A framework to construct constraint preserving and approximately well-balanced schemes

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Acknowledgements

Credit to them for the good stuff blame me for the rest

- Yogiraj Mantri, Vellore Institute of Technology, India
- Philipp Öffner, Johannes Gutenberg-University, Mainz
- Wassilij Barzukow, CNRS, Institut de Mathématiques de Bordeaux
- Davide Torlo, MathLab at SISSA, Italy

Setting: balance laws

We want to solve numerically (hyperbolic) systems of balance laws

$$\partial_t U + \nabla \cdot F(U) = S(U; \varphi(x))$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- GPR model and hyperbolic reformulations of viscous/dispersive systems
- etc.

We consider both 1D and Multi-D problems

A motivational example to fix ideas

Pseudo-1D rotating shallow water eq.s (Castro et al, SISC 31, 2008)

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ gb'(x) + c_f u - \phi v \\ c_f v + \phi u \end{pmatrix}$$

Incomplete zoology of steady states: some of the 2d richness in 1d

1. Lake at rest
2. One dimensional frictionless flows with constant energy
3. Pseudo-one dimensional rest state with transverse perturbations $v(x)$ and Coriolis effects
4. Frictionless pseudo-one dimensional flows with Coriolis effects
5. One dimensional flows with friction
6. etc. etc.

Well balanced numerics a (possibly incomplete) taxonomy

Objective: construct a will balanced scheme using as little a-priori knowledge as possible

besides the structure of the PDE itself

Well balanced how :

many meanings .. with all the (more or less) obvious connections.

(begging forgiveness in advance for any - involuntary - omissions !)

Well balanced numerics a (possibly incomplete) taxonomy

1. Preserving $V(U, \varphi(x)) = V_0$ via WB differencing or generalized polynomial approximations
   - Roe Lect.Not.Math. 1986, Bermúdez,& Vázquez-Cendón CAF 1994, Greenberg & LeRoux SINUM 1996; Audusse et all SISC 2004, Parés & Castro M2AN 2004, Parés SINUM 2006, Castro et al M3AS 2007, Hernádez-Duenas & Karni JSC 2011, Ricchiuto JSC 2011, Xing JCP 2014, Cheng & Kurganov Comm.Math.Sci. 2016 etc.etc.
2. Reconstruction/evolution of fluctuations wrt a given equilibrium $U^*(x)$
   - Castro et al SINUM 2008, Gaburro et al MNRAS 2018, Klingenberg et al SISC 2019, Berberich et al CAF 2021, etc.etc.
3. Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):
   - Castro & Parés J.Sci.Comp. 2020, Gómez-Bueno et al, Appl.Math.Comp. 2021, Gómez-Bueno et al Mathematics 2021, Guerrero Fernández et al Mathematics 2021, Gómez-Bueno et al, Appl.Num.Math. 2023, etc. etc.
4. Fully well balanced Riemann solver with 0-wave to enforce integral steady balance
   - Berthon & Chalons Math.Comp. 2016, Michel-Dansac et al JCP 2017, Castro et al SINUM 2018, Bulteau et al Calcolo 2021, etc.etc.
5. Well balanced via integration of the source term and global fluxes
   - Gascón & Corberán JCP 2001, Donat & Martinez-Gavara J.Sci.Comp. 2011, Chertock et al JCP 2018, Cheng et al J.Sci.Comp. 2019, Mantri&Noelle JCP 2021, Parés&Parés-Pulido JCP 2021, Carrillo et al JCP 2023, Ciallella et al. J.Sci.Comp. 2023, etc. etc.

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.. few of Carlos' contributions ...

Well balanced numerics a (possibly incomplete) taxonomy

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3. Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):

   $\dot{U}_i(t) + R_i(U_{\text{h}}) = R_i(U^*_{\text{h}})$ with $U^*(x)$ approx.sol. of the Cauchy pb $U' = J^{-1}S(U,\varphi)$
4. Fully well balanced Riemann solver with 0-wave to enforce integral steady balance
   - Berthon & Chalons Math.Comp. 2016, Michel-Dansac et al JCP 2017, Castro et al SINUM 2018, Bulteau et al Calcolo 2021, etc.etc.
5. Well balanced via integration of the source term and global fluxes

   $$\partial_x F = S \Rightarrow \partial_x F(U^*) = -\partial_x R^* \;, \quad R \approx \int S$$

Objective

1. Elaborate on the GF idea and provide a link with the approximate Cauchy solver method
2. Propose a MultiD constraint preserving generalization

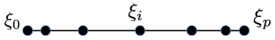
1 DG-SEM and Global Flux Quadrature in 1D

with Y. Mantri and P. Öffner

DG-SEM - Main notation

Discontinuous Galerkin Spectral Element Method

- Reference element $\xi \in [0,1]$
- $x(\xi)$ linear mapping $[0,1] \mapsto K$, for simplicity: $|K| = \mathrm{h}$
- $\{\xi_i\}_{i=0,p}$ the $p+1$ Gauss-Lobatto (GL) points
- $\{\phi_i(\xi)\}_{i=0,p}$ degree p Lagrange bases
- Set $U_\mathrm{h} = \sum_{i=0}^{p} \phi_i(x(\xi))U_i$



DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

On an element K, start from discrete approximation arising from the variational form

$$\mathrm{h} \int_0^1 \varphi_i(\xi) \partial_t U_{\mathrm{h}} - \int_0^1 \partial_\xi \varphi_i(\xi) F_{\mathrm{h}} + (\varphi_i \hat{F}_{\mathrm{h}}(U_{\mathrm{h}}^-, U_{\mathrm{h}}^+))_{\xi=1} - (\varphi_i \hat{F}_{\mathrm{h}}(U_{\mathrm{h}}^-, U_{\mathrm{h}}^+))_{\xi=0} = \mathrm{h} \int_0^1 \varphi_i(\xi) S_{\mathrm{h}}$$

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM : quadrature based on the same GL nodes used for the polynomial expansion[1]

$$\mathcal{M}\dot{\mathbf{U}} - D_x^T \mathbf{F} + \mathcal{B}\widehat{\mathbf{F}} = \mathcal{M}\mathbf{S}$$

with the notation:

- $\mathcal{M} = \mathrm{diag}(\{w_i\}_{i=0,p})$ with $w_i = \mathrm{h}\int_0^1 \phi_i(\xi)d\xi$ the quadrature weights
- with $(D_x)_{ij} = w_i \partial_\xi \phi_j(\xi_i)$
- $\mathcal{B} = \mathrm{diag}(-1, \ldots, 1)$ the matrix sampling boundary values
- $\mathbf{U}$, $\mathbf{F}$, $\widehat{\mathbf{F}}$, $\mathbf{S}$: elemental arrays of nodal solution/flux/num. flux/source values

[1] Kopriva & Gassner *J.Sci.Comp.* 44, 2010 ; Hesthaven & Warburton, Springer 2008

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM discrete equations in strong form (SBP property[2])

$$\mathcal{M}\dot{\mathbf{U}} + D_x \mathbf{F} + \mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F}) = \mathcal{M}\mathbf{S}$$

[2] Kopriva & Gassner *J.Sci.Comp.* 44, 2010; Gassner et al. *J.Comput.Phys.* 327, 2016

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM discrete equations in strong form

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F}) = \mathcal{M}\mathbf{S}$$

Setting $\widehat{\mathbf{F}}(\mathbf{U}^-, \mathbf{U}^+) = \alpha\mathbf{F}^+ + (1-\alpha)\mathbf{F}^- - \mathcal{D}[\![\mathbf{U}]\!]$, we get the fully discrete method

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathcal{M}\mathbf{S}$$

This is our "reference" non well-balanced method

DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x G(U; \varphi(x)) = 0$$

with $G = F(U) + R(U; \varphi(x))$ and

$$R(U; \varphi(x)) = R_0 - \int_{x_0}^{x} S(U; \varphi(s)) ds$$

DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x F(U) = -\partial_x R(U; \varphi(x))$$

If we have the source primitive R at all GL nodes, we can readily write the DG-SEM scheme:

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = -D_x\mathbf{R} + \mathcal{B}(\alpha[\![\mathbf{R}]\!])$$

The key now is to define R at all GL nodes

DG-SEM - Discrete variational form using global fluxes

We compute nodal values of R:

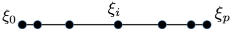
$$\partial_x R = -S$$

Choose

$$R_\mathrm{h} = \sum_{i=0,p} \varphi_i(x(\xi))R_i \ , \quad S_\mathrm{h} = \sum_{i=0,p} \varphi_i(x(\xi))S_i$$

and integrate on each element

1. For the (local) initial value we set $R_0 = R^-$
2. For all $i \in 1,p$ we compute $R_i = R_{i-1} - \mathrm{h} \sum_{l=0,p} \int\limits_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$



Inria

DG-SEM - Discrete variational form using global fluxes

We compute nodal values of R:

$$\partial_x R = -S$$

Remark. *In compact notation we have*

$$\mathbf{R} = \mathbf{R}^{-} - \mathrm{h}\mathcal{I}\mathbf{S}$$

with $\mathcal{I}$ is the tableau of the $p+1$ stages RK-LobattoIIIA implicit collocation method[3]

[3] A. Prothero & A. Robinson, Math.Comp. 28, 1974

Source integration: initial value

The initial value R^- can be related to the last value on the neighbouring element:

$$R^- = [R(x_p)]^{K^-} + [\![R]\!] = [R(x_p)]^{K^-} + \lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} S$$

The critical term is for the shallow water example

$$\lim_{\epsilon\to 0}\int_{x_0-\epsilon}^{x_0+\epsilon} g h_{\mathrm{h}} \partial_x b_{\mathrm{h}}$$

For simplicity, let us rule out discontinuous bathymetry and set

$$R^- = [R(x_p)]^{K^-}\ , \quad [\![R]\!] = 0$$

DG-SEM - Discrete variational form using global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x))$$

for all K evolve in time

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathrm{h}D_x\mathcal{I}\mathbf{S} \qquad \text{GF-DG}$$

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathcal{M}\mathbf{S} \qquad \text{DG}$$

Main result

Proposition (Discrete steady state). *The DG-SEM with global flux quadrature preserves exactly continuous discrete steady states* $U_i^* = U(F_i)$ *with* F *obtained by integrating the ODE*

$$F' = S(U(F), \varphi(x))$$

using the implicit continuous collocation RK-LobattoIIIA method on spatial slabs of size h.

As long as $U(F)$ *is a one to one mapping* $U^*(x)$ *verifies the consistency estimates of the LobattoIIIA method*[4]*: Element endpoints are* $2p$*-order accurate, internal nodes have accuracy* h^{p+2}.

[4] See e.g. Theorem 7.10 in Hairer, Wanner and Norset, Solving Ordinary Differential Equations I., Springer 1993

Global flux quadrature

$$\int_K \phi_i f_\mathrm{h} \to \mathrm{h} D_x \mathcal{I}\mathbf{f}$$

Main results

- Characterization of discrete equilibria: solution of LobattoIIIA continuous collocation method
- For steady states: super-convergence property of the LobattoIIIA ODE integrator
- Same approximate well balanced notion as in[5], however:
  - ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
  - ✓ Considerable accuracy enhancements at steady state with minor change in code
  - × Order/type of collocation method not arbitrary

[5] Gómez-Bueno, Díaz, Parés, Russo - *Mathematics* 9, 2021

Numerical examples in 1d

Pseudo-1D rotating shallow water eq.s (Castro et al, SISC 31, 2008)

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u - \phi v \\ c_f v + \phi u \end{pmatrix}$$

Notation.

h water depth

ζ free surface level

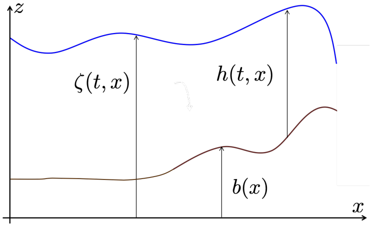
$\mathrm{v} = (u, v)$ horizontal velocity

$P = gh^2/2$ hydrostatic pressure (g gravity acceleration)

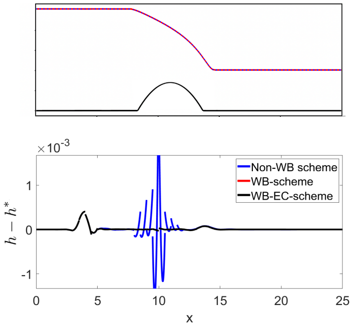
$\varphi = gb$ gravitational potential ($b(x, y)$ bottom topography)

$c_f = c_f(h, \mathrm{v})$ friction coefficient

ϕ Coriolis coefficient



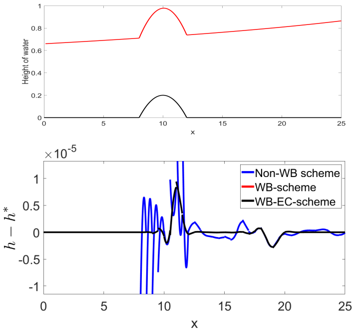
Numerical examples: perturbations of moving equilibria



Trans-critical with

$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0$$

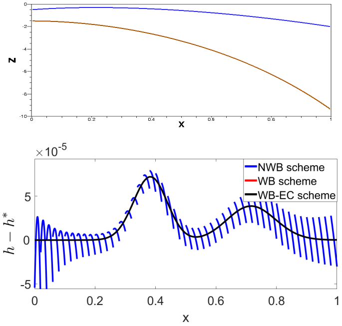
Depth perturbation $\xi_h = 10^{-3}$



Super-critical with

$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u$$

Depth perturbation $\xi_h = 10^{-5}$

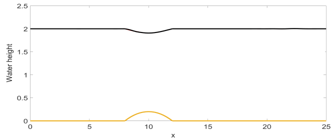


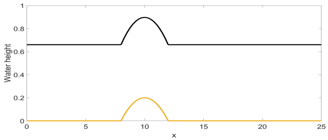
Sub-critical with $v = \phi x$ and

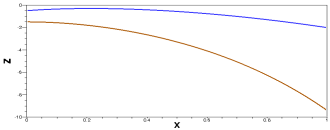
$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi u v$$

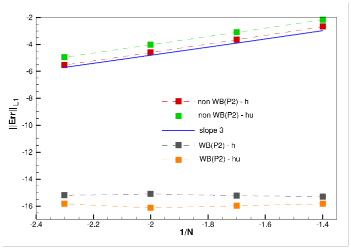
Depth perturbation $\xi_h = 10^{-5}$

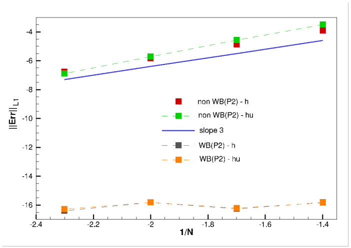
Numerical examples: error wrt ODE solver

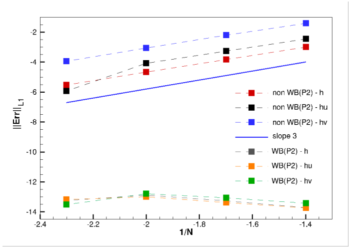












Sub-critical with

$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0$$

Super-critical with

$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u$$

Sub-critical with $v = \phi x$ and

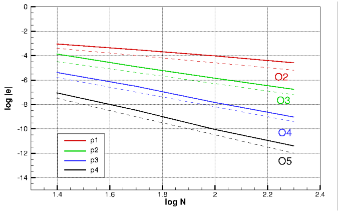
$$hu = q_0$$
$$g\zeta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi u v$$

Numerical examples: (super)convergence, DG-SEM

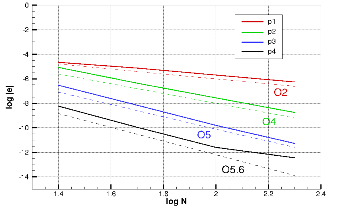
Theory for GF: internal points $= \mathcal{O}(\mathrm{h}^{p+2})$ - endpoints $= \mathcal{O}(\mathrm{h}^{2p})$

Super-critical channel with bump (no friction, no Coriolis)

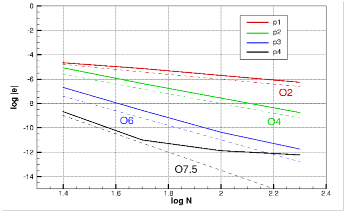
std DG-SEM



GF DG-SEM, all pts

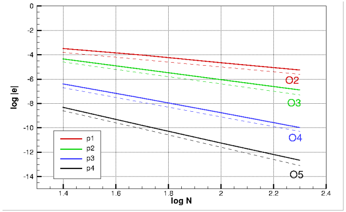


GF DG-SEM, end-pts

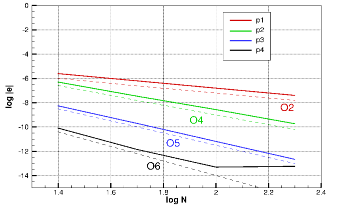


Sub-critical with Coriolis force

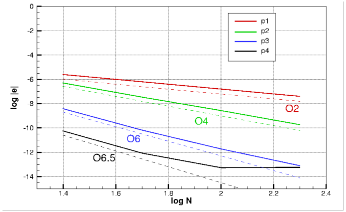
std DG-SEM



GF DG-SEM, all pts



GF DG-SEM, end-pts



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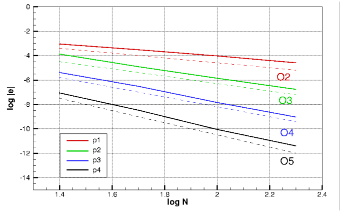
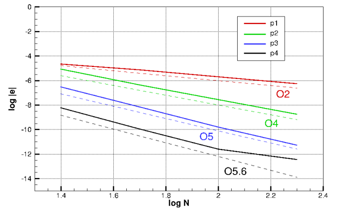
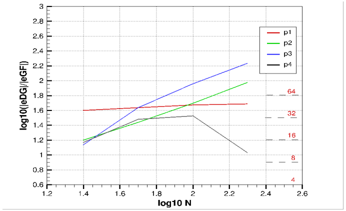
std DG-SEM



GF DG-SEM, all pts

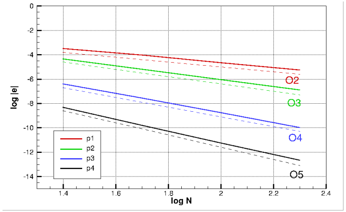


Error ratio $e_{\text{DG-SEM}}/e_{\text{GFq}}$, all-pts

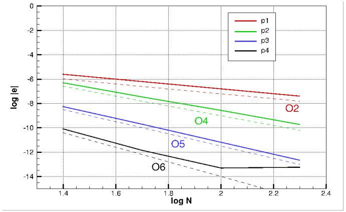


Sub-critical with Coriolis force

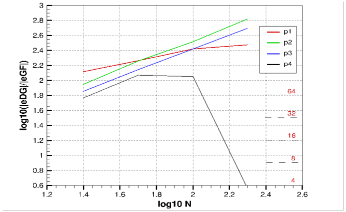
std DG-SEM



GF DG-SEM, all pts



Error ratio $e_{\text{DG-SEM}}/e_{\text{GFq}}$, all-pts



2 MultiD Contraint Preserving via Global Flux Quadrature

with W. Barsukow and D. Torlo

Well balanced in multi-D: structure/constraint preserving

$$\nabla \cdot F(U) = S(U; \varphi(x))$$

The form of the tensor F determines the type of differential constraints to preserve.

Well balanced in multi-D: structure/constraint preserving

We are going to work with the following example (with $\mathsf{v} = (u, v)$)

$$\partial_t P + \nabla \cdot \mathsf{v} = s(x, y)$$
$$\partial_t u + \partial_x P = \phi v - c_f u + \tau_x$$
$$\partial_t v + \partial_y P = -\phi u - c_f v + \tau_y$$

Linear waves with Coriolis, friction, mass source

P pressure

$\mathsf{v} = (u, v)$ velocity

$s(x, y)$ mass source

c_f friction coefficient

ϕ Coriolis coefficient

$\tau = (\tau_x, \tau_y)$ momentum forcing (e.g. wind for free suface waves)

Well balanced in multi-D: structure/constraint preserving

We are going to construct schemes preserving (simultaneously) the constrains

$$\nabla \cdot \mathsf{v} = s(x, y)$$

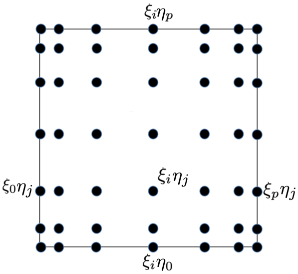
$$\nabla P = \mathsf{q}$$

- one multi-D constraint : $\nabla \cdot \mathsf{v} = s(x, y)$ (hard)
- two (pseudo-)1D constraints : $\nabla P = \mathsf{q}$ (easyer)

Discrete Framework: SEM

Tensor product spectral finite element method (SEM)

- $\mathrm{x}(\xi) = (x(\xi), y(\eta))$ mapping $[0,1]^2 \mapsto K$, for simplicity: $|K| = \mathrm{h}^2$
- $\{\xi_i\}_{i=0,p}$ and $\{\eta_j\}_{j=0,p}$ the $p+1$ Gauss-Lobatto (GL) points
- $\{\phi_i(x)\}_{i=0,p}$ and $\{\psi_j(y)\}_{j=0,p}$ 1d degree p Lagrange bases
- For node ij: $\lambda_{ij}(x(\xi), y(\eta)) = \phi_i(x)\psi_j(y)$
- Set $U_\mathrm{h} = \sum_{i,j} \lambda_{ij}(x(\xi), y(\eta)) U_{ij}$



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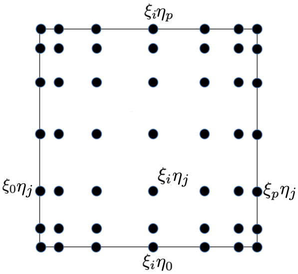

Notation: tensor product matrices

Mass matrix entries

$$\int_K \lambda_{ij}\lambda_{lm} = \int_{x_0}^{x_p} (\phi_i(x)\phi_l(x)) \times \int_{y_0}^{y_p} (\psi_j(y)\psi_m(y)) \Rightarrow M = M_x \otimes M_y = M_y \otimes M_x$$

2 Global Flux Quadrature in MultiD

Discrete Framework: SEM

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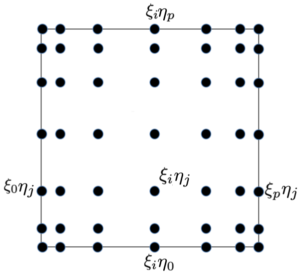

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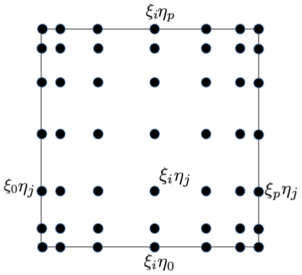

Notation: tensor product matrices

Derivative matrix entries

$$\int_K \lambda_{ij}\partial_x\lambda_{lm} = \int_{x_0}^{x_p}(\phi_i(x)\partial_x\phi_l(x)) \times \int_{y_0}^{y_p}(\psi_j(y)\psi_m(y)) \Rightarrow \quad D_x M_y = M_y D_x$$

Discrete Framework: SEM

We are going to work with the following example (with $\mathsf{v} = (u, v)$)

$$\partial_t P + \nabla \cdot \mathsf{v} = s(x, y)$$
$$\partial_t u + \partial_x P = q_x$$
$$\partial_t v + \partial_y P = q_y$$

Standard continuous SEM approximation (no stabilization) in strong form

$$M\dot{\mathbf{P}} + M_y D_x \mathbf{U} + M_x D_y \mathbf{V} = M\mathbf{S}$$
$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = M\mathbf{Q_x}$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = M\mathbf{Q_y}$$

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$$M\dot{\mathbf{P}} + M_y D_x \mathbf{U} + M_x D_y \mathbf{V} = M\mathbf{S} = \sum_K M^K \mathbf{S}^K$$
$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = M\mathbf{Q_x}$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = M\mathbf{Q_y}$$

Abuse of notation:
for continuous SEM the matrix formulation involve local elemental assembly.
We omit it for brevity, and explicitly mention only when necessary

Discrete Framework: SEM

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$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = M\mathbf{Q_y}$$

Stabilization using standard methods:
SUPG, gradient penalty, orthogonal subgrid scales etc[6].
More later (if time)

[6] see e.g. (Michel et al, J.Sci.Comp. 94, 2023) for a review

The div constraint. Homogeneous case

Consider now the multiD (steady) constraint on the div.

$$\partial_x u + \partial_y v = 0$$

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$$\left.\begin{array}{l}\partial_x u = -\partial_y v := -\sigma_x \Rightarrow D_x\mathbf{U} = -\mathrm{h}D_x\mathcal{I}_x\boldsymbol{\sigma_x} \\ \partial_y v = -\partial_x u := -\sigma_y \Rightarrow D_y\mathbf{V} = -\mathrm{h}D_y\mathcal{I}_y\boldsymbol{\sigma_y}\end{array}\right. \Rightarrow \quad \text{div} := \mathrm{h}D_y\mathcal{I}_y D_x\mathbf{P} + \mathrm{h}D_x\mathcal{I}_x D_y\mathbf{V}$$

1. We look at it as two 1D relations in one
2. We apply to each the GF quadrature as if we where working on 2 balance laws
3. We combine the two to get a discrete divergence operator

The div constraint. Homogeneous case

Consider now the multiD (steady) constraint on the div.

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Remarks

- Note that $\mathcal{I}_x = \mathcal{I}_y$, both corresponding to the LobattoIIIA tableau.
- We keep the subscripts $_x$ and $_y$ for better understanding
- Recall that the standard SEM divergence operator is

$$\text{div}_{\text{SEM}} := M_y D_x \mathbf{U} + M_x D_y \mathbf{V}$$

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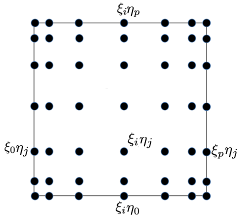
Proposition (The div residual) *The local assembly of the divergence operator can be written as*

$$\text{div} = \sum_K D_x^K D_y^K \Phi^K$$

where on each element Φ^K *is* the array of integrated divergences

$$(\Phi^K)_{lm} := \int_{y_0}^{y_m} (u_\mathrm{h}(x_l, s) - u_\mathrm{h}(x_0, s)) + \int_{x_0}^{x_l} (v_\mathrm{h}(s, y_m) - v_\mathrm{h}(s, y_0))$$



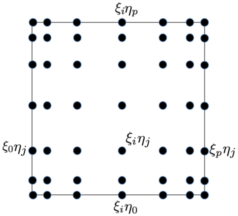
(non-stabilized) SEM-GF for the acoustics system

The SEM-GF semi-discrete approximation of the acoustics system (with abuse of notation) reads

$$M\dot{\mathbf{P}} + D_x D_y \Phi = 0$$
$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = 0$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = 0$$

Proposition (Steady states) *The SEM-GF scheme preserves exactly initial states for which* $P_i = P_0\ \forall\, i$*, and which verify within each element* K *and for every pair* $l, m \geq 1$

$$(\Phi^K)_{lm} = \int_{y_0}^{y_m} (u_\mathrm{h}(x_l, s) - u_\mathrm{h}(x_0, s)) + \int_{x_0}^{x_l} (v_\mathrm{h}(s, y_m) - v_\mathrm{h}(s, y_0)) = 0$$



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Definition (LobattoIIIA line-by-line/row-by-row projection) *Let* $\mathrm{v}_e = (u_e(x,y), v_e(x,y))$ *be a given* div *free velocity. Consider the initialization obtained by means of the LobattoIIIA method*

$$[\mathcal{I}_y \mathbf{U}(x_l)]_m = u(x_l, y_0) + \int_{y_0}^{y_m} u_e(x_l, y) \;, \quad [\mathcal{I}_x \mathbf{V}(y_m)]_l = v(x_0, y_m) + \int_{x_0}^{x_l} v_e(x, y_m)$$

with local ICs given by the last value of the previous elements, and ICs on the lowest/left boundaries $u_{\mathrm{h}}(x_i, y=0) = u_e(x_i, y=0)$ *and* $v_{\mathrm{h}}(x=0, y_j) = v_e(x=0, y_j)$.

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$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = 0$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = 0$$

Proposition (LobattoIIIA projection preservation) *Let* $\mathrm{v}_e = (u_e(x,y), v_e(x,y))$ *be a* div *free velocity, and consider the initial state consisting of the LobattoIIIA line-by-line/row-by-row projection of* v_e*, and* $P_i = P_0\ \forall i$*. The SEM-GF scheme*

1. *Preserves the initial condition within the accuracy of the evaluation of the integrals of the components of* v_e*. The IC is preserved exactly if the quadrature is exact.*
2. *It the nodal consistency order* $\mathcal{O}(\mathrm{h}^{p+2})$ *associated to the LobattoIIIA method*

(non-stabilized) SEM-GF for the full system: main ingredients

(non-stabilized) SEM-GF for the full system: main ingredients

Pressure equation: sources included using the GF recipe in x and y. Achieved setting

$$\Phi_{lm} = \int_{x_0}^{x_l} \int_{y_0}^{y_m} (\partial_x u_{\mathrm{h}} + \partial_y v_{\mathrm{h}} - s_{\mathrm{h}})$$

On each element

$$D_x D_y \Phi = \mathrm{h} D_x D_y \mathcal{I}_x \mathbf{U} + \mathrm{h} D_y D_x \mathcal{I}_x \mathbf{V} - \mathrm{h}^2 D_x \mathcal{I}_x D_y \mathcal{I}_y \mathbf{S}$$

(non-stabilized) SEM-GF for the full system: main ingredients

Velocity equations: we treat the 2 pseudo-1D constraints independently

$$\begin{aligned} \partial_x P &= q_x \\ \partial_y P &= q_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} D_x \mathbf{P} &= \mathrm{h} D_x \mathcal{I}_x \mathbf{Q_x} \\ D_y \mathbf{P} &= \mathrm{h} D_y \mathcal{I}_y \mathbf{Q_y} \end{aligned}$$

More compactly, we can write the above as

$$\begin{aligned} \partial_x P &= q_x \\ \partial_y P &= q_y \end{aligned} \quad \Rightarrow \quad \begin{aligned} D_x(\mathbf{P} - \mathrm{h}\mathcal{I}_x \mathbf{Q_x}) &= D_x \Phi_u \\ D_y(\mathbf{P} - \mathrm{h}\mathcal{I}_y \mathbf{Q_y}) &= D_x \Phi_v \end{aligned}$$

having introduced the residuals

$$(\Phi_u)_{lm} = \int_{x_0}^{x_l} (\partial_x P_\mathrm{h}(x, y_m) - q_{x\mathrm{h}}) \text{ and } (\Phi_v)_{lm} = \int_{y_0}^{y_m} (\partial_y P_\mathrm{h}(x_l, y) - q_{y\mathrm{h}})$$

(non-stabilized) SEM-GF for the full system: main ingredients

We obtain the full non-stabilized SEM-GF discretization

$$M\dot{\mathbf{P}} + D_x D_y \Phi = 0$$
$$M\dot{\mathbf{U}} + M_y D_x \Phi_u = 0$$
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All steady state preserving properties are enconded in the residual arrays $(\Phi,\, \Phi_u, \Phi_v)$

SEM-GF for the full system

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All steady state preserving properties are enconded in the residual arrays (Φ, Φ_u, Φ_v)

Stabilization

Use systematically GF quadrature to express all operators in terms of Φ, Φ_u and Φ_v

- We carry on the steady state preserving properties
- This is not the case for classical stabilization operators !

The initial solution

At the moment we can follow two approaches

1. Use a given IC (analytical or tabulated)
2. Given a div free velocity field v_0 (analytical or tabulated)
    - project v_0 on the space of discrete equilibria to obtain $\mathrm{v}_{\mathrm{h}0}$
    - Integrate the pressure in the each direction with LobattoIIIA, using $\mathrm{v}_{\mathrm{h}0}$ to evaluate the RHS
    - Use div and $curl$ conditions to combine the two and obtain a single admissible pressure initial state

Numerical examples in 2d

Linear waves with Coriolis, friction, mass source

$$\partial_t \begin{pmatrix} P \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ P \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ P \end{pmatrix} = \begin{pmatrix} s(x,y) \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}$$

Notation.

P pressure

v $= (u, v)$ velocity

$s(x, y)$ mass source

c_f friction coefficient

ϕ Coriolis coefficient

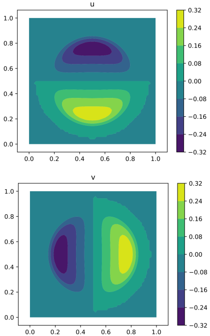
$\tau = (\tau_x, \tau_y)$ momentum forcing (e.g. wind for free suface waves)

Numerical examples in 2d

Vortex solutions

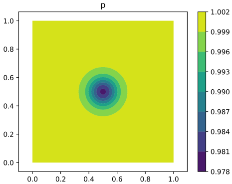
div-free exponential

$$r = \|x - x_0\|$$
$$v = (x - x_0)^{\perp} f(r)$$
$$P = 1$$



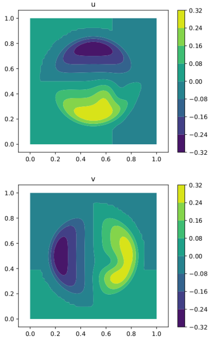
div-free exponential + Coriolis

$$r = \|x - x_0\|$$
$$v = v_{div-\text{free}} + \nabla \varphi_2$$
$$P = 1 - \phi g(r)$$



div-free exponential + mass source

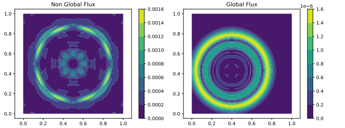
$$v = v_{div-\text{free}} + \nabla \varphi_2$$
$$P = 1$$
$$s = \Delta \varphi_2$$



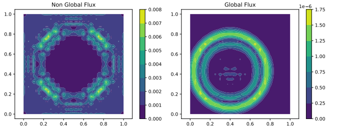
Numerical examples in 2d

Vortex solutions perturbations : $p = p_{\text{steady}} + 10^{-5}\delta_p$

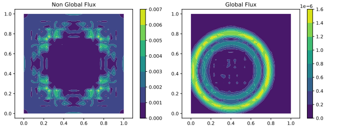
div-free exponential



$p1$ on 80×80 grid

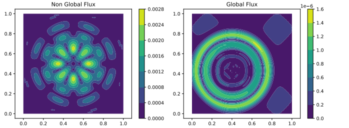


$p2$ on 20×20 grid

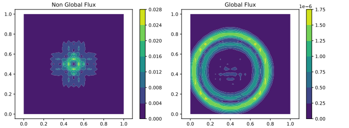


$p3$ on 13×13 grid

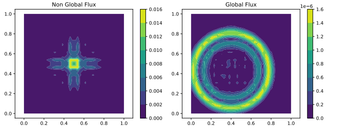
div-free exponential + Coriolis



$p1$ on 80×80 grid

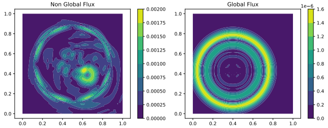


$p2$ on 20×20 grid

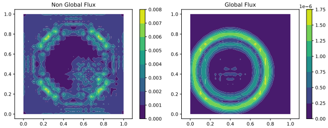


$p3$ on 13×13 grid

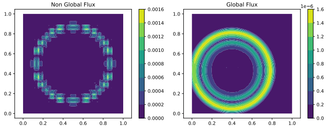
div-free exponential + mass source



$p1$ on 80×80 grid



$p2$ on 20×20 grid

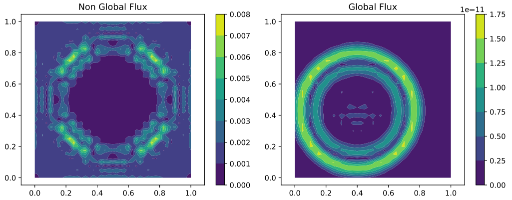


$p3$ on 13×13 grid

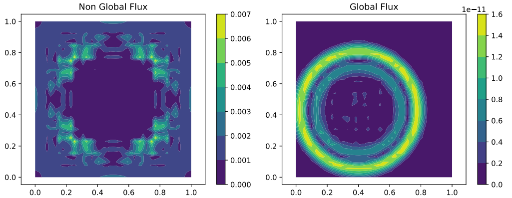
Numerical examples in 2d

Vortex solutions perturbations : $p = p_{\text{steady}} + 10^{-10}\delta_p$

div-free exponential

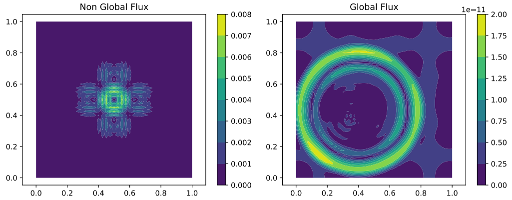


$p2$ on 20×20 grid

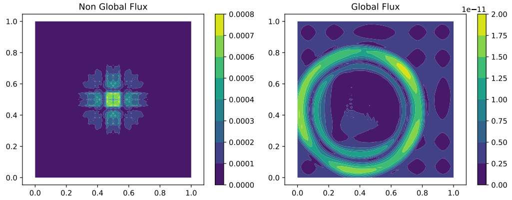


$p3$ on 13×13 grid

div-free exponential + Coriolis



$p2$ on 20×20 grid

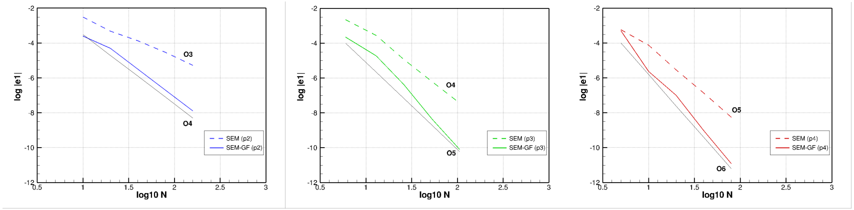


$p3$ on 13×13 grid

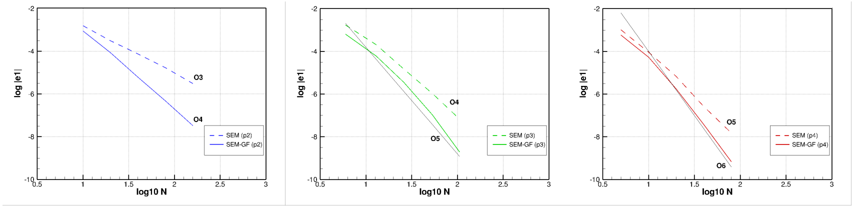
Numerical examples in 2d

Vortex solutions perturbations : super-convergence - theory GF = $\mathcal{O}(\mathrm{h}^{p+2})$

div-free exponential + Coriolis



div-free exponential + mass source



Numerical examples in 2d

The Stommel Gyre

H. Stommel, The westwards intensification of wind-driven ocean currents, Trans.Amer.Geophys.Union 29(2), 1948

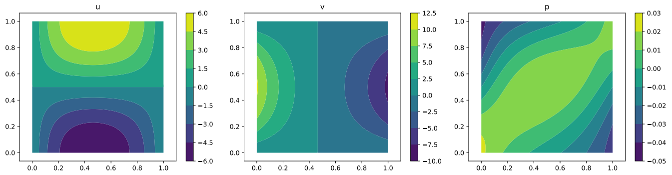
$$\partial_t \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ p \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}$$

Notation.

Friction: c_f constant

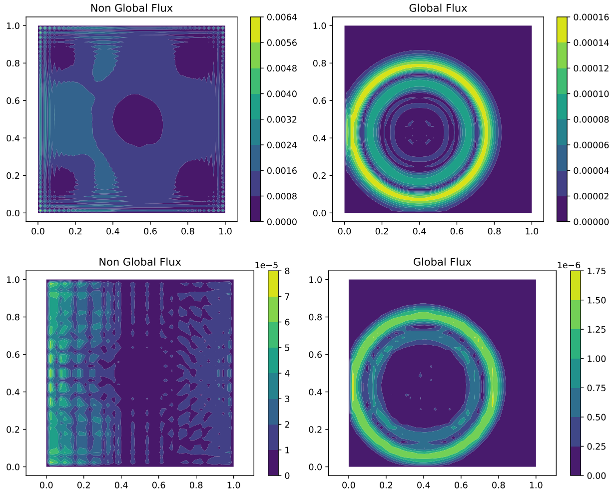
Coriolis: $\phi = \phi_0 + f_0 y$

Forcing: $\tau = \tau_0(0, \cos(\beta y))$



Numerical examples in 2d

The Stommel Gyre

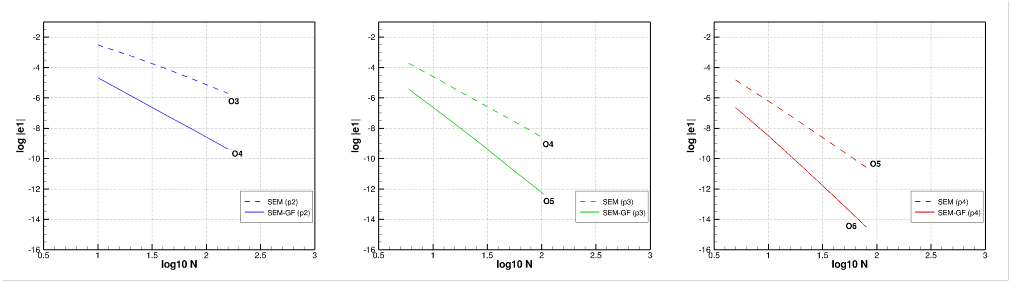


Pressure perturbation $10^{-3}\delta_p$: $p1$ on 80×80mesh

Pressure perturbation $10^{-5}\delta_p$: $p3$ on 13×13mesh

Numerical examples in 2d

The Stommel Gyre



Theory for GF: internal points $= \mathcal{O}(\mathrm{h}^{p+2})$

This is the last slide

Summary

- Global flux quadrature approach considerable accuracy enhancements at steady state for appropriate choice of the ODE integrator hidden the table $\mathcal{I}$
- In one dimension discrete equilibria can be generated a-priori if necessary
- In two dimensions fully div preserving schemes can be designed

Outlook

- FD and FV variant using ODE operators independent of the underlying method (**with Carlos**)
- MultiD with FD/FV and dime-by-dim solution of the Cauchy problem (**with Carlos, I hope :D**)
- MultiD: closer look at the curl involution
- DG-SEM with GFq for multiD scalar conservation laws and nonlinear variants

For more info/results

DG-SEM + GFq:
Mantri, Öffner, MR, Fully well-balanced entropy controlled discontinuous Galerkin spectral element method for shallow water flows: global flux quadrature and cell entropy correction, *J.Comput.Phys.* R1 in revision, `arxiv.org/abs/2212.11931` - December 2022

Continuous FEM version:
Micalizzi, MR, Abgrall, Novel well-balanced continuous interior penalty stabilizations, *J.Sci.Comp.* in revision, `arxiv.org/abs/2307.09697` - July 2023

WENO high order GF:
M. Ciallella, D. Torlo, M. Ricchiuto, Arbitrary High Order WENO Finite Volume Scheme with Flux Globalization for Moving Equilibria Preservation, *J.Sci.Comp.* 96, 2023

DG-SEM + GFq including discontinuous data/solutions:
Xu and Shu, A high-order well-balanced discontinuous Galerkin method for hyperbolic balance laws based on the Gauss-Lobatto quadrature rules, September 2023 Brown preprint

Structure preserving for acoustics:
W. Barsukow, M. Ricchiuto, and D. Torlo, Structure preserving methods via global flux quadrature: divergence preservation and curl involution with continuous finite elements, in preparation

MultiD well balanced:
W. Barsukow, M. Ricchiuto, and D. Torlo, Structure preserving methods via global flux quadrature: multidimensional well-balanced using with continuous finite elements, in preparation

.. Gracias Carlos !