General solution of the Inhomogeneous Div-Curl system and Consequences

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We will give a complete solution to the reconstruction of a vector field from its divergence and curl, i.e., the system

$$\begin{aligned} \operatorname{div} \vec{W} &= g_0, \\ \operatorname{curl} \vec{W} &= \vec{g}, \end{aligned} \tag{1}$$

for appropriate assumptions on the scalar field g_0 and the vector field $\vec{g}$ and their domain of definition in three-space.

Let $\mathbb{H}$ the non-commutative algebra of quaternions over the real field $\mathbb{R}$. Let $x = x_0 + \sum_{i=1}^{3} e_i x_i \in \mathbb{H}$, where $x_i \in \mathbb{R}$.
The subspace $\text{Vec}\mathbb{H} := \text{span}_{\mathbb{R}} \{e_1, e_2, e_3\}$ of $\mathbb{H}$ is identified with the Euclidean space $\mathbb{R}^3$ as follows

$$x_1 e_1 + x_2 e_2 + x_3 e_3 \leftrightarrow (x_1, x_2, x_3) \in \mathbb{R}^3.$$

Let $\Omega \subset \mathbb{R}^3$ be an open subset with smooth boundary. Define the Cauchy-Riemann type differential operator as

$$D = e_1 \frac{\partial}{\partial x_1} + e_2 \frac{\partial}{\partial x_2} + e_3 \frac{\partial}{\partial x_3}.$$

It acts over differentiable functions $w : \Omega \to \mathbb{H}$ of the form $w(x) = w_0(x) + \sum_{i=1}^{3} e_i w_i(x)$, where $w_i : \Omega \to \mathbb{R}$, $i = 0, 1, 2, 3$.

The following result tell us that is impossible to define an $\mathbb{H}-$holomorphic functions via the existence of the limit of the difference quotient, like in the complex case:

Theorem

Let $w \in C^1(\Omega)$ be a function defined in a domain $\Omega \subset \mathbb{H}$. If for all points in Ω the limit

$$\lim_{h \to 0} h^{-1}[w(x+h) - w(x)],$$

exists, then in Ω the function w has the form

$$w(x) = a + xb \quad a, b \in \mathbb{H}.$$

Definition

A C^1 function $w : \Omega \to \mathbb{H}$ is called left-monogenic (resp. right-monogenic) in Ω if

$$Dw = 0 \quad en \quad \Omega \quad (wD = 0 \quad in \quad \Omega).$$

We will say simply "monogenic" to refer to left-monogenic functions. Even more, since $Dw = -\text{div}\,\overrightarrow{w} + \text{curl}\,\overrightarrow{w} + \nabla w_0$, then

$$Dw = 0 \Longleftrightarrow \begin{cases} \text{div}\,\overrightarrow{w} & = 0, \\ \text{curl}\,\overrightarrow{w} & = -\nabla w_0. \end{cases}$$

Examples:

- $$E(x) := \frac{1}{4\pi}\frac{\bar{x}}{|x|^3}, \ x \neq 0,$$

  is an example of a left- and right-monogenic function, called Cauchy kernel.

- $$w(x) = -x_3 + x_1 e_2 - x_3 e_3,$$

  is left-monogenic but no right-monogenic, since $Dw = 0$ y $wD = -2e_3$.

Let us denote $\overrightarrow{\mathfrak{M}}(\Omega) = \mathrm{Sol}\,(\Omega, \mathbb{R}^3) \cap \mathrm{Irr}\,(\Omega, \mathbb{R}^3)$, where

$$\mathrm{Sol}\,(\Omega, \mathbb{R}^3) = \{\vec{w} \colon \ \mathrm{div}\,\vec{w} = 0 \text{ in } \Omega\} \subseteq C^1(\Omega, \mathbb{R}^3),$$
$$\mathrm{Irr}\,(\Omega, \mathbb{R}^3) = \{\vec{w} \colon \ \mathrm{curl}\,\vec{w} = 0 \text{ in } \Omega\} \subseteq C^1(\Omega, \mathbb{R}^3),$$

the Solenoidal e Irrotational vector fields, respectively. The elements $\vec{w} \in \overrightarrow{\mathfrak{M}}(\Omega)$ are locally the gradient of real valued harmonic functions. We write
$\mathrm{Har}\,(\Omega, A) = \{w \colon \Omega \to A, \ \Delta w = 0\}$, where $A = \mathbb{R}$, $\mathbb{R}^3$ or $\mathbb{H}$, for the corresponding sets of harmonic functions defined in A. Since $\Delta = -D^2$, then left and right monogenic functions are harmonic.

The *monogenic completion operator*

$$\vec{S}_\Omega \colon \mathrm{Har}\,(\Omega, \mathbb{R}) \to \mathrm{Har}\,(\Omega, \mathbb{R}^3)$$

is given by

$$\begin{aligned}\vec{S}_\Omega[w_0](x) &= \mathrm{Vec}\,\left(\int_0^1 -tDw_0(tx)x\,dt\right)\\ &= \int_0^1 -tDw_0(tx)\times x\,dt, \quad x\in\Omega.\end{aligned}$$

for harmonic functions w_0 defined in star-shaped open sets Ω with respect to the origin.

Let $\Omega \subset \mathbb{R}^3$ be an open set with smooth boundary and $w \in C^0(\Omega, \mathbb{H})$. The operator T_Ω defined by

$$T_\Omega[w](x) := -\int_\Omega \frac{\overline{y-x}}{4\pi|y-x|^3} w(y)dy, \quad x \in \Omega,$$

is called the Teodorescu transform, which acts like a right inverse of D. More even, if $w \in L^1(\Omega, \mathbb{H})$, then

$$DT_\Omega[w] = w,$$

weakly. Let $w \in C^1(\overline{\Omega}, \mathbb{H})$. The Cauchy-Bitsadze operator is defined as

$$F_{\partial\Omega}[w](x) := \int_{\partial\Omega} E(y-x)dy^* w(y), \quad x \in \mathbb{R}^3 \setminus \partial\Omega,$$

And $DF_{\partial\Omega}[w] = 0$.

The operator T_Ω can be descomposed in the following way

$$T_{0,\Omega}[\vec{w}](x) = \int_\Omega E(y-x)\cdot\vec{w}(y)dy,$$

$$\overrightarrow{T}_{1,\Omega}[w_0](x) = -\int_\Omega w_0(y)E(y-x)dy,$$

$$\overrightarrow{T}_{2,\Omega}[\vec{w}](x) = -\int_\Omega E(y-x)\times\vec{w}(y)dy,$$

where $\cdot$ denotes the scalar (or inner) product of vectors and $\times$ denotes the cross product. That is,

$$T_\Omega[w_0+\vec{w}] = T_{0,\Omega}[\vec{w}] + \overrightarrow{T}_{1,\Omega}[w_0] + \overrightarrow{T}_{2,\Omega}[\vec{w}].$$

The following fact is essential in the construction of the right inverse of curl:

$$T_{0,\Omega}[\vec{w}] \in \mathrm{Har}\,(\Omega, \mathbb{R}) \Leftrightarrow \vec{w} \in \mathrm{Sol}\,(\Omega, \mathbb{R}^3).$$

Theorem

Let $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set. The operator

$$\overrightarrow{T}_{2,\Omega} - \vec{S}_\Omega T_{0,\Omega} \tag{2}$$

is a right inverse for the curl on the class of functions $Sol\,(\Omega, \mathbb{R}^3)$.

Where $\vec{S}_\Omega$ is the *monogenic completion operator*. Furthermore, $(\overrightarrow{T}_{2,\Omega} - \vec{S}_\Omega T_{0,\Omega}) \colon \mathrm{Sol}\,(\Omega, \mathbb{R}^3) \to \mathrm{Sol}\,(\Omega, \mathbb{R}^3)$.

Then we can solve the homogeneous div-curl system under the assumptions $\vec{g} \in \text{Sol}\,(\Omega, \mathbb{R}^3)$:

$$\begin{aligned} \text{div}\,\vec{W} &= 0, \\ \text{curl}\,\vec{W} &= \vec{g}, \end{aligned} \tag{3}$$

And the solution is given by

$$\vec{W} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \vec{S}_{\Omega}\,T_{0,\Omega}[\vec{g}] + \nabla h,$$

where $h \in \text{Har}\,(\Omega, \mathbb{R})$ is arbitrary.

Applying a correction, and with $g_0 \in C^0(\Omega, \mathbb{R})$, and $\vec{g} \in \text{Sol}\,(\Omega, \mathbb{R}^3)$. Then a general solution to the inhomogeneous Div-Curl system

$$\begin{aligned} \text{div}\, \vec{W} &= g_0, \\ \text{curl}\, \vec{W} &= \vec{g}, \end{aligned} \tag{4}$$

is given by

$$\vec{W} = -\overrightarrow{T}_{1,\Omega}[g_0] + \overrightarrow{T}_{2,\Omega}[\vec{g}] - \vec{S}_{\Omega}\, T_{0,\Omega}[\vec{g}] + \nabla h, \tag{5}$$

where $h \in \text{Har}\,(\Omega, \mathbb{R})$ is arbitrary.

More even, $\vec{W}$ is a weak solution of the div-curl system (4) when $g_0 \in L^2(\Omega, \mathbb{R})$, $\vec{g} \in L^2(\Omega, \mathbb{R}^3)$ and $\text{div}\, \vec{g} = 0$ weakly.

Let $W : \Omega \to \mathbb{H}$, $f : \Omega \to \mathbb{R}$ a non-vanishing function. Define the **main Vekua equation** by

$$DW = \frac{Df}{f}\overline{W}.$$

The operator $D - \frac{Df}{f}C_H$ corresponding to this equation appears in different factorizations, for example when u is scalar

$$\begin{aligned} \bigtriangledown \cdot \sigma \bigtriangledown u &= \sigma^{1/2}\left(\Delta - \frac{\Delta\sigma^{1/2}}{\sigma^{1/2}}\right)\sigma^{1/2}u \\ &= -\sigma^{1/2}(D + M^{\frac{Df}{f}})\left(D - \frac{Df}{f}C_H\right)\sigma^{1/2}u, \end{aligned}$$

where $\sigma = f^2$, C_H is the quaternion conjugate operator and $M^{\frac{Df}{f}}$ is the right multiplication operator.

We said that $W = W_0 + \overrightarrow{W} = W_0 + \sum_{i=1}^{3} e_i W_i$ satisfies the main Vekua equation if and only if the scalar part W_0 and the vector part $\overrightarrow{W}$ satisfy

$$\operatorname{div}(f\overrightarrow{W}) = 0,$$
$$\operatorname{curl}(f\overrightarrow{W}) = -f^2\nabla\left(\frac{W_0}{f}\right).$$

In other words,

$$D(f\overrightarrow{W}) = -f^2\nabla\left(\frac{W_0}{f}\right).$$

And we have a div-curl system, we will give an explicit solution to solve them.

Let W such that $DW = \frac{Df}{f}\overline{W}$. Then

- $$\nabla \cdot f^2 \nabla \frac{W_0}{f} = 0.$$
- W_0 is a solution of the stationary Schrödinger equation
$$-\Delta W_0 + r_0 W_0 = 0, \text{ with } r_0 = \frac{\Delta f}{f}.$$
- $$\text{rot}\left(f^{-2}\text{rot}\,(f\vec{W})\right) = 0.$$
- $u = \frac{W_0}{f} + f\overrightarrow{W}$ is a solution of the $\mathbb{R}$-linear Beltrami equation
$$Du = \frac{1-f^2}{1+f^2}D\overline{u}.$$

In the book *Applied pseudoanalytic function theory* from the author Vladislav V.

In the complex case, $\Omega \subset \mathbb{C}$, $W = W_0 + iW_1 : \Omega \to \mathbb{C}$ $f : \Omega \to \mathbb{R}$, the main Vekua equation is given by

$$\overline{\partial} W = \frac{\partial f}{f} \overline{W},$$

where $\overline{\partial} = \frac{1}{2}(\partial_x + i\partial_y)$.
And the imaginary part W_1 also satisfies a conductivity and a Schrödinger equation:

$$\bigtriangledown \cdot f^{-2} \bigtriangledown (fW_1) = 0,$$
$$-\Delta W_1 + r_1 W_1 = 0,$$

where $r_1 = \Delta \left(\frac{1}{f}\right) f$. And the corresponding *conjugate* Beltrami equation is

$$\overline{\partial} W = \frac{1 - f^2}{1 + f^2} \overline{\partial W}.$$

If the scalar part $W_0 \colon \Omega \to \mathbb{R}$ is known, how to costruct f^2-hyperconjugate pairs?

Theorem

Let f^2 be a conductivity of class C^2 in an open star-shaped set $\Omega \subseteq \mathbb{R}^3$. Suppose that $W_0 \in C^2(\Omega, \mathbb{R})$ satisfies the conductivity equation $\nabla \cdot f^2 \nabla (W_0/f) = 0$ in Ω. Then exists a function $\vec{W}$ such that $W_0 + \vec{W}$ such that

$$DW = \frac{Df}{f}\overline{W}.$$

The function $f\vec{W}$ is unique up to the gradient of a real harmonic function.

The assumptions of the above Theorem can be relaxed to say that $f, W_0 \in H^1(\Omega, \mathbb{R})$ and satisfy the conductivity equation weakly.

The following system of equations corresponds to the **static Maxwell system**, in a medium when just the permeability f^2 is variable:

$$\begin{aligned} \operatorname{div}(f^2\vec{H}) &= 0, \\ \operatorname{div}\vec{E} &= 0, \\ \operatorname{curl}\vec{H} &= \vec{g}, \\ \operatorname{curl}\vec{E} &= f^2\vec{H}. \end{aligned} \tag{6}$$

Here $\vec{E}$ and $\vec{H}$ represent electric and magnetic fields, respectively. We will apply our results to this system and to the double curl equation

$$\operatorname{curl}(f^{-2}\operatorname{curl}\vec{E}) = \vec{g}, \tag{7}$$

which is immediate from the last two equations of (6).

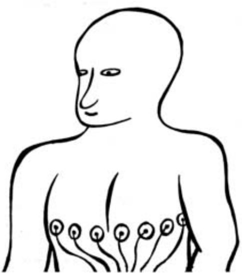
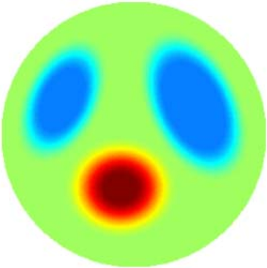
Theorem

Let the domain $\Omega \subseteq \mathbb{R}^3$ be a star-shaped open set, and assume that f^2 is a continuous proper conductivity in Ω. Let $\vec{g} \in L^2(\Omega, \mathbb{R}^3)$ satisfy $div\, \vec{g} = 0$. Then there exists a generalized solution $(\vec{E}, \vec{H})$ to the system (6) and its general form is given by

$$\vec{E} = \overrightarrow{T}_{2,\Omega}[f^2(\vec{B} + \nabla h]) - \vec{S}_\Omega[T_{0,\Omega}[f^2(\vec{B} + \nabla h]] + \nabla h_1,$$
$$\vec{H} = \vec{B} + \nabla h, \tag{8}$$

where h_1 is an arbitrary real valued harmonic function, h satisfy the inhomogeneous conductivity equation $div(f^2 \nabla h) = -\nabla f^2 \cdot \vec{B}$ and $\vec{B} = \overrightarrow{T}_{2,\Omega}[\vec{g}] - \vec{S}_\Omega[T_{0,\Omega}[\vec{g}]]$.

In impedance tomography one aims to determine the internal structure of a body from electrical measurements on its surface. Such methods have a variety of different applications for instance in engineering and medical diagnostics.

Sobolev spaces

In particular, we are interested in the Sobolev spaces $H^1(\Omega)$ and $H^{1/2}(\partial\Omega)$, where

$$H^1(\Omega) = \left\{u \in L^2(\Omega) : \bigtriangledown u \in L^2(\Omega)\right\},$$
$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2.$$

While $H^{1/2}(\partial\Omega)$ are the "boundary value" functions in $H^1(\Omega)$:

$$H^{1/2}(\partial\Omega) = \left\{u \in L^2(\partial\Omega) \mid \exists \overline{u} \in H^1(\Omega) \text{ con } \overline{u}|_{\partial\Omega} = u\right\},$$
$$\|u\|_{H^{1/2}} = \inf\left\{\|\overline{u}\|_{H^1} \mid \overline{u}|_{\partial\Omega} = u\right\}.$$

In 1980 A. Calderón showed that the impedance tomography problem admits a clear ans precise mathematical formulation:

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with connected complement and $\sigma : \Omega \to (0, \infty)$ is measurable, with σ and $1/\sigma$ bounded. Given the boundary values $\phi \in H^{1/2}(\partial\Omega)$ exists a unique solution $u \in H^1(\Omega)$ to

$$\bigtriangledown \cdot \sigma \bigtriangledown u = 0 \text{ in } \Omega,$$
$$u|_{\partial\Omega} = \phi.$$

This so-called conductivity equation describes the behavior of the electric potential in a conductive body.

Theorem

(Plemelj-Sokhotski formula)
Let f be Hölder continuous on a sufficiently smooth surface $\partial\Omega$. Then at any regular point $t \in \partial\Omega$ we have

$$n.t.\text{-}\lim_{x\to t} F_{\partial\Omega}[w](x) = \frac{1}{2}[\pm w(t) + S_{\partial\Omega}[w](t)],$$

where $x \in \Omega^{\pm}$, with $\Omega^{+} = \Omega$ and $\Omega^{-} = \mathbb{R}^3 \setminus \overline{\Omega}$. The notation $n.t.\text{-}\lim_{x\to t}$ means that the limit should be taken non-tangential.

Holomorphic functions in the Plane and n-dimensional Space from the authors Klaus Gürlebeck, Klaus Habetha and Wolfgang Sprößig, 2008.

Define the following principal value integral obtained from the Cauchy-Bitsadze integral

$$S_{\partial\Omega}[w](x) := 2\mathrm{PV}\int_{\partial\Omega} E(y-x)dy^* w(y), \quad x \in \partial\Omega,$$

From now on, $\Omega = B^3 = \left\{x \in \mathbb{R}^3 : |x| < 1\right\}$ and $\partial\Omega = S^2 = \left\{x \in \mathbb{R}^3 : |x| = 1\right\}$. Since the unitary normal vector to $\partial\Omega$ is $\eta(y) = y$, then the Cauchy kernel multiplied by the normal vector is reduced to

$$\frac{\overline{y-x}}{4\pi|y-x|^3}y = \frac{1}{4\pi}\left(\frac{1}{2|y-x|} + \frac{x \times y}{|y-x|^3}\right).$$

The following operators appear in the decomposition of the singular integral $S_{\partial\Omega}$

$$M[w](x) := \frac{1}{4\pi}\int_{\partial\Omega}\frac{w(y)}{|y-x|}ds_y,$$

$$M^1[w](x) := \text{PV}\ \frac{1}{2\pi}\int_{\partial\Omega}\frac{x_2y_3 - x_3y_2}{|y-x|^3}w(y)ds_y,$$

$$M^2[w](x) := \text{PV}\ \frac{1}{2\pi}\int_{\partial\Omega}\frac{x_3y_1 - x_1y_3}{|y-x|^3}w(y)ds_y,$$

$$M^3[w](x) := \text{PV}\ \frac{1}{2\pi}\int_{\partial\Omega}\frac{x_1y_2 - x_2y_1}{|y-x|^3}w(y)ds_y.$$

$$S_{\partial\Omega} = M + \sum_{i=1}^{3} e_i M^i.$$

According to the article [3], we define the *Hilbert transform* $\mathcal{H}$ of u as

$$\mathcal{H} : L^2(\partial\Omega, \mathbb{R}) \to L^2(\partial\Omega, \mathbb{R}^3)$$

$$\mathcal{H}(u) := \left(\sum_{i=1}^{3} e_i M^i \right) (I + M)^{-1} u.$$

The authors of the article noticed that if we define the following strategic $\mathbb{R}$-valued function

$$h := 2(I + M)^{-1} u = 2 \left(I + \mathrm{Sc} \left(S_{\partial\Omega} \right) \right)^{-1} u.$$

[3] *Hilbert Transforms on the Sphere with the Clifford Algebra Setting*, 2009, from the authors Tao Qian and Yan Yang.

Thus

$$\mathcal{H}(u) := \frac{1}{2}\text{Vec}\,(S_{\partial\Omega}h).$$

And the monogenic extension in Ω is given by the Cauchy operator $F_{\partial\Omega}(h)$. On the other hand, if we take the non-tangential limit then

$$\begin{aligned}
\text{n.t.}\ \lim_{x\to t} F_{\partial\Omega}(h)(x) &= \frac{1}{2}\left(h(t) + S_{\partial\Omega}h(t)\right) \\
&= \frac{1}{2}h(t) + \frac{1}{2}\text{Sc}(S_{\partial\Omega}h(t)) + \frac{1}{2}\text{Vec}(S_{\partial\Omega}h(t)) \\
&= \frac{1}{2}(I + M)h(t) + \mathcal{H}(u)(t) \\
&= \left(u + \mathcal{H}(u)\right)(t).
\end{aligned}$$

Now, we are interested in to define the Hilbert transform associated to the main Vekua equation $DW = \frac{Df}{f}\overline{W}$:
From now on the conductivity $\sigma = f^2 \in H^1(\Omega, \mathbb{R})$. Suppose that $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R})$ is known, and $\sigma, \frac{1}{\sigma} : \Omega \to (0, \infty)$ are measurables and bounded. Then there exists an unique extension $W_0 \in H^1(\Omega)$ such that

$$\nabla \cdot f^2 \nabla \left(\frac{W_0}{f} \right) = 0 \text{ in } \Omega,$$

$$W_0\Big|_{\partial\Omega} = \varphi \text{ in } \partial\Omega.$$

Using the decomposition of the Teodorescu operator $T_\Omega \colon L^2(\Omega) \to H^1(\Omega)$, we have that

$$T_\Omega\left[-f^2\nabla(W_0/f)\right] = T_{0,\Omega}\left[-f^2\nabla(W_0/f)\right] + \overrightarrow{T}_{2,\Omega}\left[-f^2\nabla(W_0/f)\right].$$

And the corresponding boundary value function of the scalar and the vectorial part of T_Ω are denoted as

$$u_0(t) = \lim_{x\to t} T_{0,\Omega}\left[-f^2\nabla(W_0/f)\right],$$
$$\overrightarrow{u}(t) = \lim_{x\to t} \overrightarrow{T}_{2,\Omega}\left[-f^2\nabla(W_0/f)\right]$$

for $t \in \partial\Omega$.
Notice that $g \in H^{1/2}(\partial\Omega)$, since the trace operator $\gamma \colon H^1(\Omega) \to H^{1/2}(\partial\Omega)$.

Definition

Finally, define the *Hilbert transform* $\mathcal{H}_f$ *associated to the main Vekua equation* as

$$\mathcal{H}_f : H^{1/2}(\partial\Omega, \mathbb{R}) \to L^2(\partial\Omega, \mathbb{R}^3)$$
$$\varphi \to \overrightarrow{u} - \mathcal{H}(u_0),$$

where u_0 and $\vec{u}$ are built as above using the extension for φ as solution of the conductivity equation.

Analogously, like in the monogenic case

$$h_f := 2(I + M)^{-1} u_0.$$

Then

$$\mathcal{H}_f(\varphi) = \overrightarrow{u} - \frac{1}{2}\mathrm{Vec}(S_{\partial\Omega} h_f).$$

- In particular, when $f \equiv 1$ we have that $\mathcal{H}_f = \mathcal{H}$.
- The Hilbert transform associated to the main Vekua equation $\mathcal{H}_f \colon H^{1/2}(\partial\Omega, \mathbb{R}) \to L^2(\partial\Omega, \mathbb{R}^3)$ is a bounded operator.
- More even $\mathcal{H}_f(u) \in \mathrm{Sol}\,(\partial\Omega, \mathbb{R}^3)$, for all $u \in H^{1/2}(\partial\Omega, \mathbb{R})$.

Theorem

Let $\Omega = B^3$. Let $f \in H^1(\Omega, \mathbb{R})$ a proper conductivity. Suppose that $\varphi \in H^{1/2}(\partial\Omega, \mathbb{R})$. Then there exists an extension $W = W_0 + \sum_{i=1}^{3} e_i W_i$ in Ω such that

$$DW = \frac{Df}{f}\overline{W} \quad in\ \Omega,$$
$$W_0\Big|_{\partial\Omega} = \varphi.$$

Sketch of the proof:

The extension of the vector part $\overrightarrow{W}$ is obtained directly from the Hilbert transform $\mathcal{H}_f$ and the integral operators T_Ω and $F_{\partial\Omega}$:

$$\begin{aligned} f\overrightarrow{W} :=& T_\Omega[-f^2\nabla(W_0/f)] + F_{\partial\Omega}[\mathcal{H}_f(\varphi)]. \\ =& T_{0,\Omega}\left[-f^2\nabla(W_0/f)\right] + \overrightarrow{T}_{2,\Omega}\left[-f^2\nabla(W_0/f)\right] \\ &+ F_{\partial\Omega}[\overrightarrow{u} - \mathcal{H}(u_0)]. \end{aligned}$$

Thus

- $W := W_0 + \overrightarrow{W}$ satisfy the main Vekua equation.
- $f\overrightarrow{W}$ is purely vectorial.

Although the above extension is not unique, it is the only one that satisfies

$$\text{n.t.} - \lim_{x \to t} f\overrightarrow{W}(x) = \mathcal{H}_f(\varphi), \quad , x \in \partial\Omega.$$

MERCI!