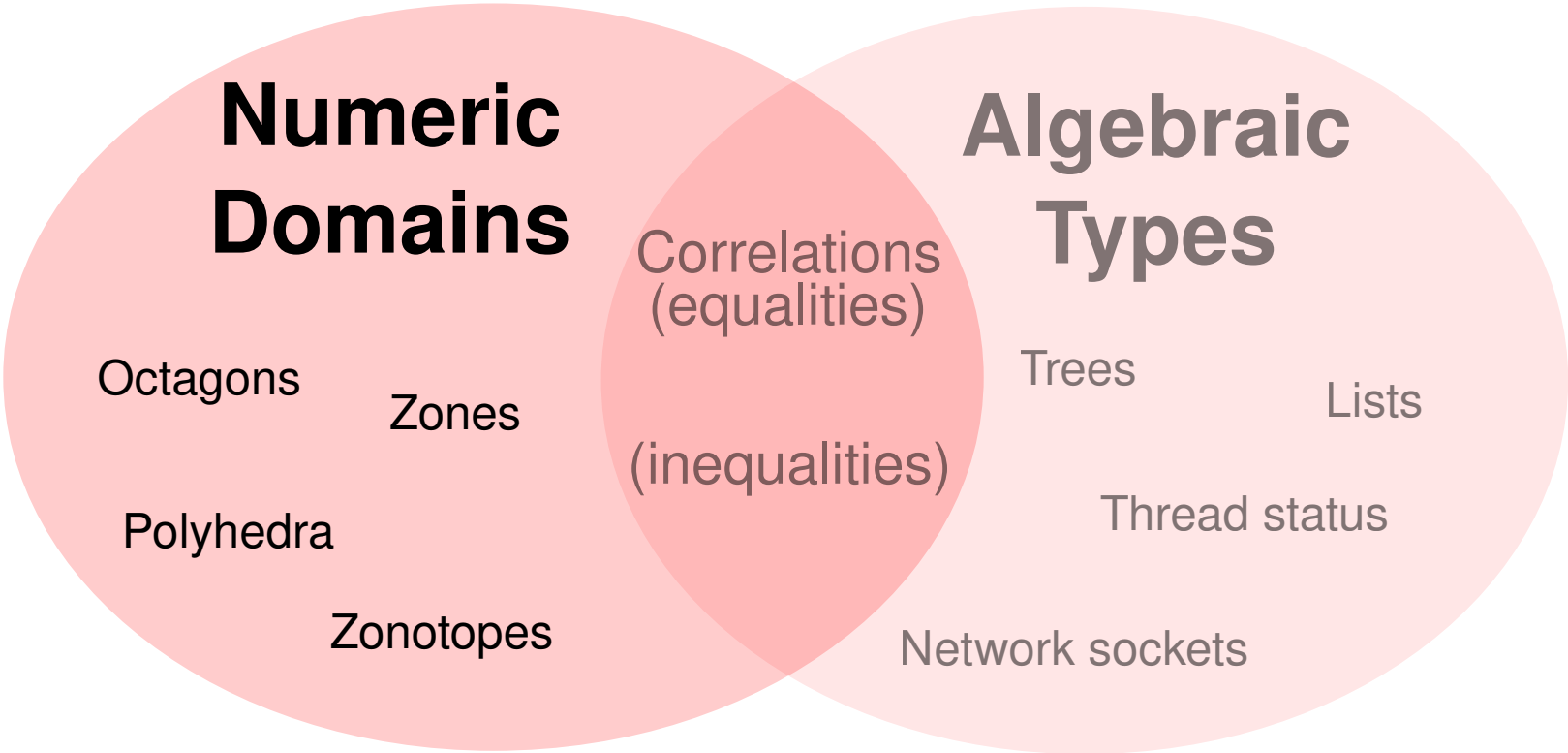
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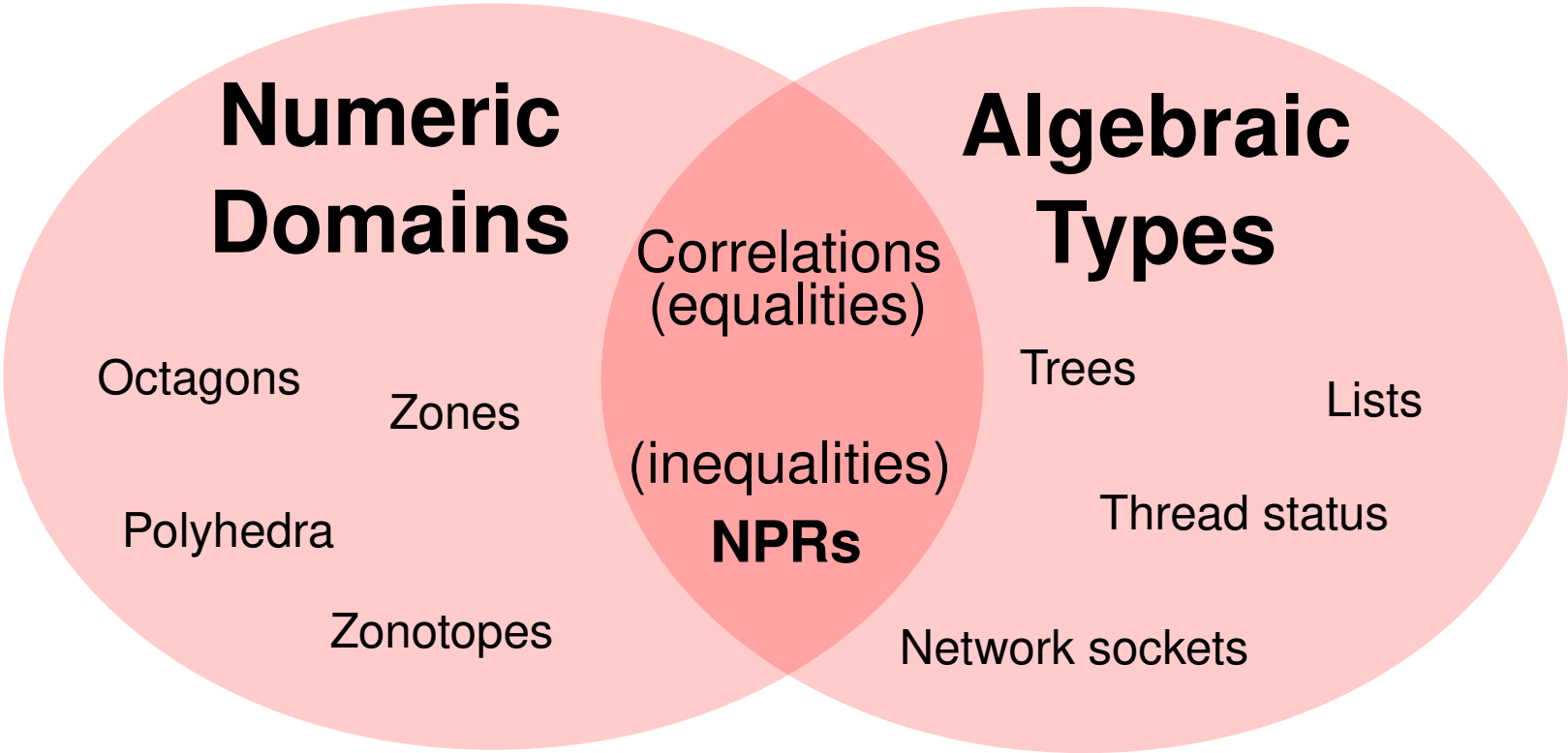
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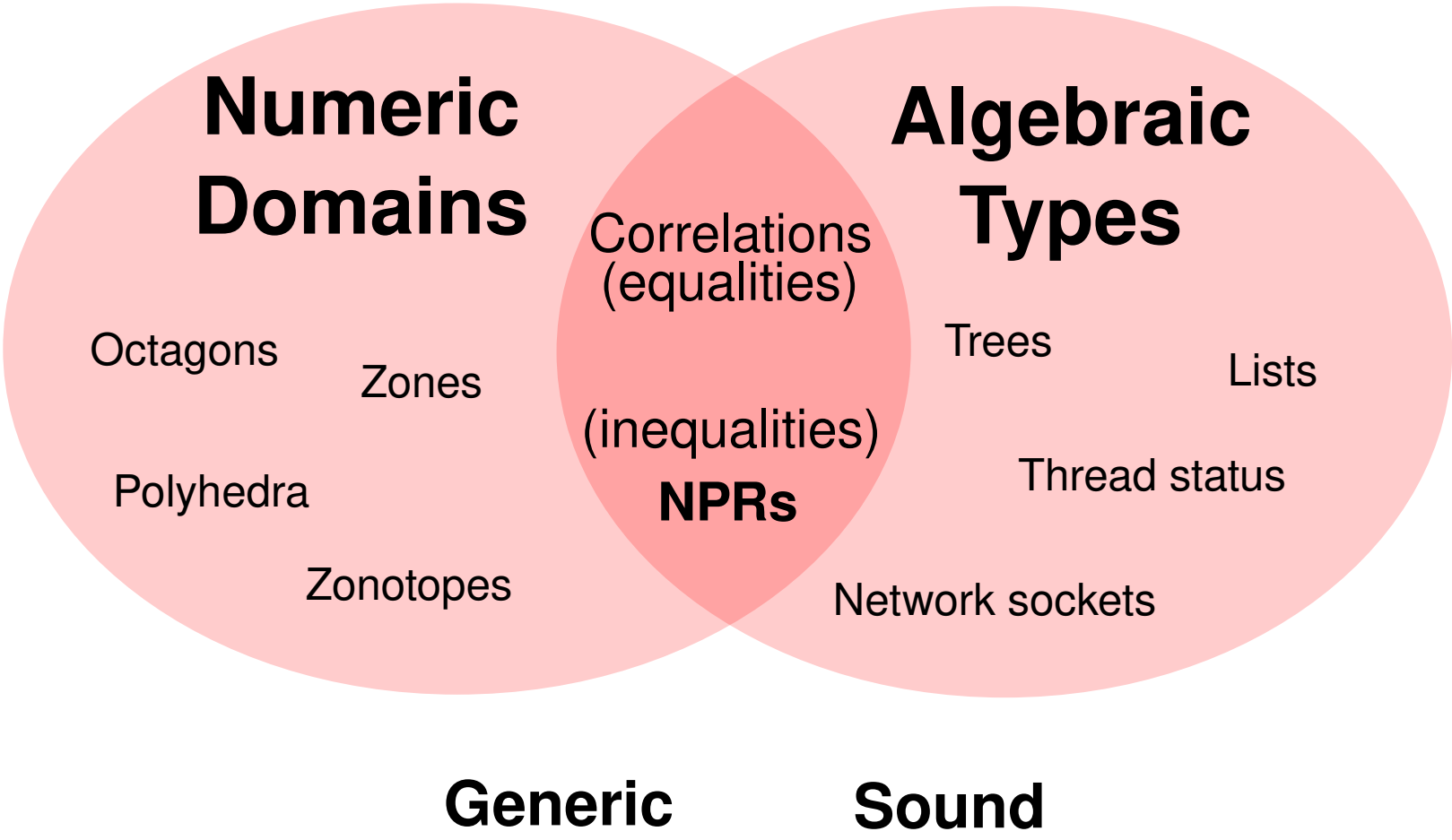
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Generic
Sound

Product types

A product type

type point =
  {absciss: number,
   ordinate: number}

A record

value origin =
  { abciss = 0,
   ordinate = 0}

Product types

$$\left\{\overline{f_i \to \tau_i}^{\,i \in I}\right\}$$

Sum types

A sum type

```
type card =
    | Spades(Number)
    | Hearts(Number)
    | Clubs(Number)
    | Diamonds(Number)
```

A variant

```
value hearts_queen =
    Hearts(12)
```

Sum types

$$\left[\overline{A_i \to \tau_i}^{i \in I}\right]$$

Algebraic Types

Example

type thread_status =
  | Running {thread_id: Number, priority: Number}
  | Asleep {seconds : Number, thread_id: Number, priority: Number}

Definition

Arbitrary nesting of sum types and product types

$$\tau ::= \text{Number} \quad | \quad \left\{ \overline{f_i \to \tau_i}^{i \in I} \right\} \quad | \quad \left[\overline{A_i \to \tau_i}^{i \in I} \right]$$

Algebraic Types

Example

type thread_status =
  | Running {thread_id: Number, priority: Number}
  | Asleep {seconds : Number, thread_id: Number, priority: Number}

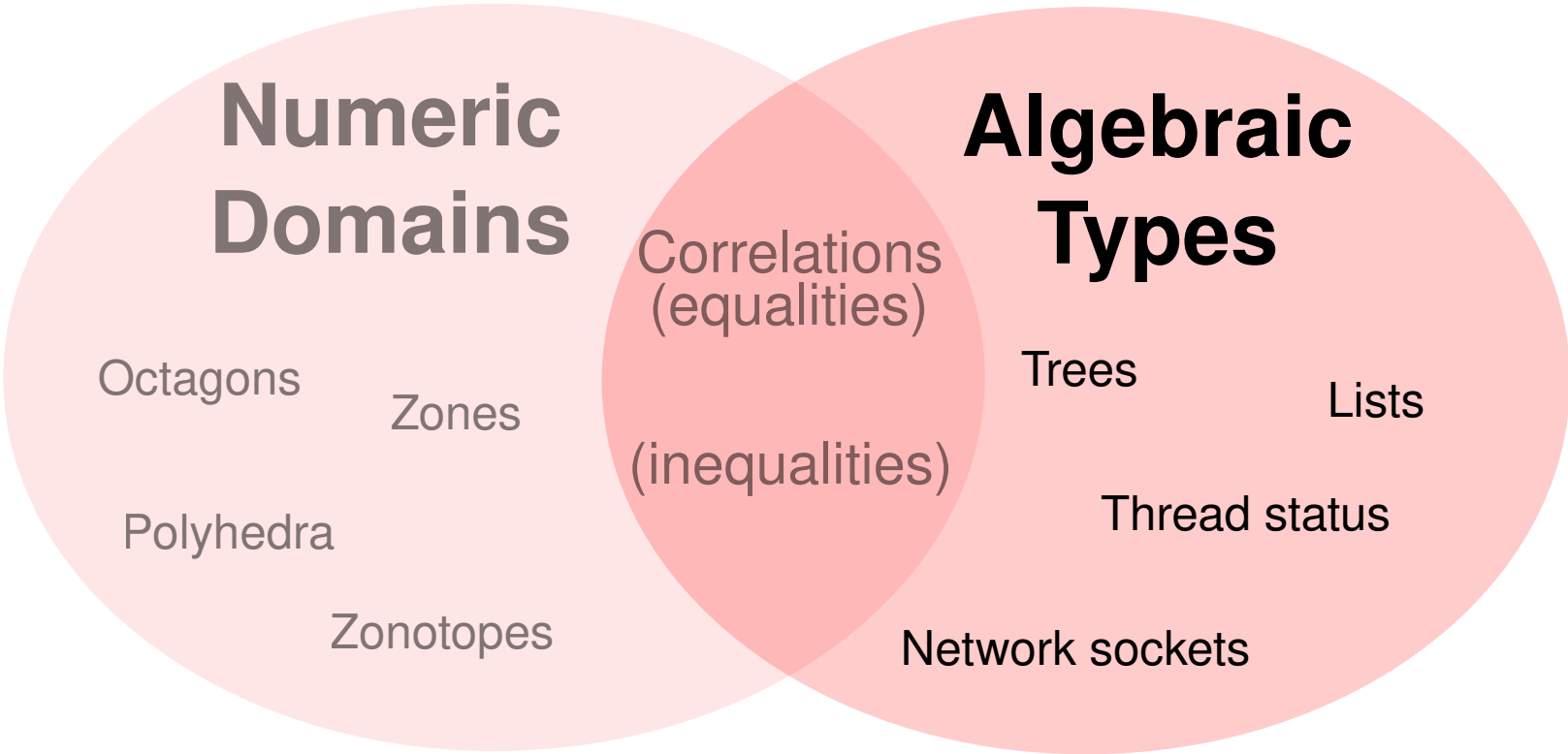
Definition

Arbitrary nesting of sum types and product types

$$\tau ::= \text{Number} \quad | \quad \left\{ \overline{f_i \to \tau_i}^{i \in I} \right\} \quad | \quad \left[\overline{A_i \to \tau_i}^{i \in I} \right]$$

Limitation

We do not handle recursive types.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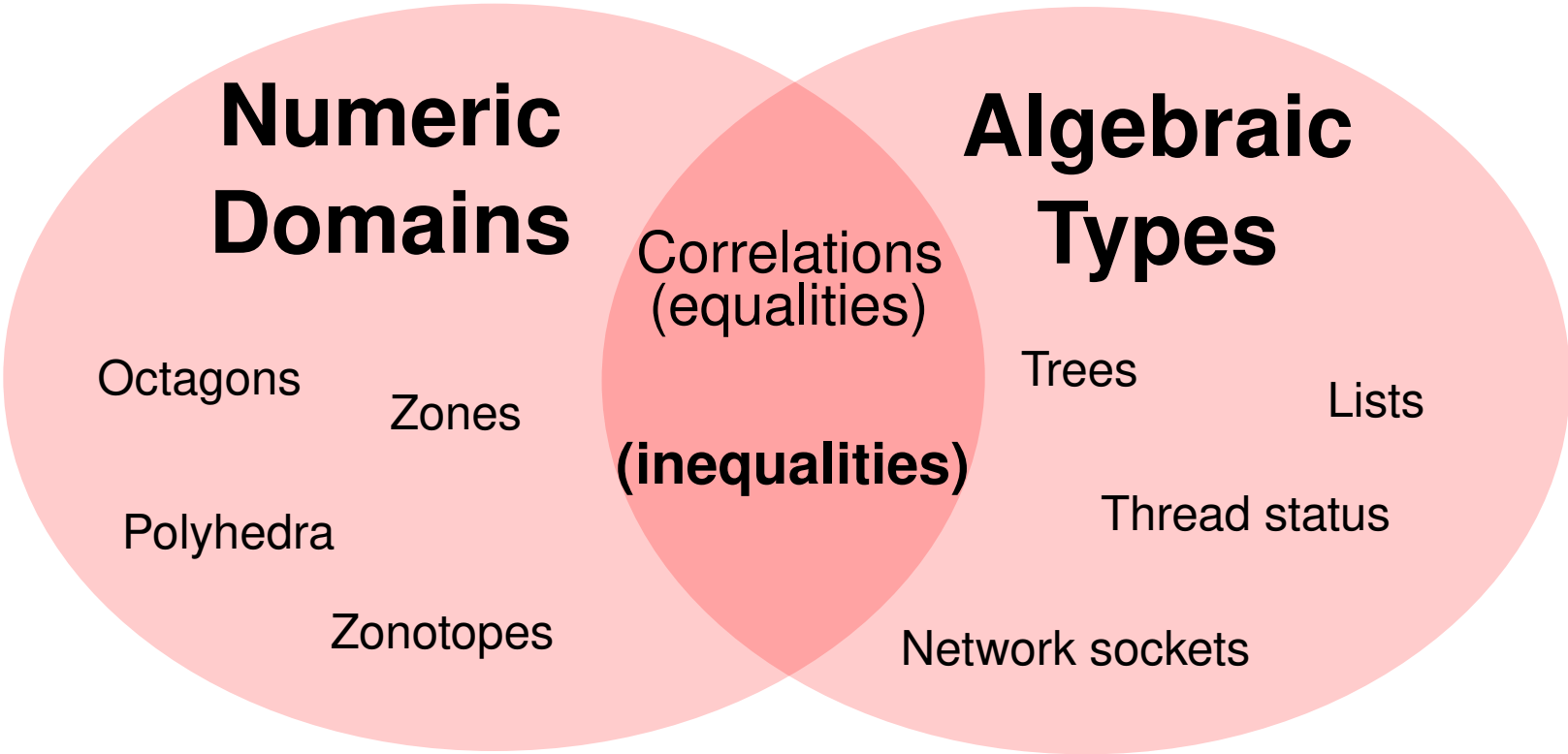
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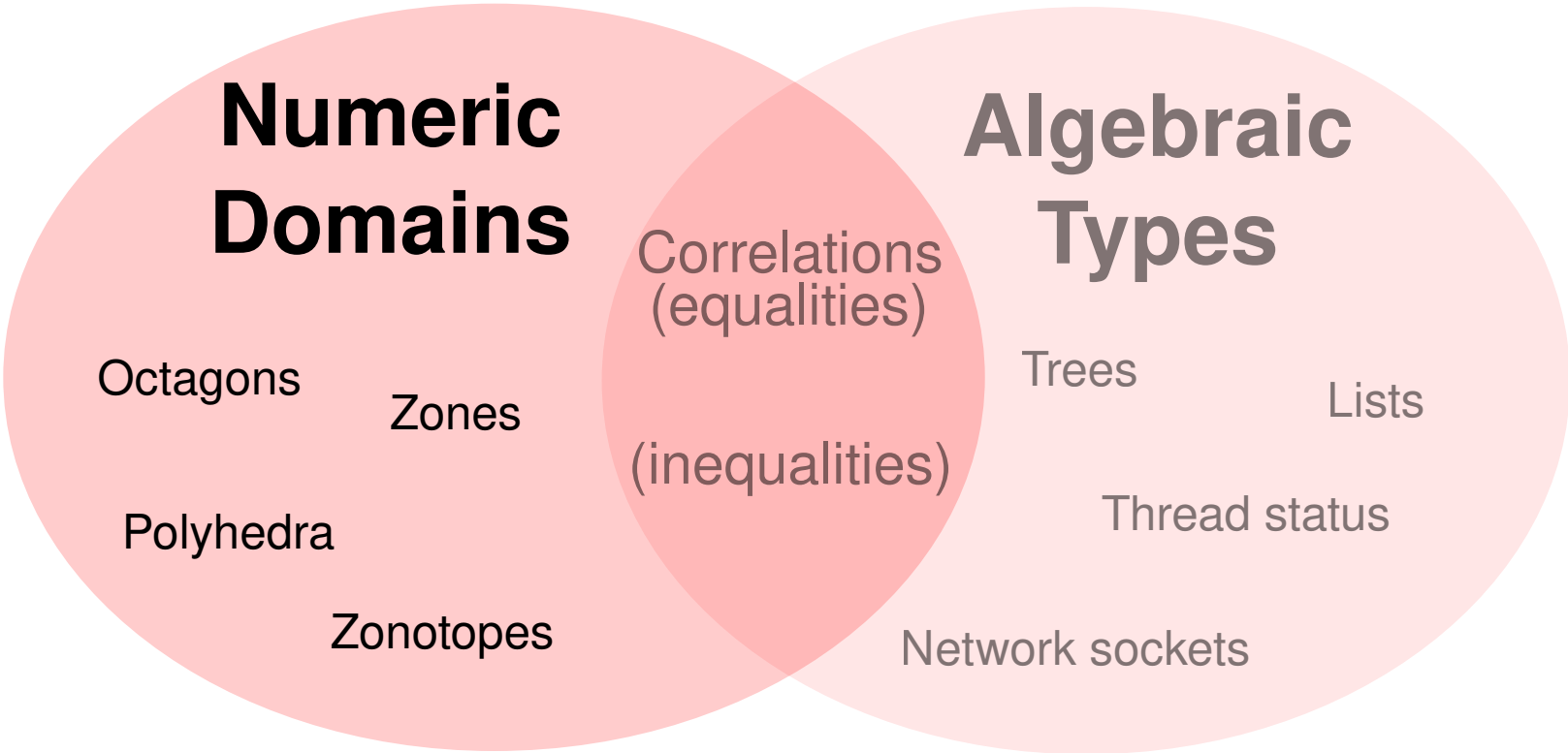
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Numeric domains

Abstract domains

- Polyhedra
  - Octagons
- Zonotopes

Numeric domains

Abstract domains

Polyhedra

Octagons

Zonotopes

Abstract values

Constraints (Variables)

Numeric domains

Abstract domains

Polyhedra

Octagons

Zonotopes

Abstract values

Constraints (Variables)

Program

```
r := 1 ;
while (n > 1) {
   r := n * r ;
   n := n – 1 ;
}
```

Numeric domains

Abstract domains

Polyhedra

Octagons

Zonotopes

Abstract values

Constraints (Variables)

Program

```
r := 1 ;
while (n > 1) {
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```

Variables

n r

Numeric domains

Abstract domains

Polyhedra

Octagons

Zonotopes

Abstract values

Constraints (Variables)

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```
r := 1 ;
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◆ Program point

Variables

n r

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Abstract domains

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Abstract values

Constraints (Variables)

Program

```
r := 1 ;
while (n > 1) {
   r := n * r ;
   n := n – 1 ;
}
```

Program point

Variables

$n \quad r$

Constraint

$r \geq n + 1$

Numeric domains

Abstract domains

Polyhedra

Octagons

Zonotopes

Abstract values

Constraints (Variables)

$\{r \geq n + 1\}$

Program

```
r := 1 ;
while (n > 1) {
   r := n * r ;
   n := n – 1 ;
}
```

Program point

Variables

$n \quad r$

Constraint

$r \geq n + 1$

The Apron interface

Numeric domain D

Abstract values $\quad$ Constraints(Variables)

The Apron interface

Numeric domain D

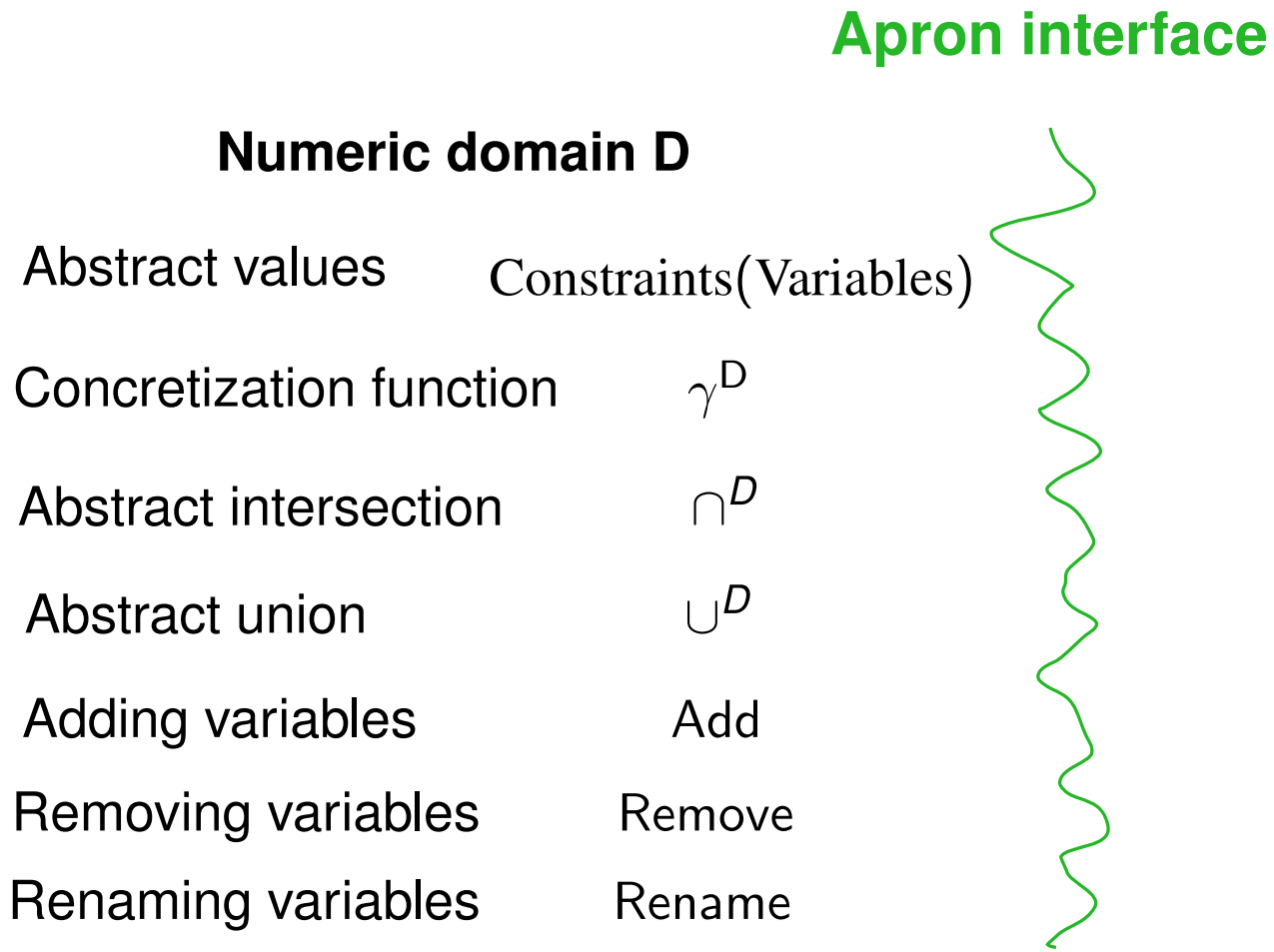
Abstract values	$\mathrm{Constraints}(\mathrm{Variables})$
Concretization function	γ^{D}
Abstract intersection	$\cap^{D}$
Abstract union	$\cup^{D}$

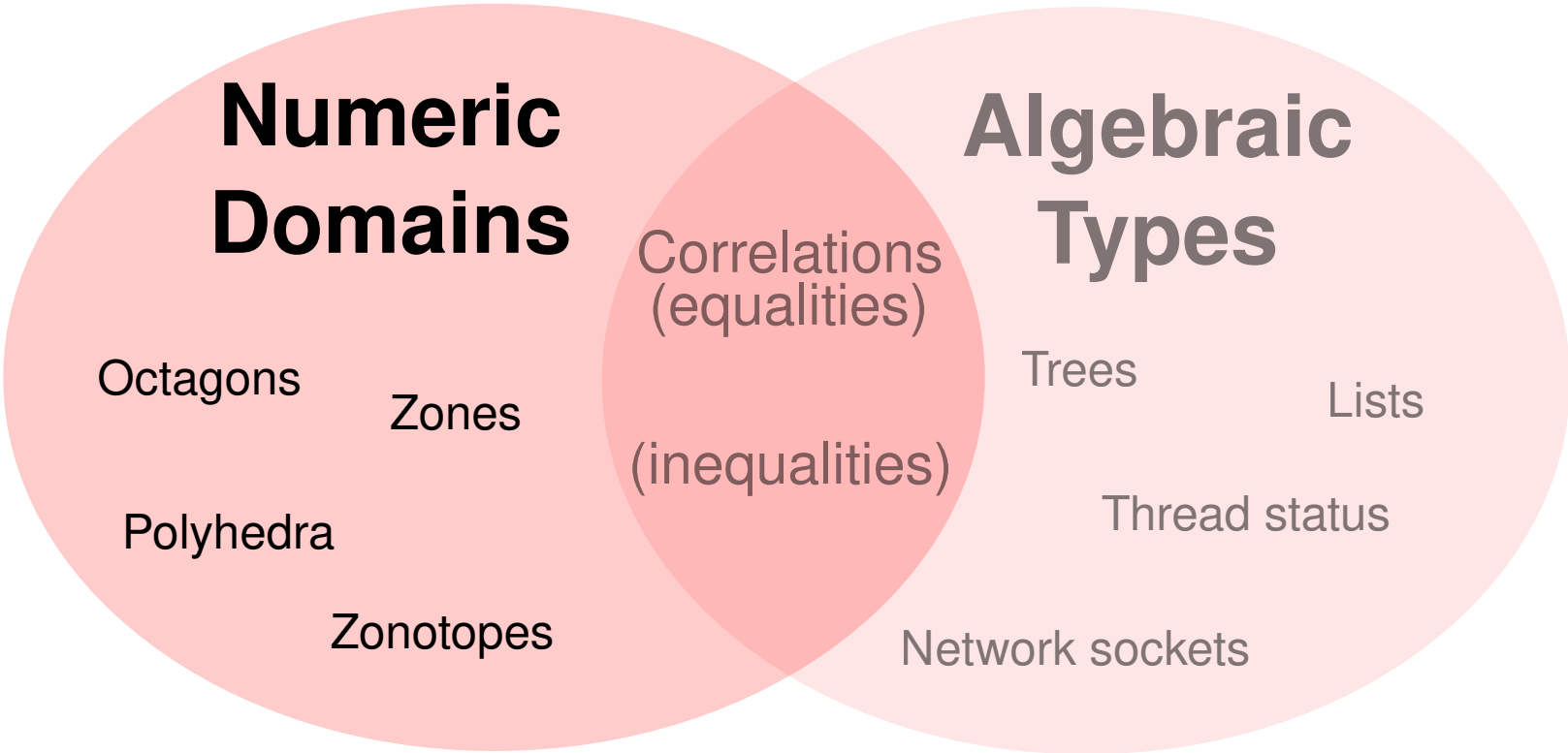
The Apron interface

Numeric domain D

Abstract values	$\mathrm{Constraints}(\mathrm{Variables})$
Concretization function	γ^{D}
Abstract intersection	$\cap^{D}$
Abstract union	$\cup^{D}$
Adding variables	Add
Removing variables	Remove
Renaming variables	Rename

The Apron interface

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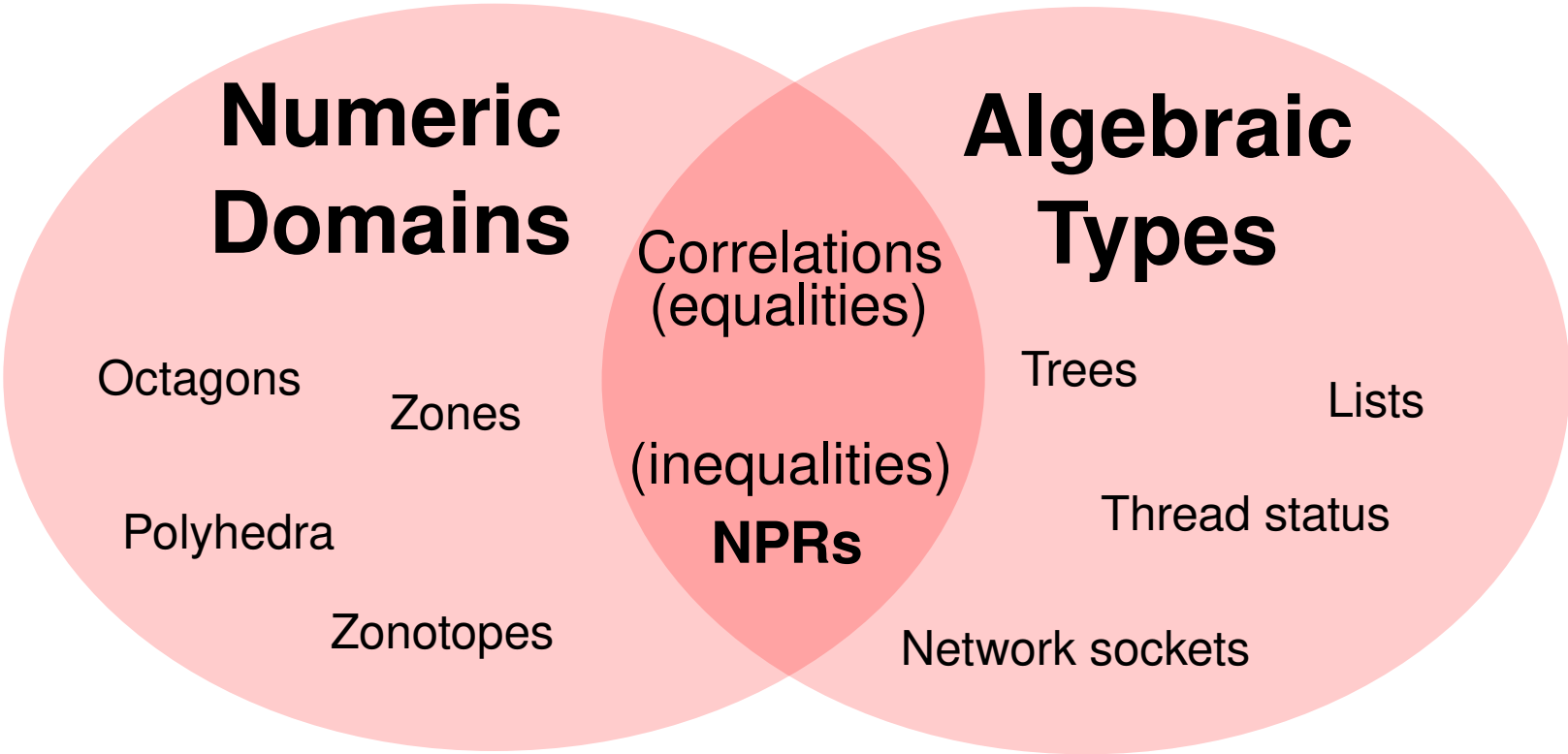
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Numeric path relations

Apron interface

Numeric domain D

Abstract values	$\text{Constraints}(\text{Variables})$
Concretization function	γ^{D}
Abstract intersection	$\cap^{D}$
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Adding variables	Add
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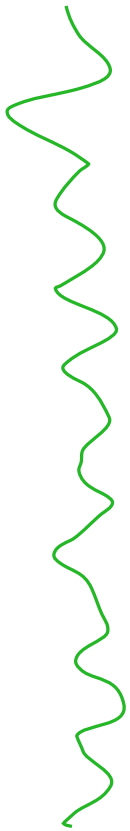
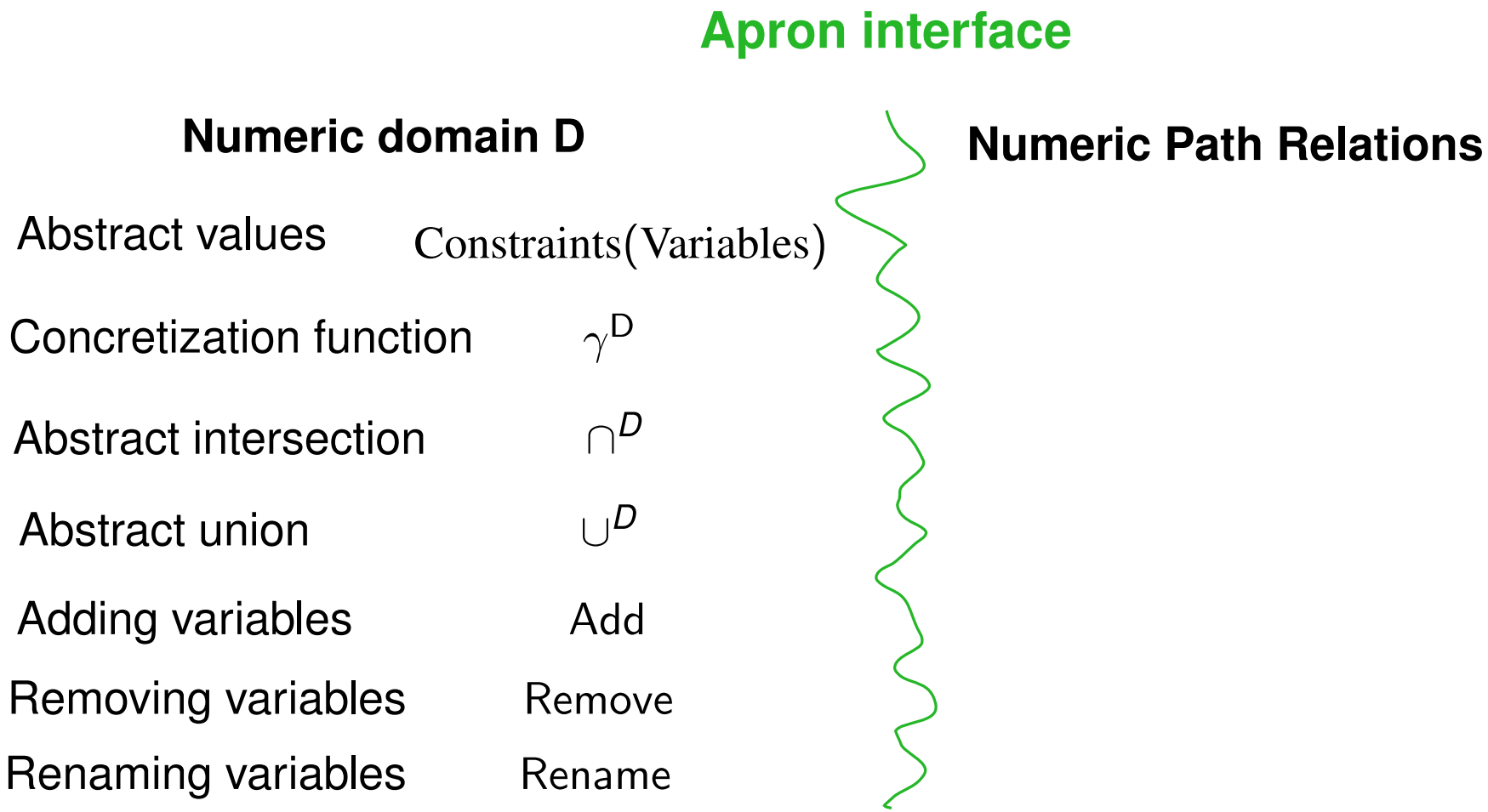
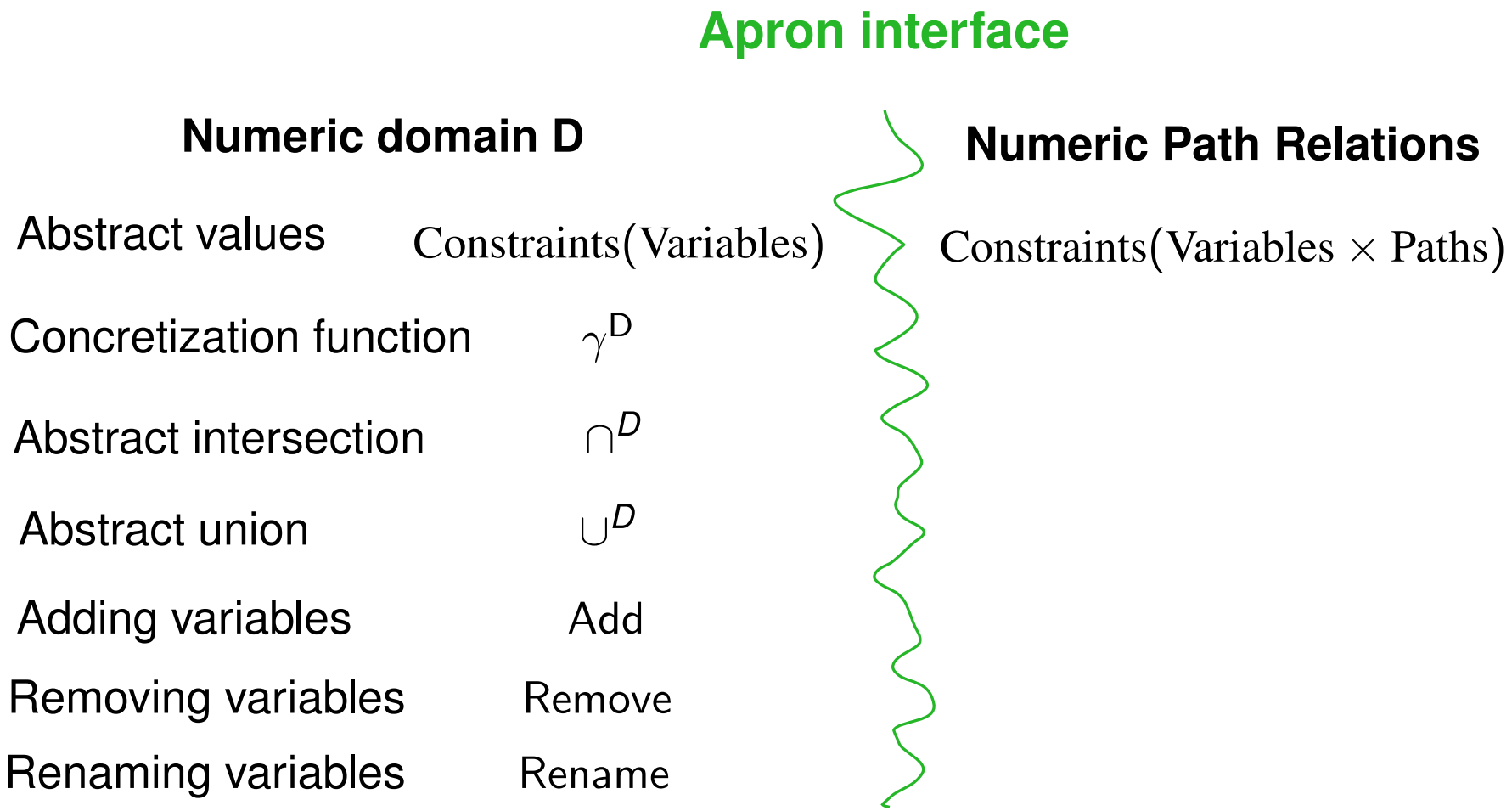


Numeric path relations



Numeric path relations

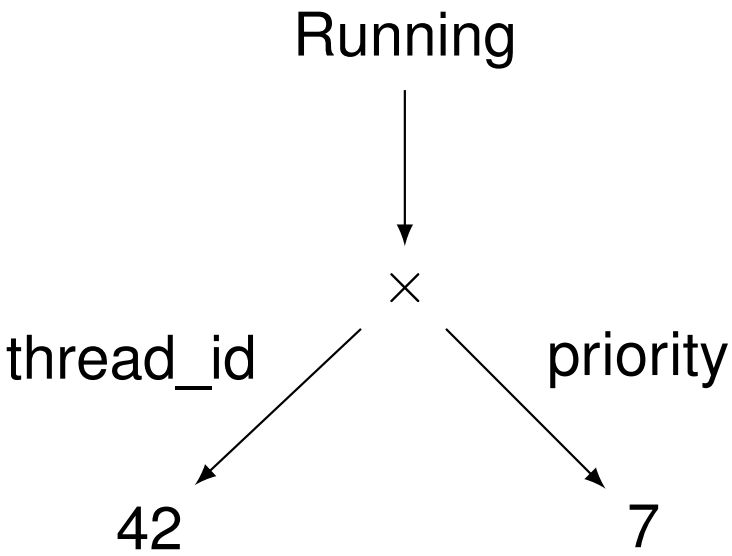


Numeric path relations

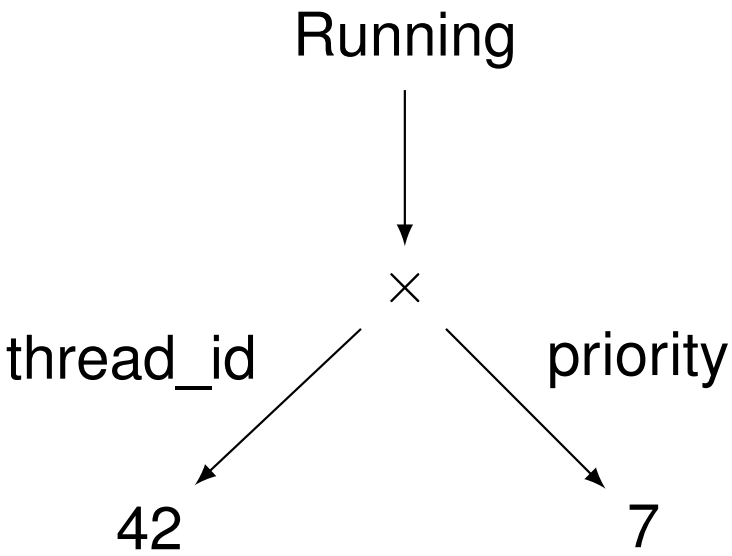
Apron interface

	Numeric domain D	**Numeric Path Relations**
Abstract values	$\text{Constraints}(\text{Variables})$	$\text{Constraints}(\text{Variables} \times \text{Paths})$
Concretization function	γ^{D}	γ^{NPR}
Abstract intersection	$\cap^{D}$	$\cap^{\text{NPR}}$
Abstract union	$\cup^{D}$	$\cup^{\text{NPR}}$
Adding variables	Add	
Removing variables	Remove	
Renaming variables	Rename	

Projection over a path



Projection over a path



$$\mathrm{th} \Downarrow^{\mathsf{val}} @\mathrm{Running.thread\_id} = 7$$

Projection over extended variables

Extended variables

Pair of a variable name and a path

Examples

$(x, \text{@Running.thread\_id})$ and $(x, \text{@Running.priority})$

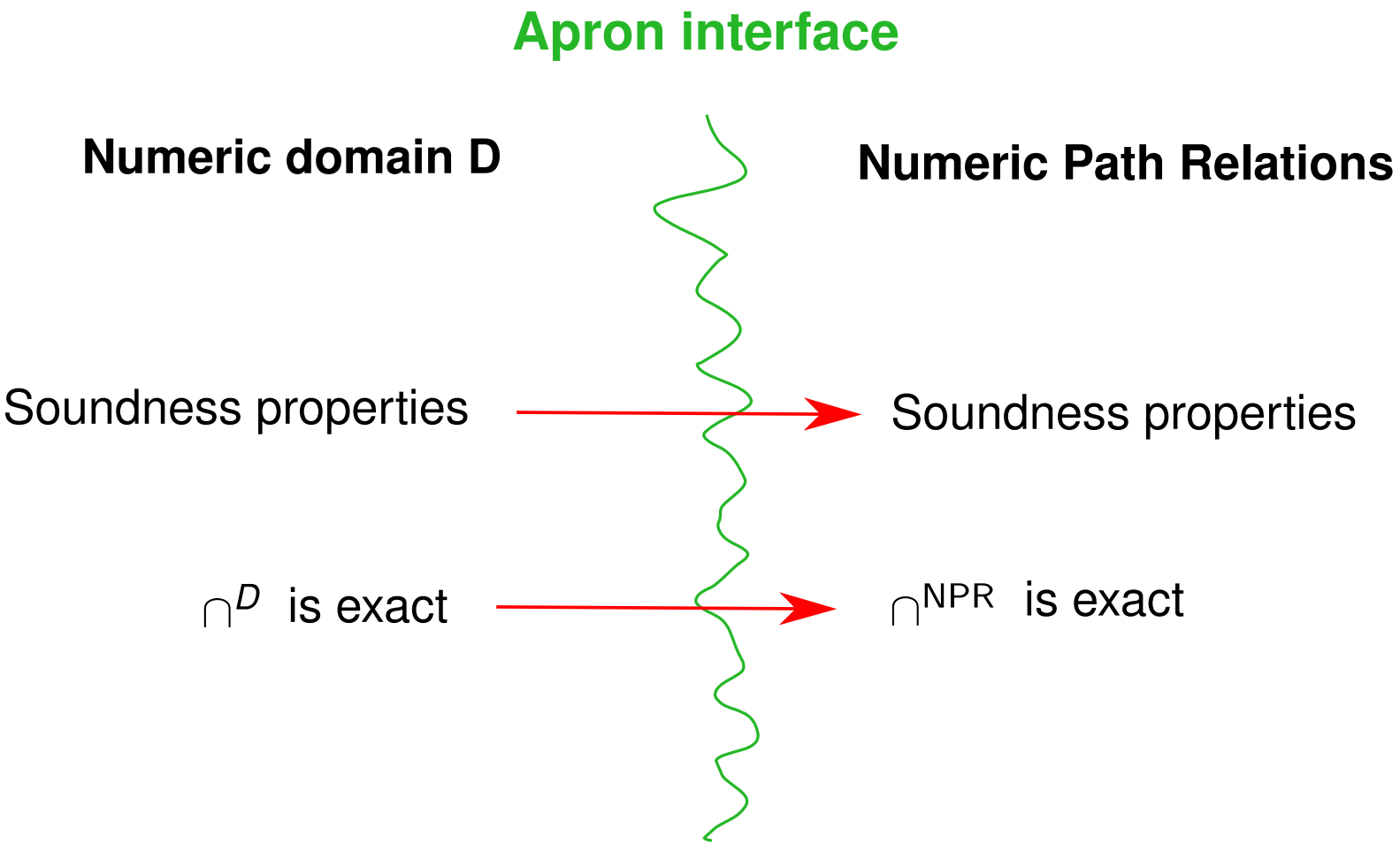
$$[x \mapsto \text{th}] \Downarrow^{\text{env}} \{(x, \text{@Running.id}), (x, \text{@Running.priority})\} = \begin{bmatrix} (x, \text{@Running.thread\_id}) & \mapsto & 42 \\ (x, \text{@Running.priority}) & \mapsto & 7 \end{bmatrix}$$

Numeric Path Relations (NPRs)

```
function tick (x) =
  match x with
    | Asleep {seconds = s, thread_id = id, priority = p} →
      if (seconds == 1)
      then r := Running {thread_id = id, priority = p}
      else r := Asleep {seconds = s – 1, thread_id = id, priority = p}
   | _ →r := x ;
 return r
```

$$\left\{ \begin{array}{l} \left\{ \begin{array}{lcl} (r, \text{@Asleep.seconds}) & = & (x, \text{@Asleep.seconds}) - 1, \\ (r, \text{@Asleep.thread\_id}) & = & (x, \text{@Asleep.thread\_id}), \\ (r, \text{@Asleep.priority}) & = & (x, \text{@Asleep.priority}) \end{array} \right\}, \\ \left\{ \begin{array}{lcl} (r, \text{@Running.thread\_id}) & = & (x, \text{@Asleep.thread\_id}), \\ (r, \text{@Running.priority}) & = & (x, \text{@Asleep.priority}) \end{array} \right\} \end{array} \right\}$$

Transferred properties



Take away and future work

- Numeric Path Relations :
  - Extend any numeric domain (given the Apron Interface)
  - Preserve soundness properties
  - Preserve exactness of abstract intersection
- Future work
  - Recursive types
  - Implementation

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