Faithful Model Reduction of Discrete Biological Systems

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Motivation

Case Study I

Birth and death model

Coarse-graining the BD model

A Generic Framework

Concrete Semantics

Abstract Semantics

Chemical Goals

Case Study II

Competition for resources

Perspectives

Conclusion

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2 Case Study I
- Birth and death model
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3 A Generic Framework
- Concrete Semantics
- Abstract Semantics
- Chemical Goals

4 Case Study II
- Competition for resources

5 Perspectives
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- Acknowledgements

Discrete Models in Biology

- Our research focuses on understanding the foundation of modeling paradigms.

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- In the thesis, we focus on assessing the rigorousness of logical models, such as Boolean Networks.
  - Signalling pathways in breast cancer cells (Chifman et al. 2017),
  - Cell division process in mammalian (Faure et al. 2006).

Discrete Models in Biology

- Our research focuses on understanding the foundation of modeling paradigms.
- In the thesis, we focus on assessing the rigorousness of logical models, such as Boolean Networks.
  - Signalling pathways in breast cancer cells (Chifman et al. 2017),
  - Cell division process in mammalian (Faure et al. 2006).
- To do so, we invoke Abstract Interpretation.
  - A framework successful in approximating mathematical structures at varying levels of granularity (see P. Cousot and R. Cousot 1977).

Related works

Two major lines of work

- Logical models: interval sampling the quantities of chemical components
  - Choices of updating strategies (synchronous, asynchronous, parallel, etc) (Paulevé and Sené 2022)
  - Yet, intermediate transitions may also be considered (Chatain, Haar, and Paulevé 2018)

Related works

Two major lines of work

- Logical models: interval sampling the quantities of chemical components
  - Choices of updating strategies (synchronous, asynchronous, parallel, etc) (Paulevé and Sené 2022)
  - Yet, intermediate transitions may also be considered (Chatain, Haar, and Paulevé 2018)
- Synthesis of discrete transition systems from reaction network dynamics
  - Either by piecewise affine systems (Dejong et al. 2004)
  - Sound abstractions on reaction dynamics (Fages and Soliman 2008)
  - Can be made more precise by exploiting properties of the concrete semantics (Abou-Jaoudé, Thieffry, and Jérôme Feret 2016)

Today's discussion

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1. formally reason on behaviors from (stochastic) reaction networks;
2. abstracting such behaviors by intervals (P. Cousot and R. Cousot 1977);
3. re-introduce probabilities between abstract transitions;
4. evaluate tendencies between refined abstract transitions.
   - Formally evaluate time-scale separation hypotheses by Faure et al. 2006

Case Study I: the birth and death model

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A logical BD model

The BD process is modeled by a reaction couple:

$$r_1 \ : \ \emptyset \xrightarrow{k_A} A \qquad\qquad r_2 \ : \ A \xrightarrow{k_{A'}} \emptyset$$

A logical BD model

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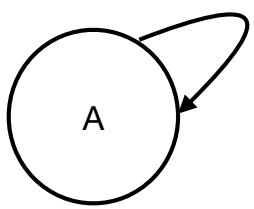
Following the focal point principle of Réné Thomas (Thomas 1973), we obtain

Directed graph	Logical function	Transition system
	$f_{x_v} = \begin{cases} \{0,1,2\} \to \{0,1,2\} \\ x_v \mapsto \begin{cases} x_v - 1 & \text{if } x_v > 1 \\ x_v + 1 & \text{if } x_v < 1 \\ 1 & \text{otherwise.} \end{cases} \end{cases}$	$\begin{array}{ccc} x_A(t) & \to & x_A(t+1) \\ 0 & \to & 1 \\ 1 & \to & 1 \\ 2 & \to & 1 \end{array}$

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A formal BD model

Consider the same reaction network:

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A formal BD model

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$$r_1 \ : \ \emptyset \xrightarrow{k_A} A \qquad\qquad r_2 \ : \ A \xrightarrow{k_{A'}} \emptyset.$$

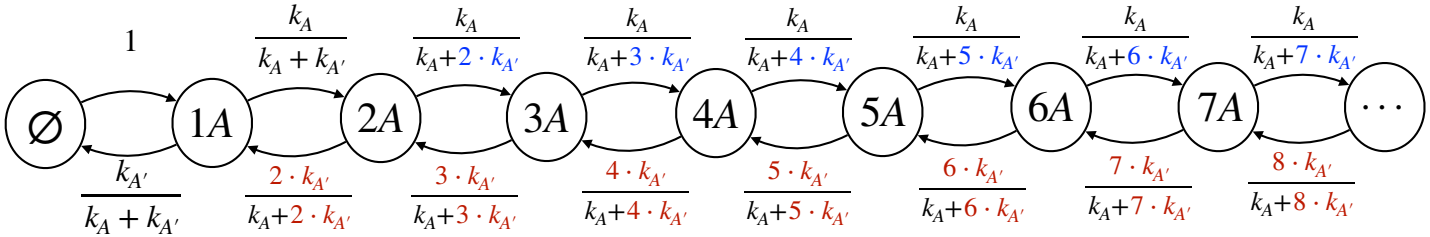
The BD probability functions, μ_r, are derived as an instance of a Discrete-time Markov Chain (DTMC):

$$\mu_{r_1}(q) = \frac{k_A}{k_A + q(A)\cdot k_{A'}} \quad \text{and} \quad \mu_{r_2}(q) = \frac{q(A)\cdot k_{A'}}{k_A + q(A)\cdot k_{A'}}$$

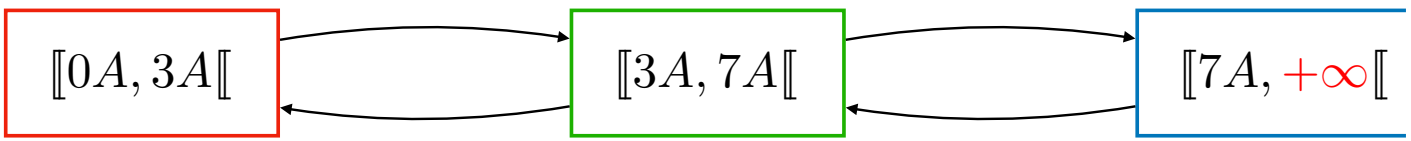
such that q is a one-dimensional vector containing the value of the species A.

Intuition for coarse-graining the BD dynamics

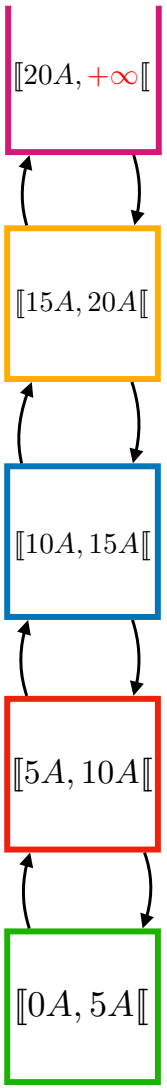
Concrete transition system

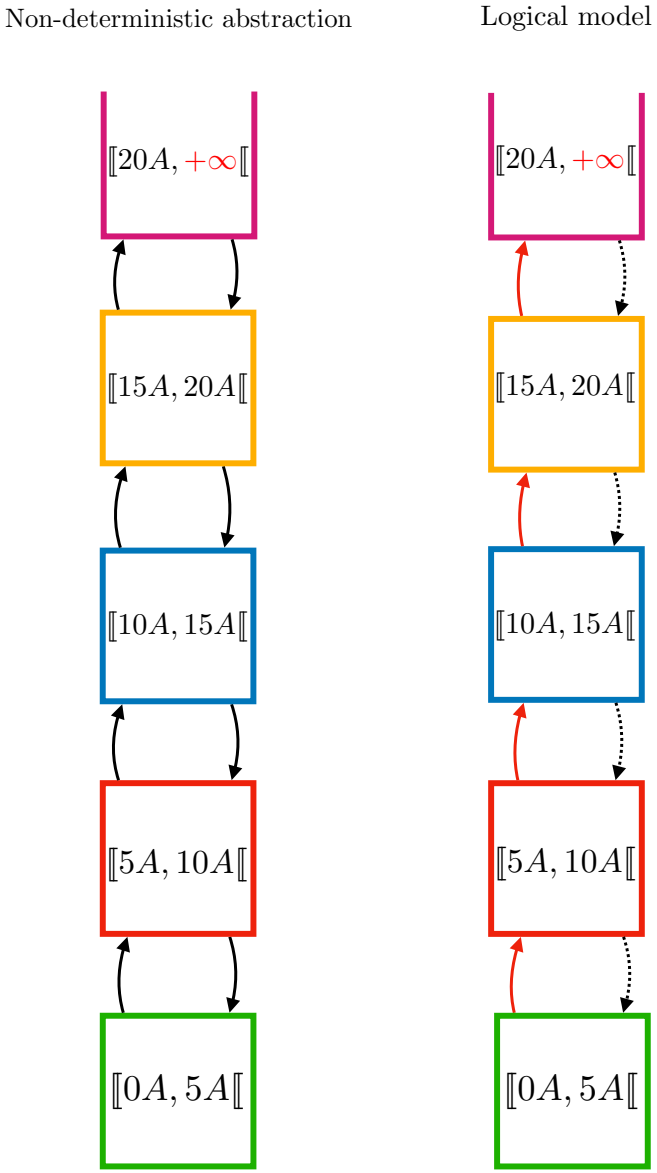


Non-deterministic abstraction with non-overlapping intervals



Coarse-graining BD dynamics with non-overlapping intervals




Non-deterministic abstraction
Logical model
$[\![20A, +\infty[\![$
$[\![15A, 20A[\![$
$[\![10A, 15A[\![$
$[\![5A, 10A[\![$
$[\![0A, 5A[\![$
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Refining a coarse-grained BD model with probabilities

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$\mathcal{Q}$ be the set of states.

$\mathcal{G} \in \wp(\wp(\mathcal{Q}))$

$g \in \mathcal{G}$

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$\mathcal{Q}$ be the set of states.
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For example, for the BD model,
$[A \mapsto 13]$.
$\{\{q \mid q(A) \geq 15\}, \{q \mid q(A) \leq 9\}\}$.
$\{q \mid q(A) \geq 15\}$.

Refining a coarse-grained BD model with probabilities

$\mathcal{Q}$ be the set of states.
$\mathcal{G} \in \wp(\wp(\mathcal{Q}))$
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For example, for the BD model,
$[A \mapsto 13]$.
$\{\{q \mid q(A) \geq 15\}, \{q \mid q(A) \leq 9\}\}$.
$\{q \mid q(A) \geq 15\}$.

Proposition

The probabilities $P_g^{\mathcal{G}}(q)$ for every state $q \in \mathcal{Q}$ are related by the following three conditions:

1. $P_g^{\mathcal{G}}(q) = 1$ *whenever* $q \in g$;
2. $P_g^{\mathcal{G}}(q) = 0$ *whenever* $q \in \bigcup \mathcal{G} \setminus g$;
3. $P_g^{\mathcal{G}}(q) = \sum_{q \xrightarrow{r_i} q'} \mu_{r_i}(q) \cdot P_g^{\mathcal{G}}(q')$ *whenever* $q \notin \bigcup \mathcal{G}$.

Note: probabilities could also be computed by PRISM (Kwiatkowska 2015).

Coarse-grained BD dynamics refined with probabilities

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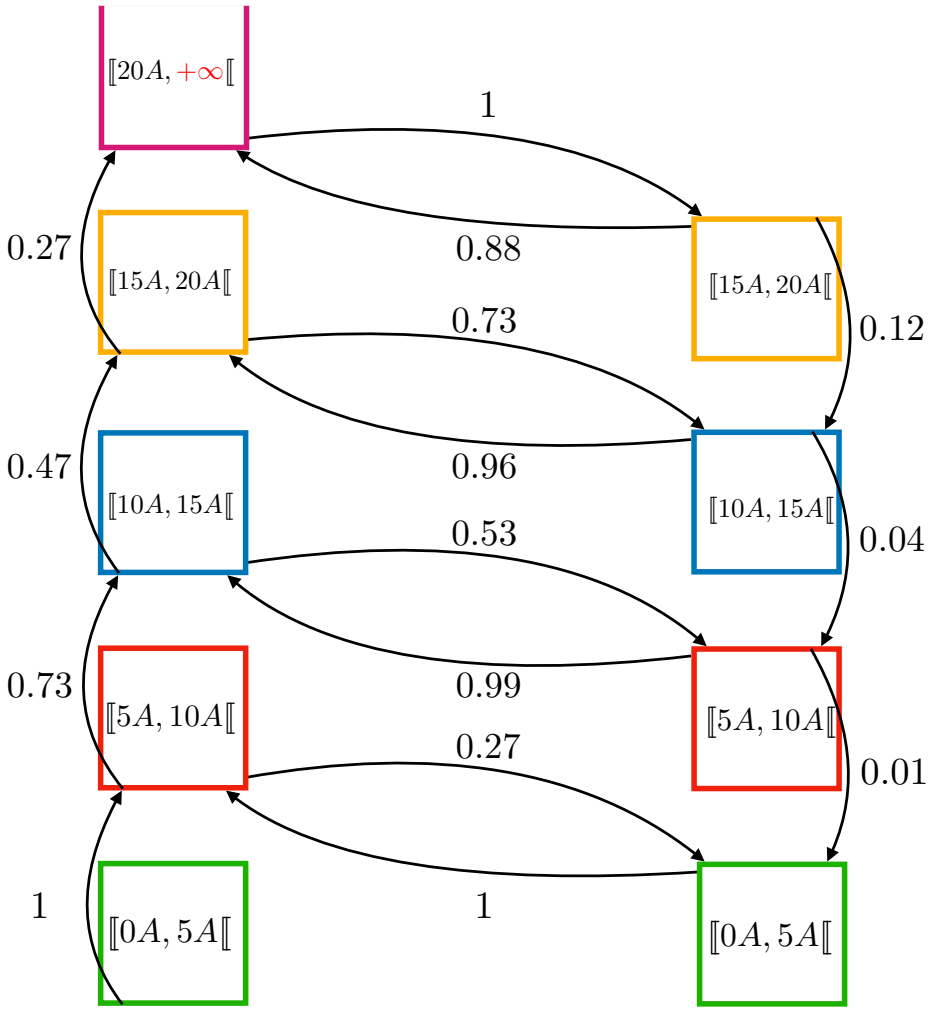

Coarse-graining BD dynamics with a minimal effort system

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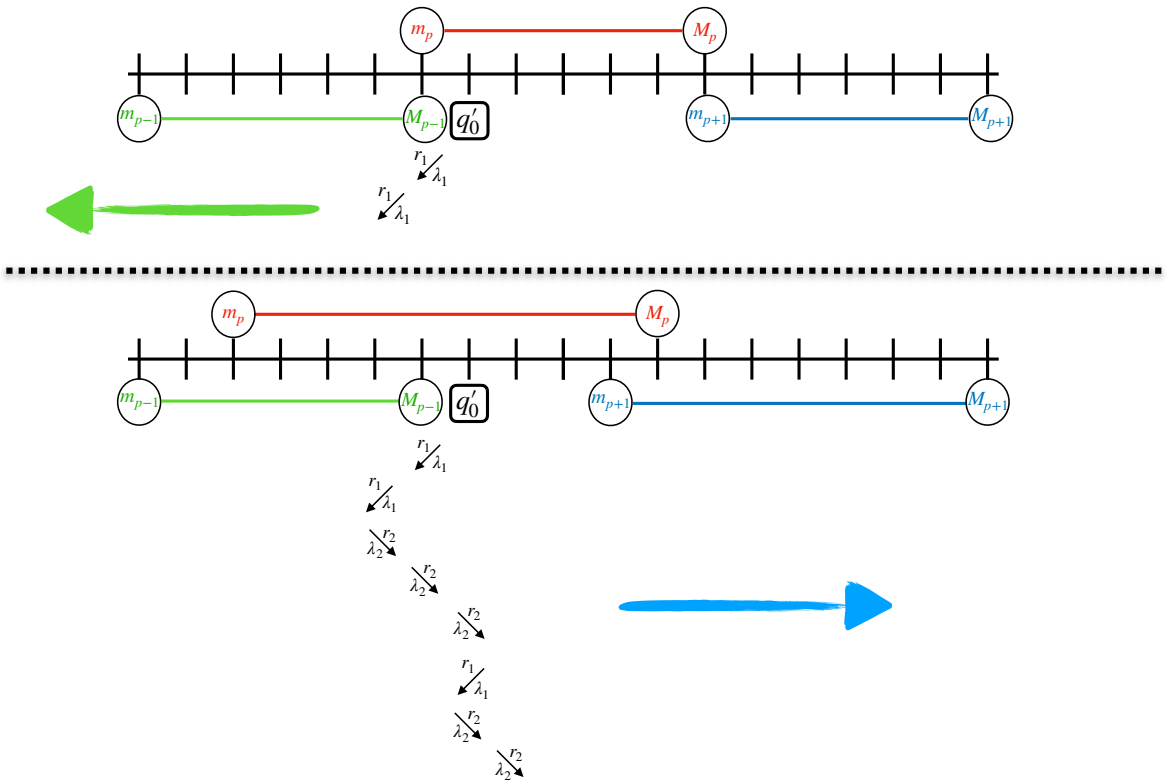
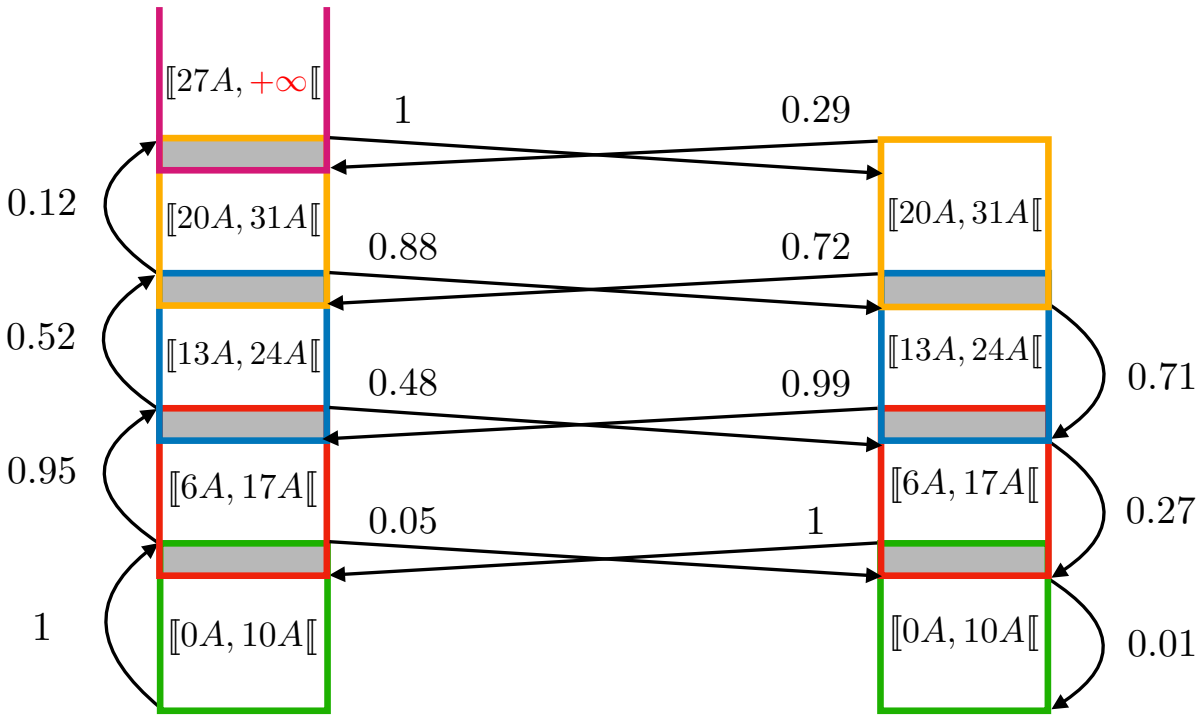


Coarse-graining BD dynamics with overlapping intervals



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A generic framework for coarse-graining (stochastic) reaction networks

We begin to derive the language of our problem: the concrete and abstract semantics.

The concrete semantics as the most-precise abstraction.

A hierarchy of mathematical objects for expressing chemical behaviors

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$$\mathcal{Q} \times (T \times [0,1])^k \qquad \text{(chemical traces)}$$

$$\mathcal{Q} \times \{r_i \mid 1 \leq i \leq n\} \times \mathcal{Q} \qquad \text{(chemical transitions)}$$

$$\mathcal{Q} \qquad \text{(chemical states)}$$

$$\mathcal{V}, \mathbb{N}, \mathcal{R} \qquad \text{(chemical variables, values, reaction networks)}$$

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Concrete semantics: chemical reaction

Definition (Chemical Reaction)

Let $\mathcal{V}$ be a set of chemical species. A reaction over the set of species $\mathcal{V}$ is defined as a triple $r = (M, V, k)$ such that:

1. $M : \mathcal{V} \to \mathbb{N}$,
2. $V : \mathcal{V} \to \mathbb{Z}$,
3. and $k : \mathbb{N}^{\mathcal{V}} \to \mathbb{R}_{\geq 0}$

such that if $\exists v \in \mathcal{V}, q(v) < M(v)$, then $k(q) = 0$.

Concrete semantics: chemical reaction networks

Definition (Chemical reaction network)

A reaction network $\mathcal{R}$ is defined as a pair $(\mathcal{V}, (r_j)_{1 \leq j \leq n})$ such that:

1. $\mathcal{V}$ is the set of chemical species;
2. $(r_i)_{1 \leq i \leq n}$ is a set of n reactions over the set $\mathcal{V}$ indexed with an integer i between 1 and n.

For each integer i between 1 and n, the reaction r_i is also denoted as $(M_{r_i}, V_{r_i}, k_{r_i})$.

$\mathcal{Q} \times (T \times [0,1])^k$ (chemical traces)

$\mathcal{Q} \times \{r_i \mid 1 \leq i \leq n\} \times \mathcal{Q}$ (chemical transitions)

$\mathcal{Q}$ (chemical states)

$\mathcal{V}, \mathbb{N}, \mathcal{R}$ (chemical variables, values, reaction networks)

Concrete semantics: chemical states

Definition (Chemical State)

A chemical state is defined as a function $q : \mathcal{V} \to \mathbb{N}$. The set of all the chemical states for a reaction network is denoted as $\mathcal{Q}$.

$$\mathcal{Q} \times (T \times [0,1])^k \qquad \text{(chemical traces)}$$

$$\mathcal{Q} \times \{r_i \mid 1 \leq i \leq n\} \times \mathcal{Q} \qquad \text{(chemical transitions)}$$

$$\mathcal{Q} \qquad \text{(chemical states)}$$

$$\mathcal{V}, \mathbb{N}, \mathcal{R} \text{ (chemical variables, values, reaction networks)}$$

Concrete semantics: chemical transitions

Definition (Chemical transition)

A chemical transition is a triple (q, r, q') relating two chemical states $q, q' \in \mathcal{Q}$ by a reaction r such that for all chemical variables $v \in \mathcal{V}$:

1. $M_r(v) \leq q(v)$;
2. $q'(v) = q(v) + V_r(v)$.

The set of the all the chemical transitions for a reaction network is denoted as T.

Concrete semantics: probabilistic transition systems

Definition (Transition probability)

Let $(q, r, q') \in T$ be a chemical transition. The probability for a chemical state $q \in \mathcal{Q}$ involved in a chemical transition is defined as:

$$\lambda_r(q) \quad = \quad \frac{k_r(q)}{\sum_{r' \in \{1,\ldots,n\}} k_{r'}(q)}$$

$\mathcal{Q} \times (T \times [0,1])^k$ (chemical traces)

$\mathcal{Q} \times \{r_i \mid 1 \leq i \leq n\} \times \mathcal{Q}$ (chemical transitions)

$\mathcal{Q}$ (chemical states)

$\mathcal{V}, \mathbb{N}, \mathcal{R}$ (chemical variables, values, reaction networks)

Concrete semantics: chemical traces

Definition (Chemical trace)

A trace of length $k \in \mathbb{N}$ is a couple $(q'_0, ((q_i, r_i, q'_i), \mu_i)_{1 \leq i \leq k}) \in \mathcal{Q} \times (T \times [0,1])^k$ that satisfies both conditions:

Concrete semantics: chemical traces

Definition (Chemical trace)

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1. for every integer i between 0 and $k-1$, we have $q'_i = q_{i+1}$;

Concrete semantics: chemical traces

Definition (Chemical trace)

A trace of length $k \in \mathbb{N}$ is a couple $(q'_0, ((q_i, r_i, q'_i), \mu_i)_{1 \leq i \leq k}) \in \mathcal{Q} \times (\mathcal{T} \times [0,1])^k$ that satisfies both conditions:

1. for every integer i between 0 and $k-1$, we have $q'_i = q_{i+1}$;
2. for every integer i between 1 and k, we have $\mu_i = \lambda_{r_i}(q_i)$.

Such a trace is usually written as $q_1 \xrightarrow[\mu_1]{r_1} \dots \xrightarrow[\mu_k]{r_k} q'_k$.

The set of all the chemical traces for a reaction network is denoted as $\mathcal{T}$.

Concrete Semantics: putting it altogether

We focus on a second example reaction network:

$$r_1 : A \xrightarrow{k_B} B \qquad\qquad r_2 : 2A \xrightarrow{k_C} C$$

Concrete Semantics: putting it altogether

We focus on a second example reaction network:

$$r_1 : A \xrightarrow{k_B} B \qquad\qquad r_2 : 2A \xrightarrow{k_C} C$$

After which, we can derive the reaction probabilities:

$$\mu_1(q) = \frac{2 \cdot k_B \cdot q(A)}{q(A) \cdot (2 \cdot k_B + k_C \cdot (q(A) - 1))}$$

$$\mu_2(q) = \frac{k_C \cdot q(A) \cdot (q(A) - 1)}{q(A) \cdot (2 \cdot k_B + k_C \cdot (q(A) - 1))}.$$

Concrete Semantics: putting it altogether

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An example of a sequence of chemical reactions (*a chemical trace*):

$$(6,0,0) \xrightarrow[0.8]{r_1} (5,1,0) \xrightarrow[0.83]{r_1} (4,2,0) \xrightarrow[0.13]{r_2} (2,2,1) \xrightarrow[0.95]{r_1}$$

$$(1,3,1) \xrightarrow[1]{r_1} (0,4,1).$$

Abstract semantics

An abstract semantics using the interval lattice domain: an over-approximation of concrete behaviors.

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(trace abstraction function)}$$

$$\Big|$$

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(transition abstraction function)}$$

$$\Big|$$

$$\overline{\beta}^{\mathcal{Q}}_{q^\sharp} \quad \text{(state abstraction function)}$$

$$\Big|$$

$$\overline{\beta}^{\mathbb{N}}_{[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![} \quad \text{(value abstraction function)}$$

$$\Big|$$

$$[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![\in \mathcal{I}^\sharp \quad \text{(abstract intervals)}$$

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$$\Big|$$

$$[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![\in \mathcal{I}^\sharp \quad \text{(abstract intervals)}$$

Abstract Semantics: abstract interval domain

Definition (Intervals)

We consider a family $(\llbracket \underline{q}_p^\sharp, \overline{q}_p^\sharp \rrbracket)_{1 \leq p \leq n}$ of n couple of values in $\mathbb{N} \cup \{+\infty\}$ where n is a natural number in $\mathbb{N}$ such that the following properties are all satisfied:

1. for every natural number p between 1 and n, $\underline{q}_p^\sharp < \overline{q}_p^\sharp$;
2. for every natural number p between 2 and n, $\underline{q}_p^\sharp \leq \overline{q}_{p-1}^\sharp$;
3. and $\underline{q}_1^\sharp = 0$ and $\overline{q}_n^\sharp = +\infty$.

We denote the set of intervals by $\mathcal{I}^\sharp$.

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(trace abstraction function)}$$

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$$[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![\in \mathcal{I}^\sharp \quad \text{(abstract intervals)}$$

Abstract semantics: value abstraction function

Definition (Value abstraction function)

Let $[\![\underline{q}^\sharp_\star, \overline{q}^\sharp_\star[\![\in \mathcal{I}^\sharp$ be a reference interval. The value abstraction function $\overline{\beta}^{\mathbb{N}}_{[\![\underline{q}^\sharp_\star, \overline{q}^\sharp_\star[\![} : \mathbb{N} \to \mathcal{I}^\sharp$ maps each value $x \in \mathbb{N}$ to the unique abstract interval $[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![\in \mathcal{I}^\sharp$, such that both of the following properties are satisfied:

1. $\underline{q}^\sharp_p \leq x < \overline{q}^\sharp_p$;
2. for any $[\![\underline{q}^\sharp_{p'}, \overline{q}^\sharp_{p'}[\![\in \mathcal{I}^\sharp$ such that $\underline{q}^\sharp_{p'} \leq x < \overline{q}^\sharp_{p'}$, we have: $|p_\star - p| \leq |p_\star - p'|$.

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(trace abstraction function)}$$

$$\Big|$$

$$\overline{\beta}^{T}_{q^\sharp} \quad \text{(transition abstraction function)}$$

$$\Big|$$

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$$\Big|$$

$$[\![\underline{q}^\sharp_p, \overline{q}^\sharp_p[\![\in \mathcal{I}^\sharp \quad \text{(abstract intervals)}$$

Abstract Semantics: abstract state function

Definition (Macro-states)

Let $\mathcal{V}$ be a set of variables. A macro-state is a function $q^\sharp : \mathcal{V} \to \mathcal{I}^\sharp$.
The set of all macro-states is denoted $\overline{\mathcal{Q}}^\sharp$.

Abstract Semantics: abstract state function

Definition (Macro-states)

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The set of all macro-states is denoted $\overline{\mathcal{Q}}^\sharp$.

Definition (State Abstraction Function)

Let $q^\sharp_*$ be an macro-state in $\overline{\mathcal{Q}}^\sharp$. The abstract state function $\overline{\beta}^{\mathcal{Q}}_{q^\sharp_*} : \mathcal{Q} \to \overline{\mathcal{Q}}^\sharp$ maps each chemical state q to the macro-state $\left[v \in \mathcal{V} \mapsto \overline{\beta}^{\mathbb{N}}_{q^\sharp_*(v)}(q(v)) \in \mathcal{I}^\sharp \right]$.

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(trace abstraction function)}$$

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Abstract semantics: macro-transitions

Definition (Macro-transitions)

An element $(q^\sharp, r, q^{\sharp\prime}) \in \overline{\mathcal{Q}}^\sharp \times [\![1, \sigma]\!] \times \overline{\mathcal{Q}}^\sharp$ is called an abstract transition.
The set of all abstract transitions is denoted $\overline{T}^\sharp$.

Abstract semantics: transition abstraction function

Definition (Chemical transition abstraction)

Let $q^\sharp \in \overline{\mathcal{Q}}^\sharp$ be a reference macro-state. The chemical transition abstraction function

$$\overline{\beta}^{T}_{q^\sharp} : \{(q, r, q') \in T \mid \overline{\beta}^{\mathcal{Q}}_{q^\sharp}(q) = q^\sharp\} \to \{(q^{\sharp\prime\prime}, r, q^{\sharp\prime}) \in \overline{T}^\sharp \mid q^{\sharp\prime\prime} = q^\sharp\}$$

maps each chemical transition (q, r, q') such that $\overline{\beta}^{\mathcal{Q}}_{q^\sharp}(q) = q^\sharp$ to the macro-transition $\left(q^{\sharp\prime\prime}, r, \overline{\beta}^{\mathcal{Q}}_{q^\sharp}(q')\right)$ such that $q^{\sharp\prime\prime} = q^\sharp$.

$$\overline{\beta}^{\mathcal{T}}_{q^\sharp} \quad \text{(trace abstraction function)}$$

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Abstract Semantics: macro-traces

Definition (Macro-traces)

Let k be an element in $\mathbb{N}$. A macro-trace of size k is an element $(q_0^{\sharp\prime}, (q_i^{\sharp}, r_i, q_i^{\sharp\prime})_{1\leq i\leq k}) \in \overline{\mathcal{Q}}^{\sharp} \times T^{\sharp k}$ such that $q_i^{\sharp} = q_{i-1}^{\sharp\prime}$ for every integer i such that $1 \leq i \leq k$.
The set of all macro-traces is denoted $\overline{\mathcal{T}}^{\sharp}$.

Abstract semantics: abstract trace function

Definition (Chemical trace abstraction function)

Let $q^\sharp \in \overline{\mathcal{Q}}^\sharp$ be a reference macro-state. The chemical trace abstraction function $\overline{\beta}^{\mathcal{T}}_{q^\sharp} : \mathcal{T} \rightharpoonup \overline{\mathcal{T}}^\sharp$ maps each chemical trace $(q'_0, ((q_i, r_i, q'_i), \mu_i)_{1 \leq i \leq k})$ to the macro-trace:

1. $\overline{\beta}^{\mathcal{T}}_{q^\sharp}(q'_0, ()) = (\overline{\beta}^{\mathcal{Q}}_{q^\sharp}(q'_0), ())$;

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1. $\overline{\beta}^{\mathcal{T}}_{q^\sharp}(q'_0, ()) = (\overline{\beta}^{\mathcal{Q}}_{q^\sharp}(q'_0), ())$;
2. By induction, $\overline{\beta}^{\mathcal{T}}_{q^\sharp}(q'_0, ((q_i, r_i, q'_i), \mu_i)_{1 \leq i \leq k})$ outputs the macro-trace

$$\left(q_0^{\sharp\prime}, (q_i^\sharp, l'_i, q_i^{\sharp\prime})_{1 \leq i < k}\right) \frown \overline{\beta}^{\mathcal{T}}_{q_{k-1}^{\sharp\prime}}((q_k, r_k, q'_k), \mu_k)$$

where $(q_0^{\sharp\prime}, (q_i^\sharp, l'_i, q_i^{\sharp\prime})_{1 \leq i < k}) = \overline{\beta}^{\mathcal{T}}_{q^\sharp}(q'_0, (q_i, r_i, q'_i)_{1 \leq i < k})$.

Refining the abstract semantics with probabilities

Now we introduce the notion of chemical goals to propagate probabilities to abstract transitions.

Intuition for chemical goals

Consider the previous reaction network:

$$r_1 : A \xrightarrow{k_B} B \qquad\qquad r_2 : 2A \xrightarrow{k_C} C$$

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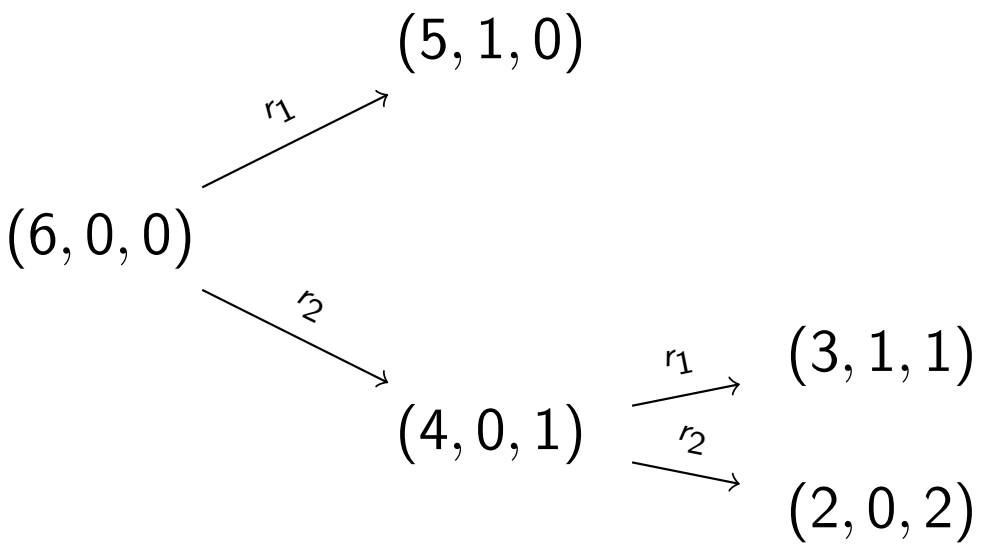
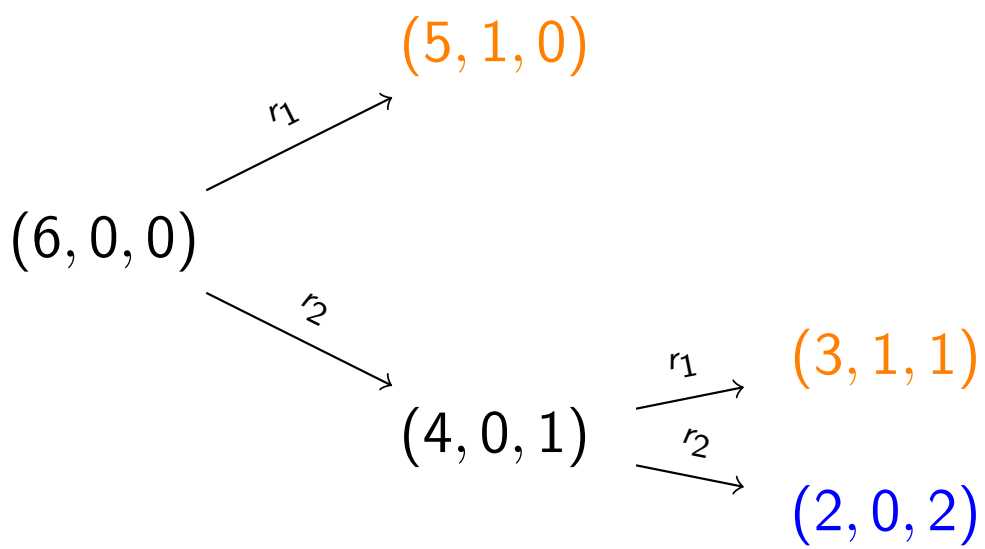


Intuition for chemical goals

A goal is when the value of B is equal to 1 before C is equal to 2, or vice versa.



Chemical goals

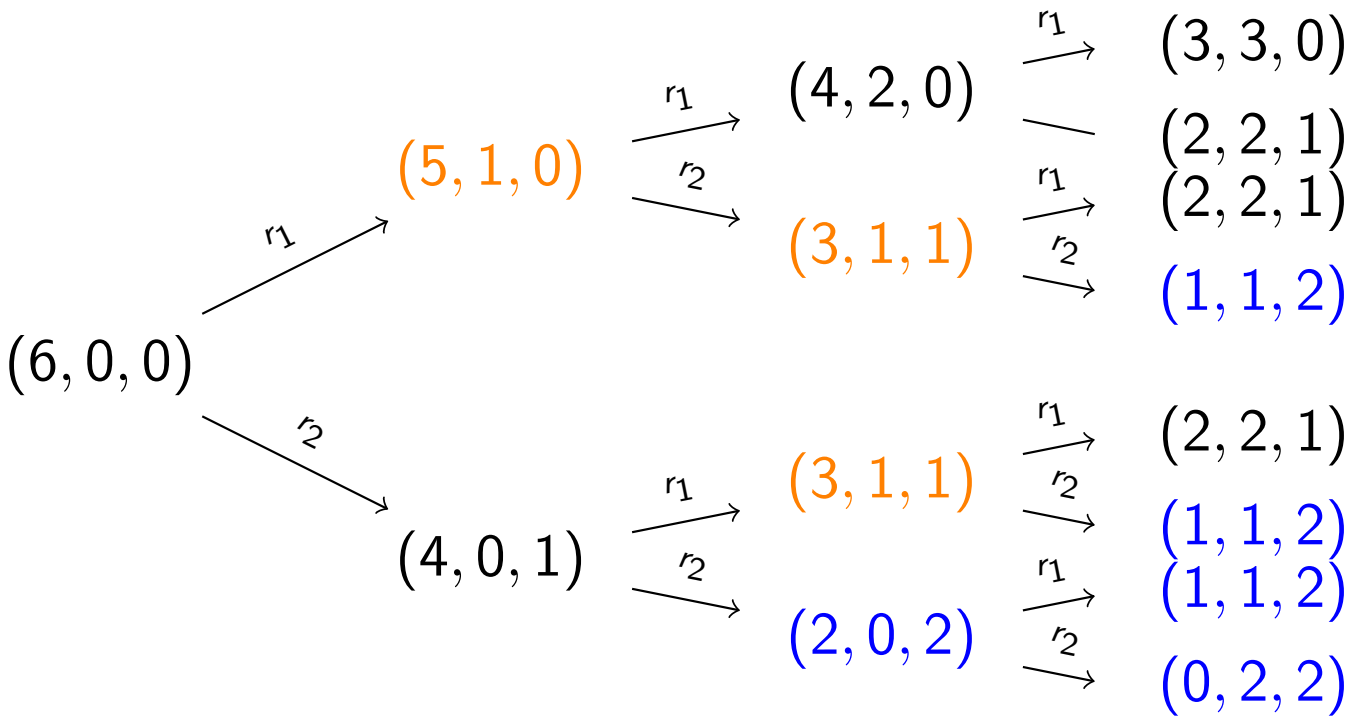
Definition (Target region)

A target region is a set of chemical states of the form

$$\{q \in \mathcal{Q} \mid q(v) \underline{\square} b\},$$

where v is a chemical species in $\mathcal{V}$, $\underline{\square}$ a binary relation in the set $\{\leq, \geq\}$, and b a natural number in $\mathbb{N}$.
This target region is denoted as $g_{v,\underline{\square},b}$. The set of target regions is denoted as $\mathcal{G}$.

We must not overcount traces



Cut the trace when it has reached a goal

Definition (Minimum successful traces)

We denote as $\chi_{(q_\bullet,\mathcal{G},g)}$ the set of the chemical traces $(q'_0,((q_i,r_i,q'_i),\mu_i)_{1\leq i\leq k})$ such that the following conditions are satisfied:

1. $q'_0 = q_\bullet$;
2. $q'_k \in g$; and
3. $\forall i \in \mathbb{N}$, such that $0 \leq i < k$, $q'_i \notin \bigcup \mathcal{G}$.

Probability for the traces that have reached a goal

Definition (Probability to reach a specific goal first)

The probability that the system reaches the target region g before any other target regions in $\mathcal{G}$ when starting in the state $q_\bullet$ is defined as:

$$P_g^{\mathcal{G}}(q_\bullet) = \sum_{\tau \in \chi(q_\bullet, \mathcal{G}, g)} P(\tau \mid q_\bullet).$$

Probability to reach a goal prior to another

Proposition

The probabilities $P_g^{\mathcal{G}}(q)$ for every state $q \in \mathcal{Q}$ are related by the following three conditions:

1. $P_g^{\mathcal{G}}(q) = 1$ *whenever* $q \in g$;
2. $P_g^{\mathcal{G}}(q) = 0$ *whenever* $q \in \bigcup \mathcal{G} \setminus g$;
3. $P_g^{\mathcal{G}}(q) = \sum_{q \xrightarrow{r_i} q'} \mu_{r_i}(q) \cdot P_g^{\mathcal{G}}(q')$ *whenever* $q \notin \bigcup \mathcal{G}$.

Motivation

Case Study I

Birth and death model

Coarse-graining the BD model

A Generic Framework

Concrete Semantics

Abstract Semantics

Chemical Goals

Case Study II

Competition for resources

Perspectives

Conclusion

Acknowledgements

Case Study II

Now for a second case study: competition for resources.

Competition for resources

This system is represented by the following reaction network:

$$r_1 : A \xrightarrow{k_B} B \qquad r_2 : 2A \xrightarrow{k_C} C$$

Competition for resources

This system is represented by the following reaction network:

$$r_1 : A \xrightarrow{k_B} B \qquad\qquad r_2 : 2A \xrightarrow{k_C} C$$

And, the following probabilities:

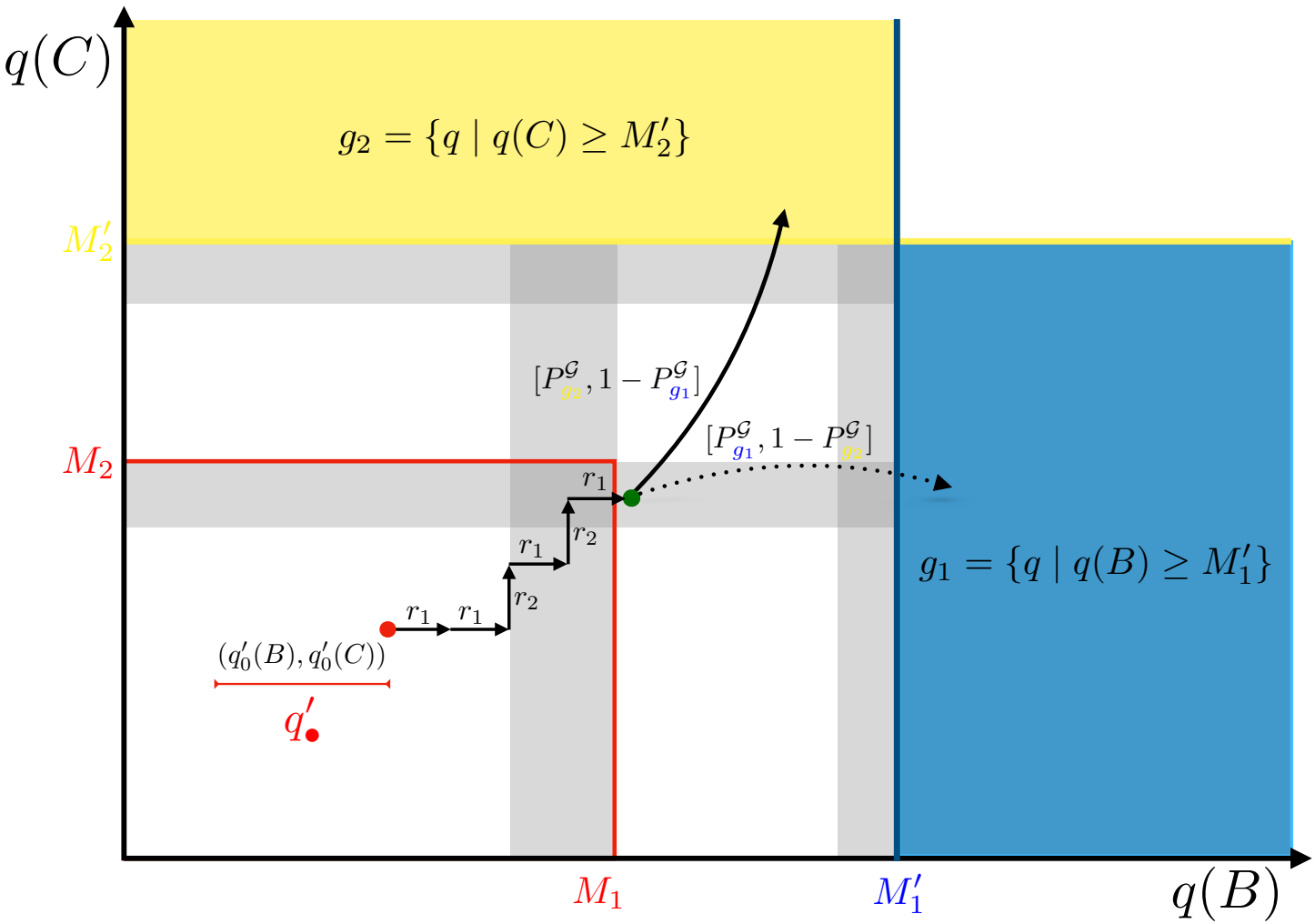
$$\mu_1(q) = \frac{2 \cdot k_B \cdot q(A)}{q(A) \cdot (2 \cdot k_B + k_C \cdot (q(A) - 1))}$$

$$\mu_2(q) = \frac{k_C \cdot q(A) \cdot (q(A) - 1)}{q(A) \cdot (2 \cdot k_B + k_C \cdot (q(A) - 1))}.$$

It is possible to identify a mass invariant:

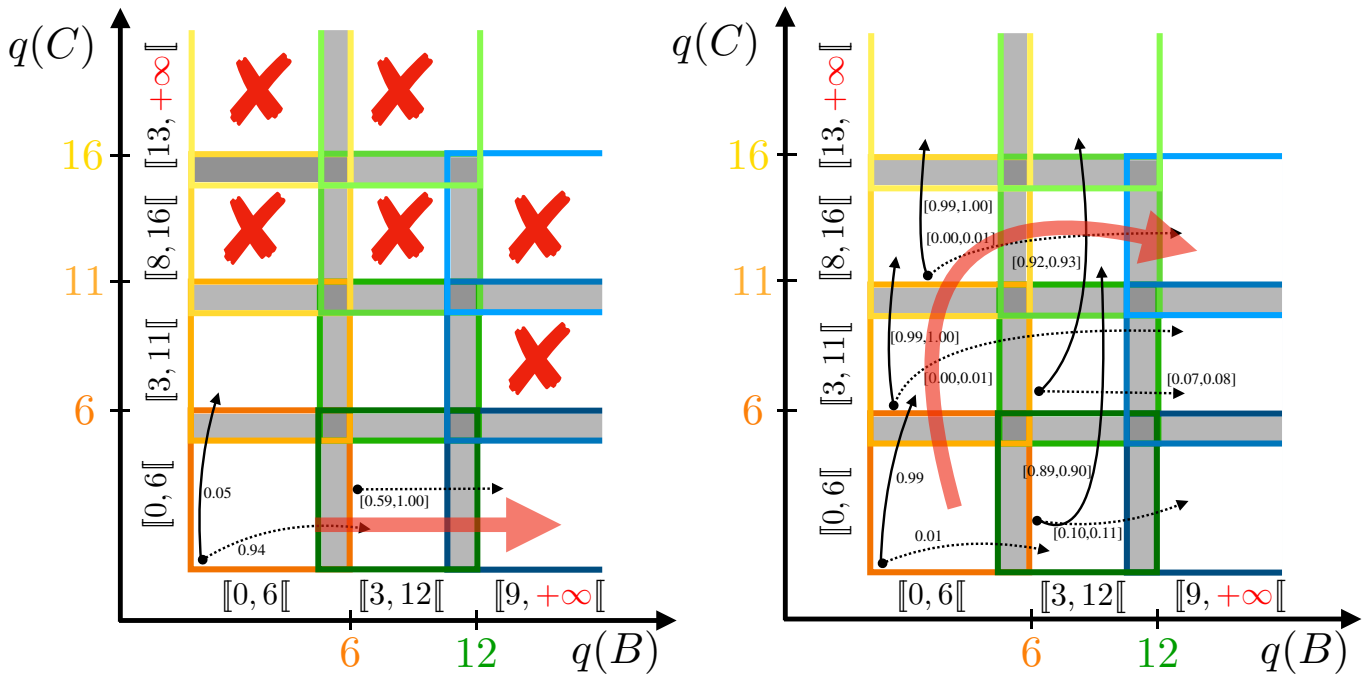
$$q(A) = q'_0(A) - (q(B) - q'_0(B)) - 2(q(C) - q'_0(C)).$$

A system competing for resources



2-D abstract transition system

$q'_0(A) = 15$ $\qquad$ $q'_0(A) = 100$



$\longrightarrow$ More probable behavior.

Perspectives

1. To date:
   - We derived a generic framework to build discrete models from (stochastic) reaction networks.
   - We show how our formally derived models can be used to assess logical models.
2. After:
   - Identify, exploit, and generalize intrinsic properties across diverse reaction networks in order to obtain more precise abstractions.
   - Use exact model reduction to simplify reaction networks beforehand (for example, Jérôme Feret, Koeppl, and Petrov 2013).
   - Use abstraction techniques (e.g., interval approximation, limited expansion, symbolic handling) to handle the trade-off between accuracy and efficiency in the computation of probabilities (for example, Miné 2006).

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- François Fages

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