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ON THE PAIRWISE COMPATIBILITY PROPERTY OF SOME SUPERCLASSES OF THRESHOLD GRAPHS

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A graph G is called a pairwise compatibility graph (PCG) if there exists a positive edge weighted tree T and two non-negative real numbers $d_{\min}$ and $d_{\max}$ such that each leaf l_u of T corresponds to a node $u \in V$ and there is an edge $(u, v) \in E$ if and only if $d_{\min} \leq d_T(l_u, l_v) \leq d_{\max}$, where $d_T(l_u, l_v)$ is the sum of the weights of the edges on the unique path from l_u to l_v in T. In this paper we study the relations between the pairwise compatibility property and superclasses of threshold graphs, i.e., graphs where the neighborhoods of any couple of nodes either coincide or are included one into the other. Namely, we prove that some of these superclasses belong to the PCG class. Moreover, we tackle the problem of characterizing the class of graphs that are PCGs of a star, deducing that also these graphs are a generalization of threshold graphs.

Keywords: PCG; leaf power graphs (LPG); mLPG; threshold graphs; matrogenic graphs.

Mathematics Subject Classification 2000: 68R10

1. Introduction

Given an edge weighted tree T, let $d_{\min}$ and $d_{\max}$ be two non-negative real numbers with $d_{\min} \leq d_{\max}$. For any two leaves l_1 and l_2 of the tree T, we denote by $d_T(l_1, l_2)$ the sum of the weights of the edges on the unique path from l_1 to l_2 in T. Starting from T, $d_{\min}$ and $d_{\max}$, it is possible to construct a *pairwise compatibility graph* of T, i.e., a graph $G(V, E)$ where each node $u \in V$ corresponds to a leaf l_u of T and there is an edge $(u, v) \in E$ if and only if $d_{\min} \leq d_T(l_u, l_v) \leq d_{\max}$. We will denote such a graph G by $PCG(T, d_{\min}, d_{\max})$. Consequently, we say that a graph G is a

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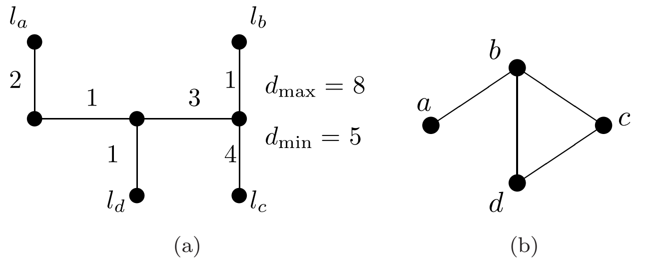

Fig. 1. (a) A pairwise compatibility tree; (b) the corresponding pairwise compatibility graph.

pairwise compatibility graph (PCG) if there exists an edge weighted tree T (called a *pairwise compatibility tree*) and two non-negative real numbers $d_{\min}$ and $d_{\max}$ such that $G = PCG(T, d_{\min}, d_{\max})$. In Fig. 1, an example of pairwise compatibility tree and the corresponding pairwise compatibility graph are depicted.

The *pairwise compatibility graph recognition problem* consists in determining whether a given graph is PCG or not. This problem seems in general difficult to tackle, and even more after the paper by Yanhaona *et al.* [19] disproving the conjecture that every graph is PCG [13].

Althought the pairwise compatibility graph recognition problem arose in a computational biology context [13] and its main application remains in phylogenetics, it has captured the interest of researchers belonging to other fields such as computational complexity and graph theory. In particular people working in computational complexity theory are fascinated from the fact that the clique problem is polynomially solvable for PCGs [13].

Due to the apparent difficulty of the pairwise compatibility recognition problem for arbitrary graphs, the research is focussed on the study of this problem for specific classes of graphs. Following this line of research many classes of graphs are proven to be in PCG such as cliques and disjoint union of cliques [1], chordless cycles and single chord cycles [20], ladder graphs [18], some particular subclasses of bipartite graphs [19] and graphs with Dilworth number two [8]. Moreover, it is proven that all graphs with 7 nodes or less are PCGs [17, 6], whereas the smallest example of a graph that is not PCG has 8 nodes [10]. Finally, in [7], the closure properties of the PCG class under some common graph operations are also studied.

In this paper we present two different contributions: The first one oriented to increase the number of specific classes of graphs that are PCGs and the other one going toward the direction of characterizing subclasses of PCGs derived from a specific topology of the pairwise compatibility tree. Both these results are related to generalizations of threshold graphs, i.e., graphs where the neighborhoods of any couple of nodes either coincide or are included one into the other.

These graphs, introduced in 1977 independently by Chavatal and Hammer [9], and Henderson and Zalcstein [12], have then found application in many fields, such as computer science, scheduling theory, modern systems biology, social sciences and psychology [15]. So in this paper, after a section recalling terminologies and concepts useful in the forthcoming work, we present the main results in Secs. 3 and 4. In particular, in Sec. 3 we prove that a wide superclass of threshold graphs is inside the PCG class. Then, in Sec. 4, the structure of the graphs that are PCGs of stars is presented, proving that the stars are pairwise compatibility trees of a new class of graphs, which we call nearly three-threshold. This class extends the class of threshold graphs At the end, some open problems derived from this work are summarized in the last section of the paper.

2. Preliminaries

In this section, we introduce some definitions and some concepts that we use in the rest of this paper.

An *edge weighted tree*, simply a *weighted tree*, is a tree with a non-negative weight assigned to each edge. In this paper we consider only weighted trees and connected graphs.

Given a connected graph G whose distinct node degrees are $\delta_1 > \cdots > \delta_r$, we define $B_i = \{v \in V(G) : deg(v) = \delta_i\}$, for any $i = 1, \ldots, r$. The sets B_i are usually referred as *boxes* and the sequence $B_1, \ldots, B_r$ is called the *degree partition of* G *into boxes*. Notice that B_1 contains all the nodes of maximum degree while B_r contains all the nodes of minimum degree and that r does not represent the maximum degree but it is the number of different degrees in the graph.

An *n-leaf star* is a tree with n nodes with distinct degrees $\delta_1 = n$ and $\delta_2 = 1$, and the cardinality of the two boxes B_1 and B_2 are 1 and $n-1$, respectively. We usually denote by c the unique node of degree n.

Given a graph G with degree partition $B_1, \ldots, B_r$, G is a *threshold graph* if and only if for all $u \in B_i$, $v \in B_j$, $u \neq v$, we have $(u, v) \in E(G)$ if and only if $i + j \leq r + 1$. As an example, see the graph in Fig. 2(a).

A *caterpillar* is a tree in which all the nodes are within distance one of a central path which is called the *spine*.

A graph $G = (K, S, E)$ is said to be *split* if there is a node partition $V = K \cup S$ such that the subgraphs induced by K and S are complete and stable, respectively.

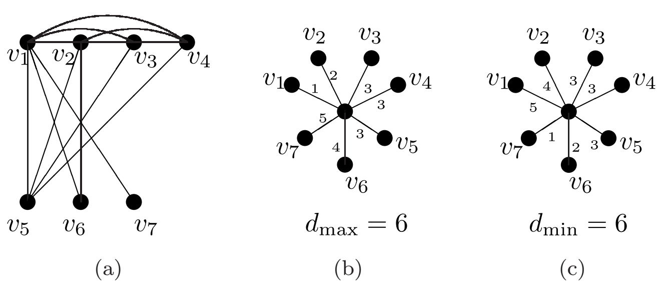

Fig. 2. (a) A threshold graph; (b) the corresponding pairwise compatibility tree that makes it a LPG; (c) the corresponding pairwise compatibility tree that makes it a mLPG.

Given two split graphs $G_1 = (K_1, S_1, E_1)$ and $G_2 = (K_2, S_2, E_2)$, their *composition* $G_1 \circ G_2$ is formed by taking the disjoint union of G_1 and G_2 and adding all the edges $\{u, v\}$ such that $u \in K_1$ and $v \in V(G_2)$. Observe that $G_1 \circ G_2$ is again a split graph.

A set M of edges is a *perfect matching* of dimension n of A onto B if and only if A and B are disjoint subsets of nodes of cardinality n and each node in A is adjacent to exactly one node in B. We say that a split graph $G = (K, S, E)$ is a *split matching* if the subset of edges in E not belonging to the clique forms a perfect matching.

An *antimatching* of dimension n of A onto B is a set of edges such that its complement is a perfect matching of dimension n of A onto B. We say that a split graph $G = (K, S, E)$ is a *split antimatching* if the subset of edges in E not belonging to the clique forms an antimatching.

A *split matrogenic* graph [15] is the composition of t split graphs $G_i = (K_i, S_i, E_i)$ with $i = 1, \ldots, t$ such that either G_i is a split matching or G_i is a split antimatching or $K_i = \emptyset$ (and G_i is called *stable graph*) or $S_i = \emptyset$ (and G_i is called *clique graph*).

It is not difficult to see that split matrogenic graphs are a super class of threshold graphs and that also split matchings and split antimatchings graphs are split matrogenic.

Before concluding this section, we introduce the definitions of two subclasses of PCGs, namely LPGs and mLPGs:

Definition 2.1. [16] A graph $G = (V, E)$ is an LPG if there exists a tree T and an integer $d_{\max}$ such that there is an edge (u, v) in E if and only if for their corresponding leaves l_u, l_v in T we have $d_T(l_u, l_v) \leq d_{\max}$. It is worth to note that LPG (leaf power graphs) have been well studied in literature since they were introduced (see e.g., 2, 3, 4, 11 and 14.)

Definition 2.2. A graph $G = (V, E)$ is an mLPG if there exists a tree T and an integer $d_{\min}$ such that there is an edge (u, v) in E if and only if for their corresponding leaves l_u, l_v in T we have $d_T(l_u, l_v) \geq d_{\min}$.

Proposition 2.3. *Let G be a graph that does not belong to some class L from $\{PCG, LPG, mLPG\}$. Then every graph H that contains G as an induced subgraph, does not belong to L either.*

3. Split Matrogenic Graphs

This section is devoted to study the relation between the class of split matrogenic graphs and PCGs. In order to prove that subclasses of split matrogenic graphs belong to the PCG class, we proceed step by step enlarging, at each step, the considered class. Let us start by proving that threshold graphs are both LPG and mLPG graphs.

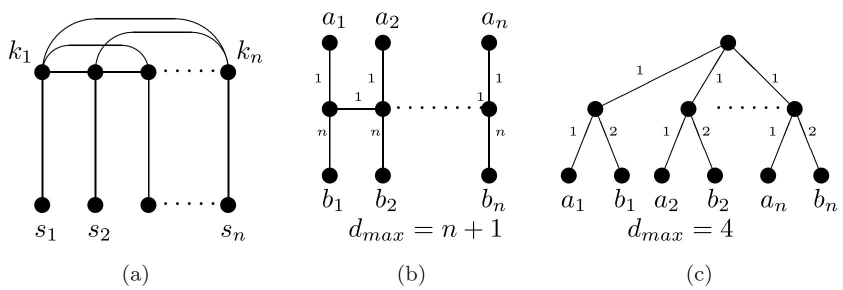

Fig. 3. (a) A split matching graph; (b) a pairwise compatibility caterpillar tree for a split matching graph; (c) a pairwise compatibility tree for a split matching graph.

Theorem 3.1. *Let G be a threshold graph, then $G \in LPG \cap mLPG$. In both of the cases a tree T and a value $d_{\min}$ or $d_{\max}$ associated to G can be found in polynomial time.*

Proof. Let G be a threshold graph on n nodes (see Fig. 2(a)) and let $B_1, \ldots, B_r$ be the degree partition of G. As tree T, we consider an n-leaf star with center at node c.

To prove that $G \in LPG$, for each node v of G, assign weight i to the edge (l_v, c) in T if $v \in B_i$. Define $d_{\max} = r + 1$. As for each $u \in B_i$, $v \in B_j$, $u \neq v$, we have $(u, v) \in E(G)$ if and only if $i + j \leq r + 1$; hence, it follows that $G = LPG(T, d_{\max})$. (See Fig. 2(b)).

On the other hand, to prove $G \in mLPG$ for any $v \in V(G)$ assign $r + 1 - i$ to the edge (l_v, c) in T if $v \in B_i$. Note that, as $i \leq r$ we assign non-negative weights to the edges of the star. Define $d_{\min} = r + 1$. For any two nodes $v \in B_i$ and $u \in B_j$, we have that if $i + j \leq r + 1$ (meaning that $(u, v) \in E(G)$) then $d_T(l_u, l_v) = 2(r+1)-(i+j) \geq r+1 = d_{\min}$. Otherwise, if $i+j > r+1$ (meaning that $(u, v) \notin E(G)$) then $d_T(l_u, l_v) = 2(r+1) - (i+j) < r+1 = d_{\min}$. (See Fig. 2(c)). This concludes the proof. $\square$

Theorem 3.2. *Let G be a split matching graph, then $G \in LPG$. A tree T and a value $d_{\max}$ associated to G can be found in polynomial time.*

Proof. Given a split matching graph $G = (K, S, E)$ with $|K| = |S| = n$ (see Fig. 3(a)), we associate a caterpillar tree T as in Fig. 3(b). The leaves a_i, corresponding to the nodes k_i of K, are connected to the spine with edges of weight 1 and the leaves b_i, corresponding to nodes $s_i \in S$, with edges of weight n. It is clear that $G = LPG(T, n+1)$. Indeed, for any two a_i, a_j it holds that $3 \leq d_T(a_i, a_j) \leq n+1$, for any two b_i, b_j we have $d_T(b_i, b_j) \geq 2n+1$, for any a_i, b_i we have $d_T(a_i, b_i) = n+1$ (hence the edge $(k_i, s_i) \in E$) and for any a_i, b_j with $i \neq j$ we have $d_T(a_i, b_j) \geq n+2$ (hence the edge $(k_i, s_j) \notin E$). $\square$

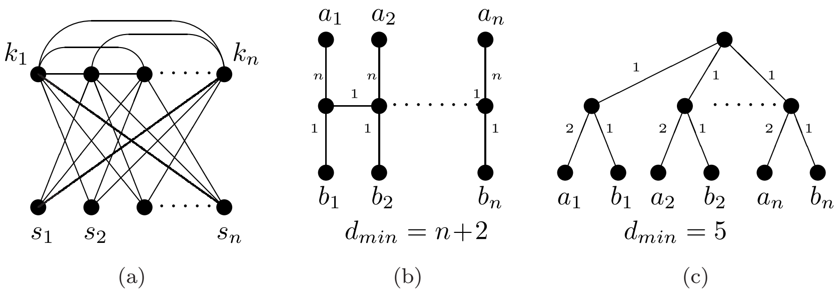

Fig. 4. (a) A split antimatching graph; (b) a pairwise compatibility caterpillar tree for a split antimatching graph; (c) a pairwise compatibility tree for a split antimatching graph.

Note that the pairwise compatibility tree provided for the split matching graph by the previous proof is not unique. Indeed, one can easily check that the binary tree T in Fig. 3(c) also is a pairwise compatibility tree of a split matching graph when $d_{\max} = 4$.

Analogously, we can show that split antimatching graphs are in $mLPG$.

Theorem 3.3. *Let G be a split antimatching graph, then $G \in mLPG$. A tree T and a value $d_{\min}$ associated to G can be found in polynomial time.*

We omit the proof of this theorem, as it uses arguments similar to those in the proof of Theorem 3.2. In Figs. 4(b) and 4(c), two possible pairwise compatibility trees associated to a split antimatching graph (Fig. 4(a)) are depicted.

We now introduce two further subclasses of split matrogenic graphs and prove that they are inside the PCG class.

Definition 3.4. Given a sequence of t split graphs $G_i = (K_i, S_i, E_i)$ with $i = 1, \ldots, t$, we say the graph $H = G_1 \circ \cdots \circ G_t$ is a *split matching (antimatching) sequence* if each of the graphs G_i is either a split matching (antimatching), or a stable graph or a clique graph.

We first prove that split matching sequences and split antimatching sequences are in PCG. In both of these proofs, in the construction of the pairwise compatibility tree, we will make use of the constructions depicted in Figs. 3(c) and 4(c), respectively. Finally, we want to point out that a clique graph (a stable graph) can be considered both as a split matching and as a split antimatching graph and in each case the pairwise compatibility tree is constructed in the same way, where only leaves a_i (respectively b_i) appear. In Fig. 5, a pairwise compatibility tree is given for an n node stable graph G when it is considered as a split matching graph (Fig. 5(a)) or as a split antimatching graph (Fig. 5(b)).

Theorem 3.5. *Let H be a split matching sequence, then $H \in LPG$. A tree T and a value $d_{\max}$ associated to H can be found in polynomial time.*

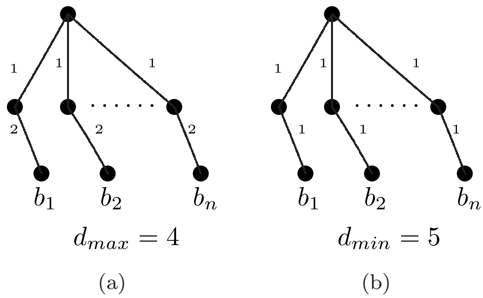

Fig. 5. The pairwise compatibility tree for a stable graph G with n nodes when it is considered as: (a) a split matching graph; (b) a split antimatching graph.

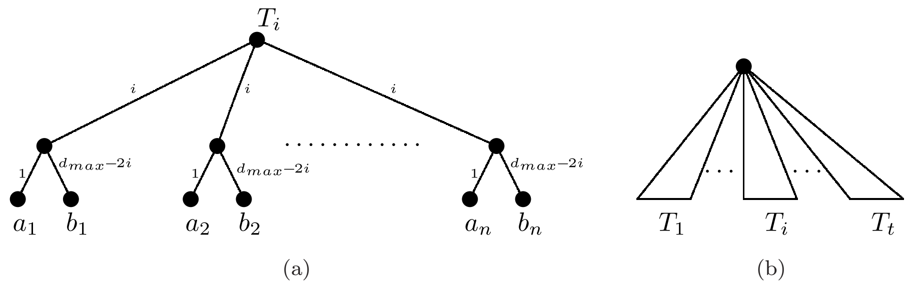

Fig. 6. (a) The pairwise compatibility tree for the split matching graph G_i; (b) the pairwise compatibility tree for the split matching sequence H.

Proof. Let $H = G_1 \circ \cdots \circ G_t$ be a split matching sequence. For each graph G_i we define a tree T_i as shown in Fig. 6(a) (where the leaves a_i (b_i) could be missing if G_i is a stable (clique) graph). It holds that $G_i = LPG(T_i, d_{\max})$ where $d_{\max}$ is a value to be defined later, but surely greater than or equal to $2(i+1)$. Indeed, let $a_1, \ldots, a_n$ be the leaves of T_i corresponding to nodes of K_i and let $b_1, \ldots, b_n$ be those corresponding to nodes of S_i. For any two leaves a_r, a_s it holds that $d_{T_i}(a_r, a_s) = 2 + 2i \leq d_{\max}$ and for any two b_s, b_r we have $d_{T_i}(b_r, b_s) = 2d_{\max} - 2i \geq d_{\max} + 2i + 2 - 2i > d_{\max}$. Finally, for any two leaves a_s, b_s that correspond to an edge of the matching, their distance is $d_{\max} - 2i + 1 \leq d_{\max}$ and for any two leaves corresponding to a non edge a_r, b_s, their distance is $d_{\max} + 1$.

In order to prove that $H \in LPG$, we define a new tree T starting from the trees $T_1, \ldots, T_t$, simply by contracting all their roots into a single node as shown in Fig. 6(b). We claim that $H = LPG(T, d_{\max})$ where we set $d_{\max} = 2(t+1)$. In order to prove it, consider two graphs G_i and G_j with $i < j$. Let a, a', b and b' be four distinct leaves corresponding to nodes in K_i, K_j, S_i and S_j respectively. Observe that the nodes in K_i are connected to all the other nodes in $K_j \cup S_j$ as the distances in T are $d_T(a, a') = 1 + i + j + 1 \leq 2(j+1) \leq d_{\max}$ and $d_T(a, b') = 1 + i + j + d_{\max} - 2j = d_{\max} + (i - j + 1) \leq d_{\max}$ (as $j \geq i+1$). Finally, any node in S_i is not connected to any node K_j and to any node S_j as in these cases, the distances are $d_T(b, a') = d_{\max} - 2i + i + j + 1 > d_{\max}$ (as $j \geq i + 1$) and $d_T(b, b') = d_{\max} - 2i + i + j + d_{\max} - 2j \geq 2d_{\max} - 2j > d_{\max}$. □

Theorem 3.6. *Let H be a split antimatching sequence, then $H \in mPCG$. A tree T and a value $d_{\min}$ associated to H can be found in polynomial time.*

We omit the details of this proof as it follows the same lines of the proof of Theorem 3.5, where the tree T_i associated to each split antimatching graph G_i is depicted in Fig. 7 and $d_{\min} = 2(t + 1) + 1$.

Now, we further enlarge the subclass of split matrogenic graphs that is inside the PCG class.

Theorem 3.7. *Let $H = G_1 \circ \cdots \circ G_t$ be a split matrogenic graph. If for each split matching graph G_i and for each split antimatching graph G_j it holds that $i < j$, then $H \in PCG$. A tree T and two values $d_{\min}, d_{\max}$ associated to H can be found in polynomial time.*

Proof. Let $H = G_1 \circ \cdots \circ G_t$. It is clear that if none of the graphs G_i is a split matching (a split antimatching) the proof trivially follows from Theorem 3.5 (Theorem 3.6). Hence, let G_q, $1 < q \leq t$, be the first occurrence of a split antimatching graph. Then, the graphs $H_1 = G_1 \circ \cdots \circ G_{q-1}$ and $H_2 = G_q \circ \cdots \circ G_t$ are a split matching sequence and a split antimatching sequence, respectively. Then, let $H_1 = LPG(T_1, M)$ where the tree is constructed the same way as in the proof of Theorem 3.5 and $M = 2(t+1)+1$ (recall that in the proof of Theorem 3.5 we only need M to be a value greater than $2q$). Similarly, according to the Theorem 3.6, $H_2 = mLPG(T_2, m)$ and $m = 2(t + 1) + 1$ (note that we choose to have $m = M$). We modify T_2 in such a way that the weights of the edges out-coming from the root start from value q and not from value 1; the other edges are modified accordingly. This is not restrictive, as T_2 results as if H_2 was the composition of t split antimatching graphs whose first $q - 1$ are empty graphs.

We construct the pairwise compatibility tree T by joining the roots of T_1 and T_2 with an edge of weight $m/2$. We set $d_{\min} = m$ and $d_{\max} = 2m$. We modify the weights of the resulting tree increasing by $m/2$ the weight of any edge incident to a

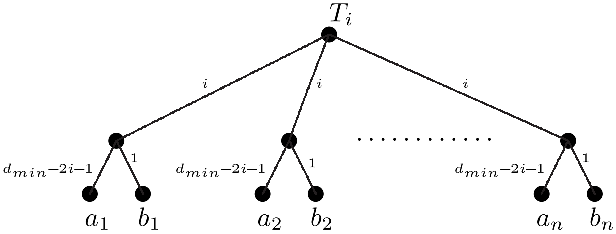

Fig. 7. The pairwise compatibility tree for the split antimatching graph G_i.

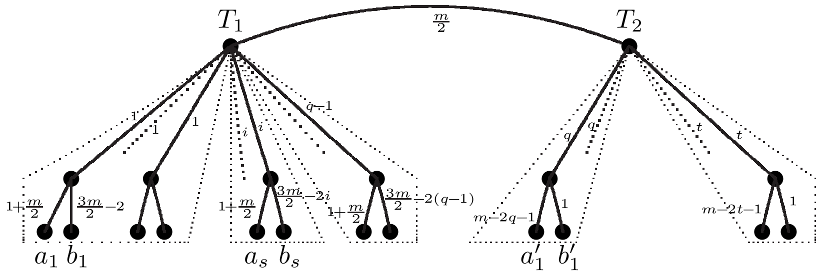

Fig. 8. The pairwise compatibility tree for the split matrogenic graph H as defined in Theorem 3.7.

leaf in T_1. Observe that in this way the distance of any two leaves in T_1 is increased by m. This means that two leaves correspond to nodes of an edge in H_1 if and only if their distance is less than or equal to $M + m = 2m$. Furthermore, the maximum distance of any two leaves in T_2 is less than or equal to $2m - 2t < 2m$ meaning that they correspond to nodes of an edge in H_2 if and only if their distance is greater than or equal to m. In Fig. 8 the pairwise compatibility tree for the split matrogenic graph H is depicted.

We claim that $H = PCG(T, 2m, m)$ (recall that $m = 2(t + 1) + 1$). We have already shown that the pairwise compatibility constraints hold for any two leaves that correspond to two nodes of the same graph H_1 or H_2. It remains to show that this constraint also holds for two leaves where one corresponds to a node in H_1 and the other one to a node in H_2. To this purpose, let a_i and b_i be two distinct leaves in T_1, connected to the root with edges of weight i and corresponding to nodes of the clique and the stable graph of H_1, respectively. Similarly let a'_j, b'_j be two distinct leaves in T_2, connected to the root with edges of weight j and corresponding to nodes in the clique and in the stable graph of H_2, respectively. The following hold:

(a) $d_T(a_i, a'_j) = 2m + i - j$ and as $i < j$ and $m > j$, then $m \leq 2m + 1 + i - j \leq 2m$. Hence, the corresponding nodes of a_i, a'_j in H are connected.
(b) $d_T(a_i, b'_j) = m + 1 + i + j + 1$ and as $m = 2t + 3 \geq i + j + 2$, then $m \leq m + i + j + 2 \leq 2m$. Hence, the corresponding nodes of a_i, b'_j in H are connected.
(c) $d_T(b_i, a'_j) = 2m - i + m - j - 1$ and as $m = 2t + 3 \geq i + j + 2$, then $2m + (m - i - j - 1) > 2m$. Hence, the corresponding nodes of b_i, a'_j in H are not connected.
(d) $d_T(b_i, b'_j) = 2m - i + j + 1$ and as $i < j$, then $2m + (i - j + 1) > 2m$. Hence, the corresponding nodes of b_i, b'_j in H are not connected.

This, concludes the proof. □

The next enlargement step would imply to prove that the composition of a split antimatching sequence followed by a split matching sequence is a PCG. Unfortunately, it does not seem possible to generalize our reasonings to this case, and we are convinced that the order of appearance of a matching or an antimatching sequence in a split matrogenic graph is some how strictly related to the pairwise compatibility property. Hence, we leave as an open problem determining whether split matrogenic graphs belong to the PCG class or not.

4. Pairwise Compatibility Graphs of Stars

In Theorem 3.1, we showed that threshold graphs are pairwise compatibility graphs of trees that are stars. It is natural to wonder how much this particular structure of the tree is connected with the structural properties of threshold graphs. Here we completely describe all the graphs that are PCGs of a star. Namely, we prove that stars are pairwise compatibility trees of a superclass of threshold graphs. To the best of our knowledge, this class of graphs has never been characterized before, so we name it as *nearly three-threshold graphs*.

Before defining this new class of graphs, we will consider another equivalent definition of threshold graphs, that is based on the concept of vicinal preorrder.

Given a graph $G = (V, E)$, let us define the *open and closed neighborhood of* x as $N(x) = \{w : w \in V, w \neq x \text{ and } (w, x) \in E\}$ and $N[x] = N(x) \cup \{x\}$.

In general, if $V' \subset V$, $N_{V'}(x)$ and $N_{V'}[x]$ are the neighborhoods (respectively open and closed) of x restricted to the graph induced by V'.

The *vicinal preorder* $\preceq$ of a graph $G = (V, E)$ on the set of nodes V guarantees that for any two nodes $u, v \in V$, $u \preceq v$ if and only if $N(u) \subseteq N[v]$. The *dual preorder* $\preceq^*$ is defined by: $u \preceq^* v$ if and only if $v \preceq u$.

A graph $G = (V, E)$ is a threshold graph if and only if the vicinal preorder on V is total, i.e., for any pair of nodes $u, v \in V$, either $u \preceq v$ or $v \preceq u$.

Definition 4.1. A graph $G = (V, E)$ is *nearly three-threshold* if it is possible to partition the set of nodes V into three classes V_K, V_{S_1}, V_{S_2} so that:

(a) The subgraph induced by $K \cup S_1$ is a threshold graph.
(b) The subgraph induced by $K \cup S_2$ is a threshold graph.
(c) The subgraph induced by $S_1 \cup S_2$ is a bipartite graph.

Furthermore, the total vicinal preorder related to the graph induced by $K \cup S_2$ is the dual of the total vicinal preorder defined by the graph induced by $K \cup S_1$ (see Fig. 9(a)).

For the subgraph induced by $S_1 \cup S_2$ we cannot deduce such a similar strong property. However, we show that under some particular conditions, even in this case, there must be a strong relationship between the neighborhoods of the nodes in $S_1 \cup S_2$.

In order to prove the next theorem, let us introduce a new definition. Consider a pairwise compatibility graph $G = PCG(T, d_{\min}, d_{\max})$ and let w be the edge-weight function for T. We define a total order $\preceq_w$ on the nodes of G such that for any $u, v \in V(G)$ it holds $v \preceq_w u$ if and only if $w(e_{l_v}) \leq w(e_{l_u})$ where, as usual, l_u, l_v denote the leaves of T corresponding to the nodes u, v and e_{l_u}, e_{l_v} denote the unique edges incident to these leaves in the tree.

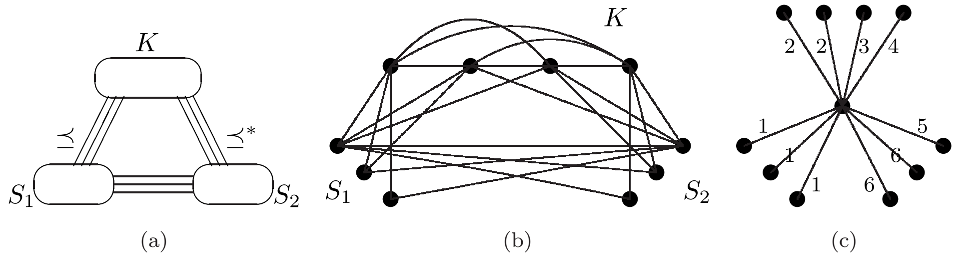

Fig. 9. (a) The structure of a PCG generated by a star; (b) an example of a PCG generated by a star.

We can now prove the following:

Theorem 4.2. *If a graph G is a PCG of a star then G is a nearly three-threshold graph.*

Proof. Let $G = PCG(T, d_{\min}, d_{\max})$ where T is a star centered in some node c and let w be the edge weight function on the tree T.

Define three subsets of the set of nodes of T, V_K, V_{S_1} and V_{S_2} as follows:

$$\begin{aligned} V_K &= \left\{ l_v \in V(T) : \frac{d_{\min}}{2} \leq w((l_v, c)) \leq \frac{d_{\max}}{2} \right\}, \\ V_{S_1} &= \left\{ l_v \in V(T) : w((l_v, c)) < \frac{d_{\min}}{2} \right\}, \\ V_{S_2} &= \left\{ l_v \in V(T) : w((l_v, c)) > \frac{d_{\max}}{2} \right\}. \end{aligned} \tag{4.1}$$

Let K, S_1 and S_2 be the sets of nodes of G whose corresponding leaves in T belong in V_K, V_{S_1} and V_{S_2}, respectively. Since the sum of the weights of two edges whose leaf extremes are in V_K is always between $d_{\min}$ and $d_{\max}$ (in view of the definition of V_K), easily K induces a clique; using similar reasonings, S_1 and S_2 induce two stable sets. From this latter consideration, it follows that the subgraph induced by $S_1 \cup S_2$ is a bipartite graph. So it remains to prove that the subgraphs induced by $K \cup S_1$ and $K \cup S_2$ are threshold graphs.

The main idea of the proof is to show that there is a strong relation between the vicinal preorder defined on $K \cup S_1$ and $K \cup S_2$ and the weights of edges incident to the corresponding leaves of the tree. More in detail, we show that $\preceq_w$ is a total vicinal preorder on $K \cup S_1$ and its dual $\preceq_w^*$ is a total vicinal preorder on $K \cup S_2$.

First we prove that $K \cup S_1$ is a threshold graph. We will show, first, that the vicinal preorder $\preceq$ defined is total on $K \cup S_1$ and then that it coincides with $\preceq_w$. To this purpose, consider any two arbitrary nodes $v, u \in K \cup S_1$ and let $w((l_v, c)) \leq w((l_u, c))$ (thus, $v \preceq_w u$). We prove that $v \preceq u$. Indeed, for any other node $x \in N_{K \cup S_1}(v)$, it must hold that $d_{\min} \leq w((l_x, c)) + w((l_v, c)) \leq d_{\max}$. Now, it is clear that $w((l_x, c)) + w((l_u, c)) \geq w((l_x, c)) + w((l_v, c)) \geq d_{\min}$. Furthermore, as $w((l_x, c))$

and $w((l_u, c))$ are both less than or equal to $d_{\max}/2$ their sum is less than or equal to $d_{\max}$. Thus, we have that $x \in N_{K\cup S_1}(u)$. Hence, $N_{K\cup S_1}(v) - \{u\} \subseteq N_{K\cup S_1}(u) - \{v\}$ meaning that the vicinal preorder $\preceq$ is total.

For the subgraph induced by $K \cup S_2$ we use similar arguments. We prove that $K \cup S_2$ is also a threshold graph by showing that the vicinal preorder $\preceq'$ defined is total on $K \cup S_1$ and moreover, it coincides with $\preceq_w^*$. To this purpose, consider two nodes $v, u \in K \cup S_2$ and suppose again that $w((l_v, c)) \leq w((l_u, c))$ (thus $v \preceq_w u$). We prove that $u \preceq' v$, i.e., $\preceq'$ coincides with $\preceq_w^*$ and $N_{K\cup S_2}(u) - \{v\} \subseteq N_{K\cup S_2}(v) - \{u\}$. For any other node $x \in N_{K\cup S_2}(u)$, it must hold that $d_{\min} \leq w((l_x, c)) + w((l_u, c)) \leq d_{\max}$. It is clear that $w((l_x, c)) + w((l_v, c)) \leq w((l_x, c)) + w((l_u, c)) \leq d_{\max}$. Furthermore as $l_v, l_x \in K \cup S_2$, then $w((l_x, c)) + w((l_v, c)) \geq d_{\max}/2 + d_{\min}/2 \geq d_{\min}$. Thus we have that $x \in N_{K\cup S_1}(v)$. Hence, $N_{K\cup S_2}(u) - \{v\} \subseteq N_{K\cup S_2}(v) - \{u\}$ meaning that the vicinal preorder $\preceq'$ is total. □

In the next claim we show that in some cases, it is possible to reveal more of the structure of the bipartite graph $S_1 \cup S_2$.

Claim 1. *Let G be a graph such that $G = PCG(T, d_{\min}, d_{\max})$ where T is a weighted star and $\frac{d_{\max}}{2} \geq d_{\min}$. Let w be the edge-weight function on T, then $G = (K, S_1, S_2)$ is a nearly three-threshold graph and $\preceq_w^*$ defines a vicinal preorder in the bipartite graph $S_1 \cup S_2$ which is total in the sets S_1 and S_2.*

Proof. Let $G = PCG(T, d_{\min}, d_{\max})$, with $d_{\max}/2 \geq d_{\min}$ and where T is a weighted star centered in some node c and let w be the edge weight function on this star. Notice that Theorem 4.2 holds for any value of $d_{\min}$ and $d_{\max} \geq d_{\min}$, so $G = (K, S_1, S_2)$ is a nearly three-threshold graph. Consider the induced bipartite graph $S_1 \cup S_2$.

We show that $\preceq_w^*$ is a total vicinal preorder on S_1, leaving to the reader the identical proof on S_2. Let us consider two arbitrary nodes $v, u \in V_{S_1}$ with $w((l_v, c)) \leq w((l_u, c))$. We prove that $N_{S_1\cup S_2}(u) \subseteq N_{S_1\cup S_2}(v)$. For any other node $x \in S_2$ such that $x \in N_{S_1\cup S_2}(u)$, we have $d_{\min} \leq w((l_x, c)) + w((l_u, c)) \leq d_{\max}$. Again $w((l_x, c)) + w((l_v, c)) \leq w((l_x, c)) + w((l_u, c)) \leq d_{\max}$. Furthermore as $w((l_x, c)) + w((l_v, c)) \geq d_{\max}/2 + w((l_v, c)) \geq d_{\min}$, we deduce that $x \in N_{K\cup S_1}(v)$ (note that here we used the fact that $d_{\min} \leq d_{\max}/2$). So, $\preceq_w^*$ is a total vicinal preorder on S_1 and the claim is proven. □

5. Conclusions and Open Problems

In this paper we present two different contributions: The first one oriented to increase the number of specific classes of graphs that are PCGs and the other one going toward the direction of characterizing subclasses of PCGs derived from a specific topology of the pairwise compatibility tree. Both these results are related to generalizations of threshold graphs.

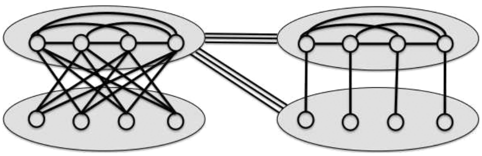

Fig. 10. The smallest split matrogenic graph for which it is still an open problem determining whether it belongs to the PCG class or not. The triple lines between the split antimatching graph and the split matching graph mean the composition operation.

For what concerns the first topic, we have proven that many split matrogenic graphs are in PCG. Nevertheless, there are some split matrogenic graphs for which we cannot say whether they are PCGs or not. In particular, it remains an open problem to understand if it is possible to find a pairwise compatibility tree and two values $d_{\min}$ and $d_{\max}$ for the split matrogenic graph $H = G_1 \circ \cdots \circ G_t$ such that for some split antimatching graph G_i and for some split matching graph G_j it holds that $i < j$. In fact, it seems that the order of appearance of a matching or an antimatching sequence in a split matrogenic graph is somehow strictly related to the pairwise compatibility property, so it would be extremely interesting to even understand whether only the split matrogenic graph in Fig. 10 is a PCG or not.

The second result presented in this paper is on the structure of graphs that are PCGs of a star. We have proven that stars are pairwise compatibility trees of a new class of graphs, the nearly three-threshold graphs, which is a superclass of threshold graphs. A natural open problem consists in completely identifying the class of graphs that are PCG of a star. Moreover, it is clear that we can ask similar questions for other particular trees. For example, we have seen that the simplest split matrogenic graphs (split matching and split antimatching graphs) are PCGs of a particular tree structure: a caterpillar. Thus, it should be interesting to determine the class of PCGs characterized by a caterpillar.

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