A framework to construct constraint preserving and approximately well-balanced schemes

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Acknowledgements
Credit to them for the good stuff
blame me for the rest

• Yogiraj Mantri, Vellore Institute of Technology, India
• Philipp Öffner, Johannes Gutenberg-University, Mainz
• Wassilij Barzukow, CNRS, Institut de Mathématiques de Bordeaux
• Davide Torlo, MathLab at SISSA, Italy
We want to solve numerically (hyperbolic) systems of balance laws

$$\partial_t U + \nabla \cdot F(U) = S(U; \varphi(x))$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- GPR model and hyperbolic reformulations of viscous/dispersive systems
- etc.

We consider both 1D and Multi-D problems
A motivational example to fix ideas

**Pseudo-1D rotating shallow water eq.s** (Castro et al, SISC 31, 2008)

\[
\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ gb'(x) + cfu - \phi v \\ cfv + \phi u \end{pmatrix}
\]

**Incomplete zoology of steady states: some of the 2d richness in 1d**

1. Lake at rest
2. One dimensional frictionless flows with constant energy
3. Pseudo-one dimensional rest state with transverse perturbations \( v(x) \) and Coriolis effects
4. Frictionless pseudo-one dimensional flows with Coriolis effects
5. One dimensional flows with friction
6. etc. etc.
Objective: construct a will balanced scheme using as little a-priori knowledge as possible

besides the structure of the PDE itself

Well balanced how:

many meanings .. with all the (more or less) obvious connections.

(begging forgiveness in advance for any - involuntary - omissions !)
Preserving $V(U, \varphi(x)) = V_0$ via WB differencing or generalized polynomial approximations


Reconstruction/evolution of fluctuations wrt a given equilibrium $U^*(x)$


Reconstruction/evolution of fluctuations wrt discrete equilibria (approximate full well balanced):


Fully well balanced Riemann solver with 0-wave to enforce integral steady balance


Well balanced via integration of the source term and global fluxes

Well balanced numerics a (possibly incomplete) taxonomy

1. Preserving $V(U, \varphi(x)) = V_0$ via WB differencing or generalized polynomial approximations

2. Reconstruction/evolution of fluctuations wrt a given equilibrium $U^*(x)$

3. Reconstruction/evolution of fluctuations wrt discrete equilibria (approximate full well balanced):

4. Fully well balanced Riemann solver with 0-wave to enforce integral steady balance

5. Well balanced via integration of the source term and global fluxes

.. few of Carlos’ contributions ...

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Carlos’ contributions are essential in advancing the field of well-balanced numerical methods.
Well balanced numerics a (possibly incomplete) taxonomy

1. Preserving $V(U, \varphi(x)) = V_0$ via WB differencing or generalized polynomial approximations

2. Reconstruction/evolution of fluctuations wrt a given equilibrium $U^*(x)$

3. Reconstruction/evolution of fluctuations wrt discrete equilibria (approximate full well balanced):
   \[ \dot{U}_i(t) + R_i(U_h) = R_i(U_h^*) \text{ with } U^*(x) \text{ approx. sol. of the Cauchy pb } U' = J^{-1} S(U, \varphi) \]

4. Fully well balanced Riemann solver with 0-wave to enforce integral steady balance

5. Well balanced via integration of the source term and global fluxes
   \[ \partial_x F = S \Rightarrow \partial_x F(U^*) = -\partial_x R^* , \quad R \approx \int S \]
Objective

1. Elaborate on the GF idea and provide a link with the approximate Cauchy solver method
2. Propose a MultiD constraint preserving generalization
DG-SEM and Global Flux Quadrature in 1D

with Y. Mantri and P. Öffner
DG-SEM - Main notation
Discontinuous Galerkin Spectral Element Method

• Reference element $\xi \in [0, 1]$
• $x(\xi)$ linear mapping $[0, 1] \mapsto K$, for simplicity: $|K| = h$
• $\{\xi_i\}_{i=0,p}$ the $p + 1$ Gauss-Lobatto (GL) points
• $\{\phi_i(\xi)\}_{i=0,p}$ degree $p$ Lagrange bases
• Set $U_h = \sum_{i=0}^{p} \phi_i(x(\xi)) U_i$
DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

On an element $K$, start from discrete approximation arising from the variational form

$$h \int_0^1 \varphi_i(\xi) \partial_t U_h - \int_0^1 \partial_\xi \varphi_i(\xi) F_h + (\varphi_i \hat{F}_h(U_h^- U_h^+))_{\xi=1} - (\varphi_i \hat{F}_h(U_h^- U_h^+))_{\xi=0} = h \int_0^1 \varphi_i(\xi) S_h$$
Global Flux Quadrature in 1D

DG-SEM - Discrete variational form
Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM : quadrature based on the same GL nodes used for the polynomial expansion

$$\mathcal{M} \dot{U} - D_x^T \mathbf{F} + \hat{\mathbf{F}} = \mathcal{MS}$$

with the notation:

- $\mathcal{M} = \text{diag}(\{w_i\}_{i=0,p})$ with $w_i = h \int_0^1 \phi_i(\xi) d\xi$ the quadrature weights
- with $(D_x)_{ij} = w_i \partial_\xi \phi_j(\xi_i)$
- $\mathbf{B} = \text{diag}(-1, \ldots, 1)$ the matrix sampling boundary values
- $\mathbf{U, F, \hat{F}, S}$: elemental arrays of nodal solution/flux/num. flux/source values

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DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM discrete equations in strong form (SBP property\(^2\))

$$\mathcal{M}\dot{U} + D_x F + B(\hat{F} - F) = \mathcal{M}S$$

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DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM discrete equations in strong form

$$\mathcal{M} \dot{U} + D_x F + B(\hat{F} - F) = \mathcal{M} S$$

Setting $\hat{F}(U^-, U^+) = \alpha F^+ + (1 - \alpha) F^- - \mathcal{D}[U]$, we get the fully discrete method

$$\mathcal{M} \dot{U} + D_x F + B(\alpha F) - B(\mathcal{D}[U]) = \mathcal{M} S$$

This is our "reference" non well-balanced method
DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x G(U; \varphi(x)) = 0$$

with \( G = F(U) + R(U; \varphi(x)) \) and

$$R(U; \varphi(x)) = R_0 - \int_{x_0}^{x} S(U; \varphi(s)) ds$$
DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x F(U) = -\partial_x R(U; \varphi(x))$$

If we have the source primitive $R$ at all GL nodes, we can readily write the DG-SEM scheme:

$$M\dot{U} + D_x F + B(\alpha[F]) - B(D[U]) = -D_x R + B(\alpha[R])$$

The key now is to define $R$ at all GL nodes
DG-SEM - Discrete variational form using global fluxes

We compute nodal values of $R$:

$$\partial_x R = -S$$

Choose

$$R_h = \sum_{i=0,p} \varphi_i(x(\xi)) R_i , \quad S_h = \sum_{i=0,p} \varphi_i(x(\xi)) S_i$$

and integrate on each element

1. For the (local) initial value we set $R_0 = R^-$

2. For all $i \in 1, p$ we compute $R_i = R_{i-1} - h \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$
DG-SEM - Discrete variational form using global fluxes

We compute nodal values of $R$:

$$\partial_x R = -S$$

Remark. *In compact notation we have*

$$R = R^- - h\mathcal{I}S$$

with $\mathcal{I}$ is the tableau of the $p + 1$ stages RK-LobattoIIIa implicit collocation method.

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3 A. Prothero & A. Robinson, Math.Comp. 28, 1974
Source integration: initial value

The initial value $R^-$ can be related to the last value on the neighbouring element:

$$R^- = [R(x_p)]^K + [R] = [R(x_p)]^K + \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} S$$

The critical term is for the shallow water example

$$\lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} gh \cdot \partial_x b_h$$

For simplicity, let us rule out discontinuous bathymetry and set

$$R^- = [R(x_p)]^K, \quad [R] = 0$$
DG-SEM - Discrete variational form using global fluxes

We seek solutions of the (hyperbolic) system of balance laws

\[ \partial_t U + \partial_x F(U) = S(U; \varphi(x)) \]

for all \( K \) evolve in time

\[
\mathcal{M} \dot{U} + D_x F + B(\alpha[F]) - B(D[U]) = h D_x IS \quad \text{GF-DG}
\]

\[
\mathcal{M} \dot{U} + D_x F + B(\alpha[F]) - B(D[U]) = \mathcal{M} S \quad \text{DG}
\]
Main result

Proposition (Discrete steady state). The DG-SEM with global flux quadrature preserves exactly continuous discrete steady states $U^*_i = U(F_i)$ with $F$ obtained by integrating the ODE

$$F' = S(U(F), \varphi(x))$$

using the implicit continuous collocation RK-LobattoIIIA method on spatial slabs of size $h$.

As long as $U(F)$ is a one to one mapping $U^*(x)$ verifies the consistency estimates of the LobattoIIIA method$^4$: Element endpoints are $2p$-order accurate, internal nodes have accuracy $h^{p+2}$.

$^4$See e.g. Theorem 7.10 in Hairer, Wanner and Norset, Solving Ordinary Differential Equations I., Springer 1993
Global flux quadrature

\[ \int_K \phi_i f_h \rightarrow hD_x I f \]

Main results

- Characterization of discrete equilibria: solution of LobattoIIIA continuous collocation method

- For steady states: super-convergence property of the LobattoIIIA ODE integrator

- Same approximate well balanced notion as in \(^5\), however:
  - ✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)
  - ✓ Considerable accuracy enhancements at steady state with minor change in code
  - × Order/type of collocation method not arbitrary

Pseudo-1D rotating shallow water eq.s (Castro et al, SISC 31, 2008)

\[
\begin{aligned}
\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} &= -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u - \phi v \\ c_f v + \phi u \end{pmatrix}
\end{aligned}
\]

Notation.

- $h$: water depth
- $\zeta$: free surface level
- $v = (u, v)$: horizontal velocity
- $P = gh^2/2$: hydrostatic pressure ($g$: gravity acceleration)
- $\varphi = gb$: gravitational potential ($b(x, y)$: bottom topography)
- $c_f = c_f(h, v)$: friction coefficient
- $\phi$: Coriolis coefficient
Numerical examples: perturbations of moving equilibria

Trans-critical with

\[ h u = q_0 \]
\[ g \zeta + u^2/2 = \mathcal{E}_0 \]

Depth perturbation \( \xi_h = 10^{-3} \)

Super-critical with

\[ h u = q_0 \]
\[ g \zeta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u \]

Depth perturbation \( \xi_h = 10^{-5} \)

Sub-critical with \( v = \phi x \) and

\[ h u = q_0 \]
\[ g \zeta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi uv \]

Depth perturbation \( \xi_h = 10^{-5} \)
Numerical examples: error wrt ODE solver

Sub-critical with

\[ hu = q_0 \]
\[ g\zeta + u^2/2 = \mathcal{E}_0 \]

Super-critical with

\[ hu = q_0 \]
\[ g\zeta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u \]

Sub-critical with \( v = \phi x \) and

\[ hu = q_0 \]
\[ g\zeta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi uv \]
Numerical examples: (super)convergence, DG-SEM

Theory for GF: internal points = $O(h^{p+2})$ - endpoints = $O(h^{2p})$

Super-critical channel with bump (no friction, no Coriolis)

Sub-critical with Coriolis force
Numerical examples: (super)convergence, DG-SEM

Theory for GF: internal points $= \mathcal{O}(h^{p+2})$ - endpoints $= \mathcal{O}(h^{2p})$

Super-critical channel with bump (no friction, no Coriolis)

Sub-critical with Coriolis force
2 MultiD Constraint Preserving via Global Flux Quadrature

with W. Barsukow and D. Torlo
Well balanced in multi-D: structure/constraint preserving

\[ \nabla \cdot F(U) = S(U; \varphi(x)) \]

The form of the tensor $F$ determines the type of differential constraints to preserve.
Well balanced in multi-D: structure/constraint preserving

We are going to work with the following example (with $v = (u, v)$)

$$
\partial_t P + \nabla \cdot v = s(x, y)
$$

$$
\partial_t u + \partial_x P = \phi v - c_f u + \tau_x
$$

$$
\partial_t v + \partial_y P = -\phi u - c_f v + \tau_y
$$

Linear waves with Coriolis, friction, mass source

$P$ pressure

$v = (u, v)$ velocity

$s(x, y)$ mass source

$c_f$ friction coefficient

$\phi$ Coriolis coefficient

$\tau = (\tau_x, \tau_y)$ momentum forcing (e.g. wind for free surface waves)
Well balanced in multi-D: structure/constraint preserving

We are going to construct schemes preserving (simultaneously) the constrains

\[ \nabla \cdot v = s(x, y) \]

\[ \nabla P = q \]

- one multi-D constraint : \( \nabla \cdot v = s(x, y) \) (hard)
- two (pseudo-)1D constraints : \( \nabla P = q \) (easier)
Global Flux Quadrature in MultiD

Discrete Framework: SEM

Tensor product spectral finite element method (SEM)

- $x(\xi) = (x(\xi), y(\eta))$ mapping $[0, 1]^2 \mapsto K$, for simplicity: $|K| = h^2$
- $\{\xi_i\}_{i=0,p}$ and $\{\eta_j\}_{j=0,p}$ the $p + 1$ Gauss-Lobatto (GL) points
- $\{\phi_i(x)\}_{i=0,p}$ and $\{\psi_j(y)\}_{j=0,p}$ 1d degree $p$ Lagrange bases
- For node $ij$: $\lambda_{ij}(x(\xi), y(\eta)) = \phi_i(x)\psi_j(y)$
- Set $U_h = \sum_{i,j} \lambda_{ij}(x(\xi), y(\eta))U_{ij}$
**Discrete Framework: SEM**

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- Set \( U_h = \sum_{i,j} \lambda_{ij}(x(\xi), y(\eta))U_{ij} \)

**Notation: tensor product matrices**

Mass matrix entries

\[
\int_K \lambda_{ij} \lambda_{lm} = \int_{x_0}^{x_p} (\phi_i(x)\phi_l(x)) \times \int_{y_0}^{y_p} (\psi_j(y)\psi_m(y)) \Rightarrow M = M_x \otimes M_y = M_y \otimes M_x
\]
Global Flux Quadrature in MultiD

Discrete Framework: SEM

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Global Flux Quadrature in MultiD

Discrete Framework: SEM

Tensor product spectral finite element method (SEM)

- $x(\xi) = (x(\xi), y(\eta))$ mapping $[0, 1]^2 \mapsto K$, for simplicity: $|K| = h^2$

- $\{\xi_i\}_{i=0}^p$ and $\{\eta_j\}_{j=0}^p$ the $p+1$ Gauss-Lobatto (GL) points

- $\{\phi_i(x)\}_{i=0}^p$ and $\{\psi_j(y)\}_{j=0}^p$ 1d degree $p$ Lagrange bases

- For node $ij$: $\lambda_{ij}(x(\xi), y(\eta)) = \phi_i(x)\psi_j(y)$

- Set $U_h = \sum_{i,j} \lambda_{ij}(x(\xi), y(\eta))U_{ij}$

Notation: tensor product matrices

Derivative matrix entries

$$\int_K \lambda_{ij} \partial_x \lambda_{lm} = \int_{x_0}^{x_p} (\phi_i(x)\partial_x \phi_l(x)) \times \int_{y_0}^{y_p} (\psi_j(y)\psi_m(y)) \Rightarrow D_x M_y = M_y D_x$$
Discrete Framework: SEM

We are going to work with the following example (with $v = (u, v)$)

$$\partial_t P + \nabla \cdot v = s(x, y)$$
$$\partial_t u + \partial_x P = q_x$$
$$\partial_t v + \partial_y P = q_y$$

Standard continuous SEM approximation (no stabilization) in strong form

$$M\dot{P} + M_y D_x U + M_x D_y V = MS$$
$$M\dot{U} + M_y D_x P = MQ_x$$
$$M\dot{V} + M_x D_y P = MQ_y$$
Discrete Framework: SEM
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\partial_t v + \partial_y P &= q_y
\end{align*}
\]

Standard continuous SEM approximation (no stabilization) in strong form

\[
\begin{align*}
M \dot{P} + M_y D_x U + M_x D_y V &= MS = \sum K M^K S^K \\
M \dot{U} + M_y D_x P &= MQ_x \\
M \dot{V} + M_x D_y P &= MQ_y
\end{align*}
\]

Abuse of notation:
for continuous SEM the matrix formulation involve local elemental assembly. We omit it for brevity, and explicitly mention only when necessary
Global Flux Quadrature in MultiD

Discrete Framework: SEM

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M \dot{V} + M_x D_y P &= MQ_y
\end{align*}
\]

**Stabilization using standard methods:**
SUPG, gradient penalty, orthogonal subgrid scales etc\(^6\).

More later (if time)

\(^6\)see e.g. (Michel et al, J.Sci.Comp. 94, 2023) for a review
The *div* constraint. Homogeneous case

Consider now the multiD (steady) constraint on the *div*.

\[ \partial_x u + \partial_y v = 0 \]
The \textit{div} constraint. Homogeneous case

Consider now the multiD (steady) constraint on the \textit{div}.

\[
\begin{align*}
\partial_x u &= -\partial_y v := -\sigma_x \Rightarrow D_x U = -h D_x I_x \sigma_x \\
\partial_y v &= -\partial_x u := -\sigma_y \Rightarrow D_y V = -h D_y I_y \sigma_y
\end{align*}
\]

\[\Rightarrow \quad \text{div} := h D_y I_y D_x P + h D_x I_x D_y V\]

1. We look at it as two 1D relations in one
2. We apply to each the GF quadrature as if we where working on 2 balance laws
3. We combine the two to get a discrete divergence operator
The \textit{div} constraint. Homogeneous case

Consider now the multiD (steady) constraint on the \textit{div}.

\[
\begin{align*}
\partial_x u &= -\partial_y v := -\sigma_x \Rightarrow D_x U = -hD_x I_x \sigma_x \\
\partial_y v &= -\partial_x u := -\sigma_y \Rightarrow D_y V = -hD_y I_y \sigma_y
\end{align*}
\]

⇒ \text{div} := hD_y I_y D_x U + hD_x I_x D_y V

Remarks

• Note that $I_x = I_y$, both corresponding to the LobattoIIIA tableau.
• We keep the subscripts $x$ and $y$ for better understanding.
• Recall that the standard SEM divergence operator is

\[
\text{div}_{\text{SEM}} := M_y D_x U + M_x D_y V
\]
The \textit{div} constraint. Homogeneous case

Consider now the multiD (steady) constraint on the \textit{div}.

\[ \partial_x u = -\partial_y v := -\sigma_x \Rightarrow D_x U = -hD_x I_x \sigma_x \]

\[ \partial_y v = -\partial_x u := -\sigma_y \Rightarrow D_y V = -hD_y I_y \sigma_y \]

**Proposition** (The div residual) The local assembly of the divergence operator can be written as

\[ \text{div} = \sum_K D^K_x D^K_y \Phi^K \]

where on each element $\Phi^K$ is the array of integrated divergences

\[ (\Phi^K)_{lm} := \int_{y_0}^{y_m} (u_h(x_l, s) - u_h(x_0, s)) + \int_{x_0}^{x_l} (v_h(s, y_m) - v_h(s, y_0)) \]
(non-stabilized) SEM-GF for the acoustics system

The SEM-GF semi-discrete approximation of the acoustics system (with abuse of notation) reads

\[ M \dot{P} + D_x D_y \Phi = 0 \]
\[ M \dot{U} + M_y D_x P = 0 \]
\[ M \dot{V} + M_x D_y P = 0 \]

**Proposition** (Steady states) *The SEM-GF scheme preserves exactly initial states for which \( P_i = P_0 \ \forall \ i \), and which verify within each element \( K \) and for every pair \( l, m \geq 1 \)

\[ (\Phi^K)_{lm} = \int_{y_0}^{y_m} (u_h(x_l, s) - u_h(x_0, s)) + \int_{x_0}^{x_l} (v_h(s, y_m) - v_h(s, y_0)) = 0 \]
Global Flux Quadrature in MultiD

(non-stabilized) SEM-GF for the acoustics system

The SEM-GF semi-discrete approximation of the acoustics system (with abuse of notation) reads

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M\dot{P} + D_x D_y \Phi &= 0 \\
M\dot{U} + M_y D_x P &= 0 \\
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\end{align*}
\]

**Definition** (LobattoIIIA line-by-line/row-by-row projection) Let \( v_e = (u_e(x, y), v_e(x, y)) \) be a given div free velocity. Consider the initialization obtained by means of the LobattoIIIA method

\[
\begin{align*}
[I_y U(x_l)]_m &= u(x_l, y_0) + \int_{y_0}^{y_m} u_e(x_l, y) \\
[I_x V(y_m)]_l &= v(x_0, y_m) + \int_{x_0}^{x_l} v_e(x, y_m)
\end{align*}
\]

with local ICs given by the last value of the previous elements, and ICs on the lowest/left boundaries

\[
\begin{align*}
u_h(x_i, y = 0) &= u_e(x_i, y = 0) \quad \text{and} \quad v_h(x = 0, y_j) = v_e(x = 0, y_j)
\end{align*}
\]
Global Flux Quadrature in MultiD

(non-stabilized) SEM-GF for the acoustics system

The SEM-GF semi-discrete approximation of the acoustics system (with abuse of notation) reads

\[
\begin{align*}
M \dot{P} + D_x D_y \Phi &= 0 \\
M \dot{U} + M_y D_x P &= 0 \\
M \dot{V} + M_x D_y P &= 0
\end{align*}
\]

Proposition (LobattoIIIA projection preservation) Let \( \textbf{v}_e = (u_e(x, y), v_e(x, y)) \) be a \textit{div} free velocity, and consider the initial state consisting of the LobattoIIIA line-by-line/row-by-row projection of \( \textbf{v}_e \), and \( P_i = P_0 \ \forall i \). The SEM-GF scheme

1. Preserves the initial condition within the accuracy of the evaluation of the integrals of the components of \( \textbf{v}_e \). The IC is preserved exactly if the quadrature is exact.

2. It the nodal consistency order \( O(h^{p+2}) \) associated to the LobattoIIIA method
Global Flux Quadrature in MultiD

(non-stabilized) SEM-GF for the full system: main ingredients
Global Flux Quadrature in MultiD

(non-stabilized) SEM-GF for the full system: main ingredients

Pressure equation: sources included using the GF recipe in $x$ and $y$. Achieved setting

$$\Phi_{lm} = \int_{x_0}^{x_i} \int_{y_0}^{y_m} (\partial_x u_h + \partial_y v_h - s_h)$$

On each element

$$D_x D_y \Phi = h D_x D_y I_x U + h D_y D_x I_x V - h^2 D_x I_x D_y I_y S$$
(non-stabilized) SEM-GF for the full system: main ingredients

Velocity equations: we treat the 2 pseudo-1D constraints independently

\[
\begin{align*}
\partial_x P &= q_x \\
\partial_y P &= q_y \\
\Rightarrow \quad D_x P &= hD_x I_x Q_x \\
D_y P &= hD_y I_y Q_y
\end{align*}
\]

More compactly, we can write the above as

\[
\begin{align*}
\partial_x P &= q_x \\
\partial_y P &= q_y \\
\Rightarrow \quad D_x (P - h I_x Q_x) &= D_x \Phi_u \\
D_y (P - h I_y Q_y) &= D_x \Phi_v
\end{align*}
\]

having introduced the residuals

\[
(\Phi_u)_{lm} = \int_{x_0}^{x_1} (\partial_x P_h(x, y_m) - q_{x_h}) \quad \text{and} \quad (\Phi_v)_{lm} = \int_{y_0}^{y_m} (\partial_y P_h(x_l, y) - q_{y_h})
\]
(non-stabilized) SEM-GF for the full system: main ingredients

We obtain the full non-stabilized SEM-GF discretization

\[
\begin{align*}
M\dot{P} + D_x D_y \Phi &= 0 \\
M\dot{U} + M_y D_x \Phi_u &= 0 \\
M\dot{V} + M_x D_y \Phi_v &= 0
\end{align*}
\]

All steady state preserving properties are enconded in the residual arrays \((\Phi, \Phi_u, \Phi_v)\)
SEM-GF for the full system

We obtain the full non-stabilized SEM-GF discretization

\[ M \dot{P} + D_x D_y \Phi = 0 \]
\[ M \dot{U} + M_y D_x \Phi_u = 0 \]
\[ M \dot{V} + M_x D_y \Phi_v = 0 \]

All steady state preserving properties are encoded in the residual arrays \((\Phi, \Phi_u, \Phi_v)\)

Stabilization

Use systematically GF quadrature to express all operators in terms of \(\Phi, \Phi_u\) and \(\Phi_v\)

- We carry on the steady state preserving properties
- This is not the case for classical stabilization operators!
The initial solution
At the moment we can follow two approaches

1. Use a given IC (analytical or tabulated)

2. Given a \( \text{div} \) free velocity field \( v_0 \) (analytical or tabulated)
   - project \( v_0 \) on the space of discrete equilibria to obtain \( v_{h0} \)
   - Integrate the pressure in the each direction with LobattoIIIA, using \( v_{h0} \) to evaluate the RHS
   - Use \( \text{div} \) and \( \text{curl} \) conditions to combine the two and obtain a single admissible pressure initial state
Linear waves with Coriolis, friction, mass source

\[
\partial_t \begin{pmatrix} P \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ P \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ P \end{pmatrix} = \begin{pmatrix} s(x, y) \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}
\]

Notation.
- \( P \) pressure
- \( v = (u, v) \) velocity
- \( s(x, y) \) mass source
- \( c_f \) friction coefficient
- \( \phi \) Coriolis coefficient
- \( \tau = (\tau_x, \tau_y) \) momentum forcing (e.g. wind for free surface waves)
Vortex solutions

**div-free exponential**

\[
\begin{align*}
    r &= \|x - x_0\| \\
    v &= (x - x_0) \perp f(r) \\
    P &= 1
\end{align*}
\]

**div-free exponential + Coriolis**

\[
\begin{align*}
    r &= \|x - x_0\| \\
    v &= v_{\text{div-free}} + \nabla \varphi_2 \\
    P &= 1 - \phi g(r)
\end{align*}
\]

**div-free exponential + mass source**

\[
\begin{align*}
    v &= v_{\text{div-free}} + \nabla \varphi_2 \\
    P &= 1 \\
    s &= \Delta \varphi_2
\end{align*}
\]
Vortex solutions perturbations: \( p = p_{\text{steady}} + 10^{-5} \delta p \)

- div-free exponential
- div-free exponential + Coriolis
- div-free exponential + mass source
Vortex solutions perturbations: \( p = p_{\text{steady}} + 10^{-10} \delta p \)

*div-free exponential*

\[ \begin{align*}
\text{Non Global Flux} & \\
\text{Global Flux} & \\
\end{align*} \]

\[ \begin{array}{c}
p2 \text{ on } 20 \times 20 \text{ grid} \\
p3 \text{ on } 13 \times 13 \text{ grid} \\
\end{array} \]

*div-free exponential + Coriolis*

\[ \begin{align*}
\text{Non Global Flux} & \\
\text{Global Flux} & \\
\end{align*} \]

\[ \begin{array}{c}
p2 \text{ on } 20 \times 20 \text{ grid} \\
p3 \text{ on } 13 \times 13 \text{ grid} \\
\end{array} \]
Vortex solutions perturbations: super-convergence - theory $GF = \mathcal{O}(h^{p+2})$

$\text{div-free exponential + Coriolis}$

$\text{div-free exponential + mass source}$
The Stommel Gyre


\[
\partial_t \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ p \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}
\]

Notation.

Friction: \( c_f \) constant

Coriolis: \( \phi = \phi_0 + f_0 y \)

Forcing: \( \tau = \tau_0(0, \cos(\beta y)) \)
**The Stommel Gyre**

Pressure perturbation $10^{-3} \delta_p$: $p1$ on $80 \times 80$ mesh

Pressure perturbation $10^{-5} \delta_p$: $p3$ on $13 \times 13$ mesh
The Stommel Gyre

Theory for GF: internal points $= \mathcal{O}(h^{p+2})$
Summary

- Global flux quadrature approach considerable accuracy enhancements at steady state for appropriate choice of the ODE integrator hidden the table $\mathcal{I}$

- In one dimension discrete equilibria can be generated a-priori if necessary

- In two dimensions fully div preserving schemes can be designed

Outlook

- FD and FV variant using ODE operators independent of the underlying method (with Carlos)

- MultiD with FD/FV and dime-by-dim solution of the Cauchy problem (with Carlos, I hope :D)

- MultiD: closer look at the curl involution

- DG-SEM with GFq for multiD scalar conservation laws and nonlinear variants
DG-SEM + GFq:

Continuous FEM version:

WENO high order GF:

DG-SEM + GFq including discontinuous data/solutions:
Xu and Shu, A high-order well-balanced discontinuous Galerkin method for hyperbolic balance laws based on the Gauss-Lobatto quadrature rules, September 2023 Brown preprint

Structure preserving for acoustics:
W. Barsukow, M. Ricchiuto, and D. Torlo, Structure preserving methods via global flux quadrature: divergence preservation and curl involution with continuous finite elements, in preparation

MultiD well balanced:
... gracias Carlos!