Beyond the Saint-Venant system

Course 5

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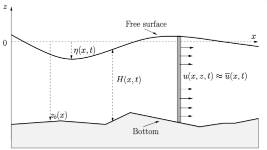
Course materials available at
https://team.inria.fr/ange/course-materials/

The Saint-Venant system



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0$$

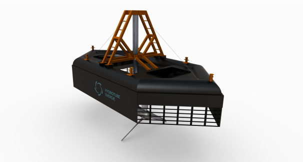
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) = -gH\frac{\partial z_b}{\partial x} - \kappa\bar{u}$$

$$\frac{\partial (H\bar{v})}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}\left(H\bar{v}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial y} - \kappa\bar{v}$$

- Derivation
- Numerical analysis
- Numerical methods
- Analysis ? Existence results ?

What to do with such models ?

- Hazardous flows
  - floodings, rogue waves, storm sturge
  - nuclear power plant
  - hydraulic energy (dams, security)
- Predictive models
  - control
  - optimization
- Renewable energies
  - complex source terms added
  - example



Beyond shallow water type models

The Saint-Venant system is

- widely used in 1d and 2d
- a good approximation of NS for long wave phenomena
- a good model for river flows,...

It can be

- enriched with several terms (curvature, stiff topography,...)
- used in many context
  - blood flows
  - traffic flows

■ Some remaining difficulties arround the Shallow Water syst. but scientific challenges and real-life applications often concern more complex models.

■ Applications to social sciences, humanities

Non-hydrostatic models

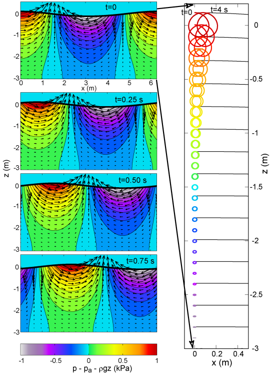
$$\left\{ \begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0 \\ \rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g \end{array} \right.$$

- Illustrations
  - (Hydro vs. non-hydro), (Wave over a beach), (Mascaret)
- An extensive literature
  - many models/results (Bona, Boussinesq, Green-Nagdhi, Peregrine / Lannes, Saut, Duchêne, Kazerani)
  - no more hyperbolic systems
  - often based on irrotational flows
- Derivation of SW non-hydrostatic models
  - a classical tool : asymptotic expansion
  - but strong limitations
- No more compressible fluid mechanics
  - p : lagrange multiplier

Water waves & hydrostatic assumption

$$\begin{cases} \dfrac{\partial u}{\partial x} + \dfrac{\partial w}{\partial z} = 0 \\ \rho_0 \left(\dfrac{\partial u}{\partial t} + u \dfrac{\partial u}{\partial x} + w \dfrac{\partial u}{\partial z} \right) + \dfrac{\partial p}{\partial x} = 0 \\ \rho_0 \left(\dfrac{\partial w}{\partial t} + u \dfrac{\partial w}{\partial x} + w \dfrac{\partial w}{\partial z} \right) + \dfrac{\partial p}{\partial z} = -\rho_0 g \end{cases}$$

- Particles trajectory
- Velocity field variations along z



Averaged Euler (M3AS 2011, DCDS 2015)

- The Euler system
$$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \end{cases}$$
- Boundary conditions
  - kinematic (bottom + free surface), dynamical ($p_s = p^a$)
- Energy equality: a constraint

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E\ dz + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u\,(E+p)\,dz = 0, \qquad E = \frac{u^2+w^2}{2} + gz$$

☐ A dispersive shallow water system

$$\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0 \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2 + H\overline{p}_{nh}\right) = -(gH + 2\overline{p}_{nh})\frac{\partial z_b}{\partial x} \\ \frac{\partial (H\bar{w})}{\partial t} + \frac{\partial (H\bar{w}\bar{u})}{\partial x} = 2\overline{p}_{nh} \\ \bar{w} = -\frac{H}{2}\frac{\partial \bar{u}}{\partial x} + \frac{\partial z_b}{\partial x}\bar{u} \end{cases}$$

Derivation

- The Euler system

$$\left\{\begin{array}{l} \frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=0 \\ \frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g \end{array}\right.$$

- Boundary conditions
  - kinematic

$$\frac{\partial z_b}{\partial x}u_b - w_b = 0, \qquad \frac{\partial \eta}{\partial t}+\frac{\partial \eta}{\partial x}u_s - w_s = 0$$

  - dynamical $p_s = p^a \approx cst$

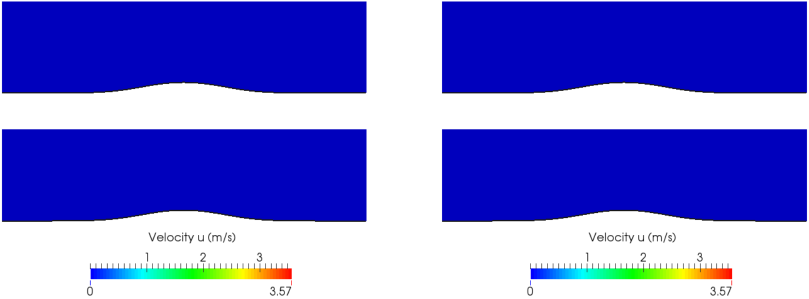
- Energy equality: a constraint

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E\ dz + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u\,(E+p)\,dz = 0, \qquad E = \frac{u^2+w^2}{2}+gz$$

□ **A "simplified/directional Galerkin approach"** $\int_{z_b}^{\eta} 1_{z\in[\eta,z_b]}(*)dz$

Typical behavior

Flows over a bump



Comparison with other dispersive models

- Green-Nagdhi (flat bottom)

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$
$$\frac{\partial (H\overline{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + \frac{g}{2}H^2 + H\overline{p}_{gn}\right) = 0$$
$$\frac{\partial}{\partial t}(H\overline{w}) + \frac{\partial}{\partial x}(H\overline{uw}) = \frac{3}{2}\overline{p}_{gn}$$
$$\bar{w} = -\frac{H}{2}\frac{\partial \bar{u}}{\partial x}$$

- Energy balance

$$\frac{\partial \overline{E}_{gn}}{\partial t} + \frac{\partial}{\partial x}\overline{u}\left(\overline{E}_{gn} + H\overline{p}_{gn}\right) = 0$$

with $\overline{E}_{gn} = \frac{H}{2}\left(\overline{u}^2 + \frac{2}{3}\overline{w}^2\right) + \frac{g}{2}H^2$

- Flows with advection vs. wave propagation

Other dispersive models

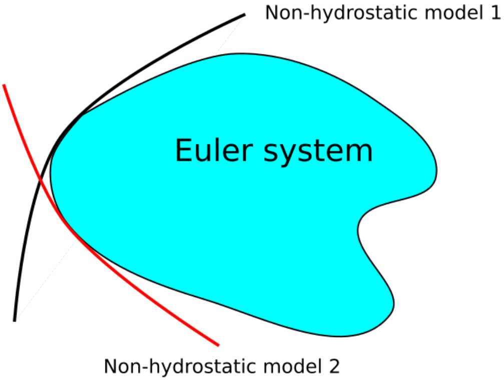
- Peregrine (1967) and related models

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$
$$\frac{\partial (H\overline{u})}{\partial t} - \frac{\tilde{H}^3}{4}\frac{\partial^3 \overline{u}}{\partial x^2 \partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + \frac{g}{2}H^2\right) = 0$$

- Korteweg-de Vries (Burgers) equation

$$\frac{\partial \overline{u}}{\partial t} + 6\overline{u}\frac{\partial \overline{u}}{\partial x} + \mu\frac{\partial^3 \overline{u}}{\partial x^3} = 0$$

Why so many dispersive models ?



Model derivation based on

- physical assumptions
- mathematical considerations
- ...

Rewriting of the model - Euler-like system

- Initial writting (flat bottom)

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$
$$\frac{\partial (H\overline{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + \frac{g}{2}H^2 + H\overline{p}_{nh}\right) = 0$$
$$\frac{\partial}{\partial t}(H\overline{w}) + \frac{\partial}{\partial x}(H\overline{uw}) = 2\overline{p}_{nh}$$
$$\bar{w} = -\frac{H}{2}\frac{\partial \bar{u}}{\partial x}$$

- New writting $\quad \overline{\mathrm{u}} = (\overline{u}, \overline{w})^T$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$
$$\frac{\partial}{\partial t}(H\overline{\mathrm{u}}) + \frac{\partial}{\partial x}(\overline{u}.H\overline{\mathrm{u}}) + \nabla\left(\frac{g}{2}H^2\right) + \nabla_{sw}\,\overline{p}_{nh} = 0$$
$$\mathrm{div}_{sw}\,\overline{\mathrm{u}} = 0$$

  - duality relation $\int \overline{p}_{nh}\mathrm{div}_{sw}\,\overline{\mathrm{u}}ds = (H\overline{up}_{nh})_x - \int \overline{\mathrm{u}}.\nabla_{sw}\,\overline{p}_{nh}ds$

Idea of the numerical scheme

- Rewriting of the model $\overline{\mathrm{u}} = (\overline{u}, \overline{w})^T$

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0$$
$$\frac{\partial}{\partial t}(H\overline{\mathrm{u}}) + \frac{\partial}{\partial x}(\overline{u}.H\overline{\mathrm{u}}) + \nabla\left(\frac{g}{2}H^2\right) + \nabla_{sw}\,\overline{p}_{nh} = 0$$
$$\mathrm{div}_{sw}\,\overline{\mathrm{u}} = 0$$

- Hyperbolic step

$$H_i^{n+1/2} = H_i^n - \sigma_i\left(\mathcal{F}_{H,i+1/2} - \mathcal{F}_{H,i-1/2}\right)$$
$$(H\mathrm{u})_i^{n+1/2} = (H\mathrm{u})_i^n - -\sigma_i\left(\mathcal{F}_{H\mathrm{u},i+1/2} - \mathcal{F}_{H\mathrm{u},i-1/2}\right)$$

- Correction step

$$\left\{\begin{array}{l} H_i^{n+1} = H_i^{n+1/2} \\ \mathrm{u}_i^{n+1} = \mathrm{u}_i^{n+1/2} - \frac{\Delta t}{H_i^{n+1}}\nabla_{sw}(p_{nh}^{n+1})_i \\ \mathrm{div}_{sw}\,(\mathrm{u}^{n+1})_i = 0 \end{array}\right. \quad \Rightarrow \quad \mathrm{div}_{sw}\left(\frac{\Delta t}{H_i^{n+1}}\nabla_{sw}(p_{nh}^{n+1})\right)_i = \mathrm{div}_{sw}\,(\mathrm{u}^{n+1/2})_i$$

Variational formulation (correction step)

$$H_i^{n+1} = H_i^{n+1/2}$$
$$\mathrm{u}_i^{n+1} = \mathrm{u}_i^{n+1/2} - \frac{\Delta t}{H_i^{n+1}} \nabla_{sw}(p_{nh}^{n+1})_i$$
$$\mathrm{div}_{sw}\left(\mathrm{u}^{n+1}\right)_i = 0$$

Variational mixed problem

Find $\mathrm{u} \in V$ and $p \in Q$ such that

$$\begin{aligned} \frac{1}{\Delta t^n} a(\mathrm{u},\mathrm{v}) + b(\mathrm{v},p) &= \frac{1}{\Delta t^n} a(\mathrm{u}^{n+1/2},\mathrm{v}), \quad \forall \mathrm{v} \in V, \\ b(\mathrm{u},q) &= 0, \quad \forall q \in Q. \end{aligned}$$

$Q = \{q \in L^2(\Omega)\}$ $V = \{\mathrm{v} = (v_1, v_2) \in (L^2(\Omega))^3 | \mathrm{div}_{sw}(\mathrm{v}) \in L^2(\Omega)\}$

$$a(\mathrm{u},\mathrm{v}) = \int_\Omega H\mathrm{u}\cdot\mathrm{v}\,dx, \quad \forall\,\mathrm{u},\mathrm{v} \in V,$$
$$b(\mathrm{v},q) = -\int_\Omega \mathrm{div}_{sw}(\mathrm{v}) q\,dx, \quad \forall \mathrm{v} \in V, \forall\, q \in Q,$$

Numerical approximation

Discrete problem

$$\begin{pmatrix} \frac{1}{\Delta t^n} A_H & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} U \\ P \end{pmatrix} = \begin{pmatrix} \frac{1}{\Delta t^n} A_H U^{n+1/2} \\ 0 \end{pmatrix},$$

- Inf-sup condition
- Conjugate gradient
- Require a preconditioning

$$M_H = \left(\int_\Omega H \varphi_i \varphi_j dx \right)_{1 \leq i,j \leq N}, \quad A_H = \begin{pmatrix} M_H & 0 & 0 \\ 0 & M_H & 0 \\ 0 & 0 & M_H \end{pmatrix}.$$

and the two matrices B^t, B defined by

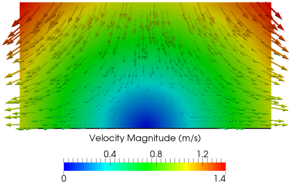
$$B^t = \left(\int_\Omega \nabla_{sw}(\phi_l) \varphi_i dx \right)_{1 \leq l \leq M, 1 \leq i \leq N}, B = - \left(\int_\Omega \text{div}_{sw}(\varphi_j) \phi_l dx \right)_{1 \leq l \leq M, 1 \leq j \leq N}$$

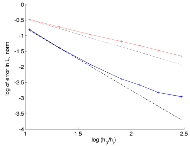
Non-stationary analytical solutions

- The Navier-Stokes system for a Newtonian fluid

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z},$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = -g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z},$$

- Analytical solution (anim)

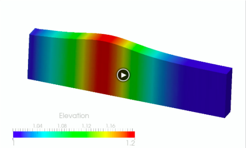


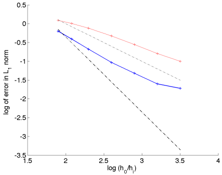


$$H(x,t) = \frac{\alpha}{t - t_0 + t_1},$$
$$u(x,z,t) = \frac{x}{t - t_0 + t_1},$$
$$w(x,z,t) = -\frac{z}{t - t_0 + t_1},$$
$$p(x,z,t) = p^a(t) - \frac{2\mu}{t - t_0 + t_1} + g(H - z) + \frac{H^2 - z^2}{(t - t_0 + t_1)^2},$$

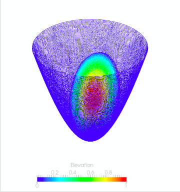
2d simulations

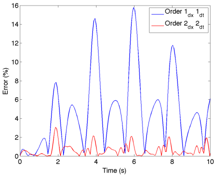
- Soliton





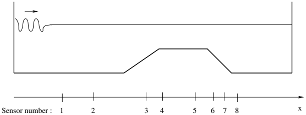
- Thacker's solution

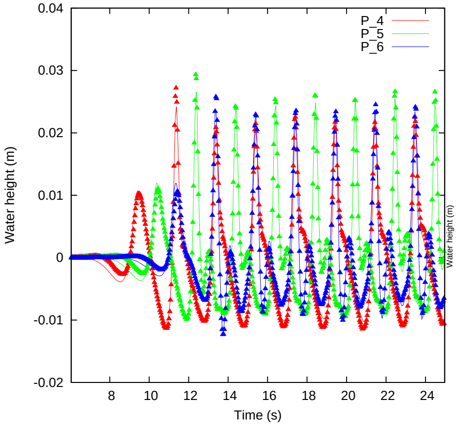


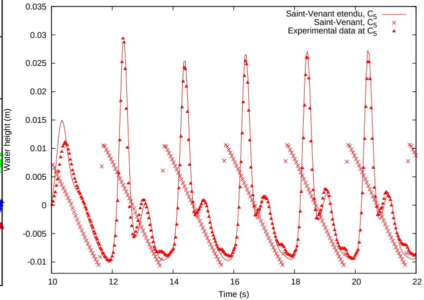


Experimental validation (I)

- Dinguemans experiments
- Animation
- Comparisons (sensors 4, 5 and 6)

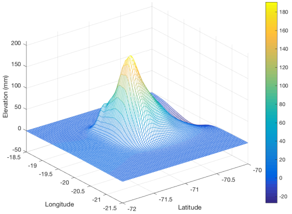


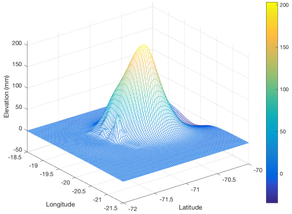




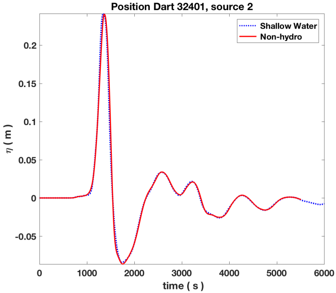
Non-hydrostatic effects : non intuitive behavior

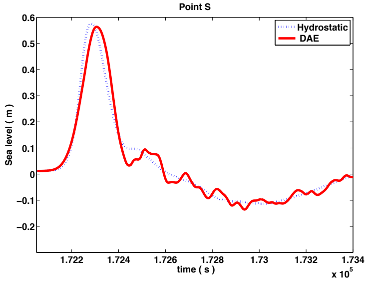
Bottom displacement





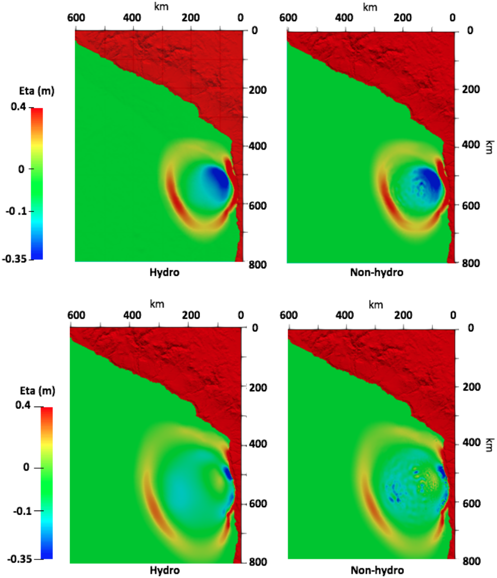
Generated waves





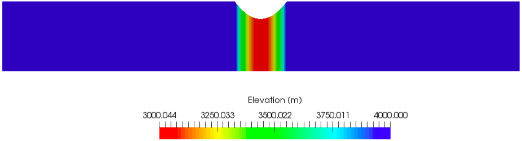
Hydrostatic vs. non-hydrostatic

Generated waves



Non-hydrostatic system : "surprising" behaviour

Asteroid impact over the ocean or air blast
collaboration with M. Berger (NYU) & R. LeVeque (Washington univ.)



- ocean with 500 meters depth, crater 125 meters depth (hydro) (non hydro)
- ocean with 4000 meters depth, crater 1000 meters depth (hydro) (non hydro)

Conclusion

- Towards Euler (Navier-Stokes) system
- Hydrostatic flows
  - numerical techniques exist
  - positivity, WB, fully discrete entropy, CV
- Non hydrostatic flows
  - models exist, (numerical) analysis is very complex
  - computational costs
  - explicit (in time) treatment of disper. terms

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◇ Ecology, hydrology & hydrodynamics-biology coupling

◇ Towards predictive models, optimization, control, FSI
  - difficult but very interesting applications

◇ **Transfer**