Numerical approximation of the Saint-Venant system

Course 3

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Course materials available at
https://team.inria.fr/ange/course-materials/

Outline

- Semi and fully discrete scheme for

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad (*)$$

- The Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1 + \frac{\kappa}{3\nu}H}$$

  - The numerical treatment of viscous / friction terms does not raise major difficulties
- Kinetic interpretation of the Saint-Venant system
  - using a Boltzmann equation
  - a scheme adapted from the one for (*)
- Numerical treatment of the topography source term

Required properties

$$\frac{\partial X}{\partial t} + \frac{\partial F(X)}{\partial x} = 0 \quad \Big| \quad X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right)$$

- Consistency

$$\mathcal{F}(X, X) = F(X).$$

Stationnary solutions. We can see that this condition guarantees obviously that if for all i, $X_i^n = cst$ then also $X_i^{n+1} = cst$.

- Invariant domain (positivity)

$$X_i^n \in \mathcal{X} \ \text{ for all } i \quad \Rightarrow \quad X_i^{n+1} \in \mathcal{X} \ \text{ for all } i.$$

- Discrete entropy inequality

$$\zeta(X_i^{n+1}) \leq \zeta(X_i^n) - \sigma_i^n \left(\mathcal{G}_{i+1/2}^n - \mathcal{G}_{i-1/2}^n \right)$$

- Convergence of the scheme (often very difficult)
- Comparison with analytical solutions

Numerical scheme

- A conservation law

$$\frac{\partial X}{\partial t} + \frac{\partial F(X)}{\partial x} = 0$$

- Notations

$$X_i^n = \frac{1}{\Delta x_i} \int_{C_i} X(t^n, x) dx.$$

- Discretization

$$\int_{t^n}^{t^{n+1}} \int_{C_i} \frac{\partial X}{\partial t} dxdt + \int_{t^n}^{t^{n+1}} \int_{C_i} \frac{\partial F(X)}{\partial x} dxdt = 0$$

leading to

$$X_i^{n+1} = X_i^n - \frac{1}{\Delta x_i} \int_{t^n}^{t^{n+1}} \big(F(X(x_{i+1/2}, t)) - F(X(x_{i-1/2}, t)) \big) dt$$

- Approximate Riemann solver

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right) \quad (1)$$

with $\mathcal{F}_{i+1/2}^n = \mathcal{F}(X_i^n, X_{i+1}^n)$, $\sigma_i^n = \frac{\Delta t^n}{\Delta x_i}$

Two well known fluxes

A general HCL

$$\frac{\partial X}{\partial t} + \frac{\partial F(X)}{\partial x} = 0$$

FV discretization

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right)$$

Fluxes

$$\lambda_1^n(X_i^n) = u_i^n + \sqrt{gH_i^n}, \qquad \lambda_2^n(X_{i+1}^n) = u_{i+1}^n - \sqrt{gH_{i+1}^n}$$

- Rusanov

$$\mathcal{F}^{Rus}(X_i^n, X_{i+1}^n) = \frac{F(X_i^n) + F(X_{i+1}^n)}{2} - \max_{k=1,2} \left(|\lambda_k^n(X_i^n)|, |\lambda_k^n(X_{i+1}^n)| \right) \frac{X_{i+1}^n - X_i^n}{2}$$

Two well known fluxes (cont'd)

FV discretization

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right)$$

Fluxes

$$\lambda_1^n(X_i^n) = u_i^n + \sqrt{gH_i^n}, \qquad \lambda_2^n(X_{i+1}^n) = u_{i+1}^n - \sqrt{gH_{i+1}^n},$$

- HLL

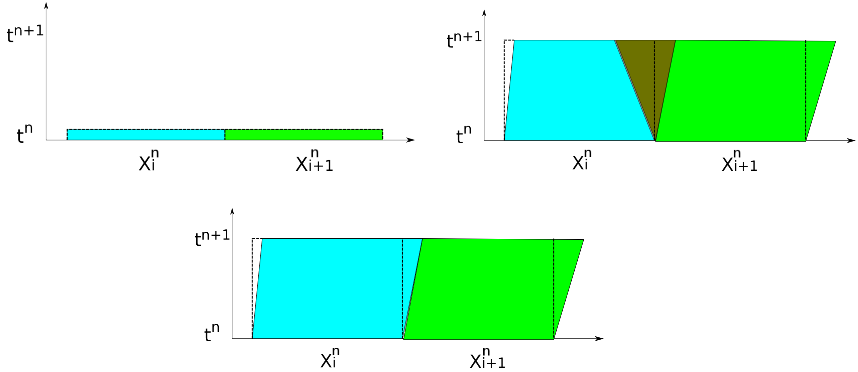
$$c_1^n = \min_{k=1,2} \left(\min_{X=X_i^n, X_{i+1}^n} \lambda_k^n(X) \right) \qquad c_2^n = \max_{k=1,2} \left(\max_{X=X_i^n, X_{i+1}^n} \lambda_k^n(X) \right)$$

$$\mathcal{F}^{HLL}(X_i^n, X_{i+1}^n) = \begin{cases} F(X_i^n) & \text{if } c_1^n \geq 0 \\ \frac{c_2^n F(X_i^n) - c_1^n F(X_{i+1}^n)}{c_2^n - c_1^n} + \frac{c_2^n c_1^n}{c_2^n - c_1^n}(X_{i+1}^n - X_i^n) & \text{if } c_1^n < 0 < c_2^n \\ F(X_{i+1}^n) & \text{if } c_2^n \leq 0 \end{cases}$$

Riemann problem

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right)$$

with $\mathcal{F}_{i+1/2}^n = \mathcal{F}(X_i^n, X_{i+1}^n)$



In 2d

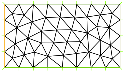
Continuous level

Starting from the conservation law

$$\frac{\partial X}{\partial t} + \nabla \cdot \underline{F}(X) = 0$$

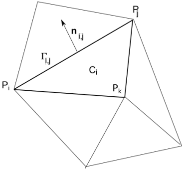
using the Green-Ostrogradski formula, an integration of the previous formula over any cell $\mathcal{C}$ leads to

$$\frac{\partial}{\partial t} \int_{\mathcal{C}} X ds + \int_{\Gamma} \underline{F}(X) \cdot \underline{n} dl = 0$$



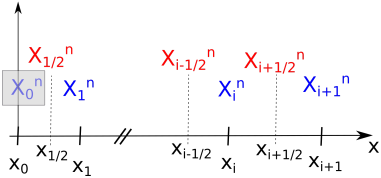
Discrete level

$$|C_i| \frac{\partial X_i}{\partial t} + \sum_{j \in \mathcal{K}_i} \int_{\Gamma_{i,j}} \underline{F}_{i,j} \cdot \underline{n}_{i,j} dl = 0$$



Boundary conditions

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right)$$
$$\mathcal{F}_{i+1/2}^n = \mathcal{F}(X_i^n, X_{i+1}^n)$$



- Impossible to impose "strongly" the boundary conditions (`run:anim/bc_weak.avi`)

$$\frac{\partial R}{\partial t} + \lambda \frac{\partial R}{\partial x} = 0, \qquad \text{with } \lambda = \lambda(t, x), \text{ and } \operatorname{sign}(\lambda) = \pm$$

- The upwinding is also required for the boundary conditions
- At each boundary, two unknowns $(H_0^n, \bar{u}_0^n)$ but one condition $Q = H\bar{u}$ or H
- In practice
  - if $H(x = 0, t)$ is given then $H_0^n = H(x = 0, t^n)$ and $\bar{u}_0^n + 2\sqrt{gH_0^n} = \bar{u}_1^n + 2\sqrt{gH_1^n}$
  - if $(H\bar{u})(x = 0, t)$ is given then $H_0^n \bar{u}_0^n = (H\bar{u})(x = 0, t^n)$ and $\bar{u}_0^n + 2\sqrt{gH_0^n} = \bar{u}_1^n + 2\sqrt{gH_1^n}$

Viscous & friction terms

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H \frac{\partial \bar{u}}{\partial x}\right) - \kappa \bar{u}$$

- In a compact form

$$\frac{\partial X}{\partial t} + \frac{\partial F(X)}{\partial x} = S_b(X) + S_{v,f}(X)$$

- Fractional time scheme

$$\begin{cases} X^{n+1/2} = X^n - \Delta t^n \left(\frac{\partial F(X^n)}{\partial x} - S_b(X^n)\right) \\ X^{n+1} = X^{n+1/2} + \Delta t^n S_{v,f}(X^{n+q}) \qquad q = 0, 1/2, 1 \end{cases}$$

- More precisely

$$X_i^{n+1/2} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right) + \sigma_i^n S_b(X_{i-1/2+}^n, X_{i+1/2-}^n)$$

$$(H\bar{u})_i^{n+1} = (H\bar{u})_i^{n+1/2} + \Delta t^n \left(4\nu \frac{H_{i+1/2}^{n+1/2}(\bar{u}_{i+1}^{n+p} - \bar{u}_i^{n+p}) + H_{i-1/2}^{n+1/2}(\bar{u}_i^{n+p} - \bar{u}_{i-1}^{n+p})}{\Delta x_i^2} - \kappa \bar{u}_i^{n+p'}\right)$$

with $p, p' = 0, 1/2, 1$

Viscous & friction terms (cont'd)

- Finite differences

$$(H\bar{u})_i^{n+1} = (H\bar{u})_i^{n+1/2} + \Delta t^n \Big(4\nu \frac{H_{i+1/2}^{n+1/2}(\bar{u}_{i+1}^{n+p} - \bar{u}_i^{n+p}) + H_{i-1/2}^{n+1/2}(\bar{u}_i^{n+p} - \bar{u}_{i-1}^{n+p})}{\Delta x_i^2} - \kappa \bar{u}_i^{n+p'} \Big)$$

with $p, p' = 1/2, 1$, warning $H^{n+1} = H^{n+1/2}$

- Finite elements

$$\mathcal{M}_H \bar{u}^{n+1} = \mathcal{M}_H \bar{u}^{n+1/2} + \Delta t^n \Big(\mathcal{K}_H \bar{u}^{n+p} - \kappa \bar{u}^{n+p'} \Big)$$

- Explicit (CFL / $H_i^{n+1/2} - 4\nu\Delta t^n \frac{H_{i+1/2}^{n+1/2} + H_{i-1/2}^{n+1/2}}{2\Delta x_i^2} \geq 0$)
- Implicit (linear system to solve)

Numerical scheme (in time)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad\qquad (c = cst)$$

Explicit vs. implicit scheme

- Semi discrete (in time) scheme

$$u^{n+1} = u^n - \Delta t^n c \frac{\partial u^{n+p}}{\partial x}$$

- Explicit Euler $p = 0$ and implicit Euler $p = 1$
- Energy balance (explicit) $(^*) \times u^n$

$$u^{n+1}u^n = (u^n)^2 - \Delta t^n c \frac{\partial u^n}{\partial x} u^n$$

or equivalently

$$\frac{(u^{n+1})^2}{2} - \frac{(u^n)^2}{2} + \Delta t^n \frac{\partial}{\partial x}\left(\frac{c}{2}(u^n)^2\right) = +\frac{1}{2}(u^{n+1} - u^n)^2$$

- Energy balance (explicit) $(^*) \times u^{n+1}$

$$\frac{(u^{n+1})^2}{2} - \frac{(u^n)^2}{2} + \Delta t^n \frac{\partial}{\partial x}\left(\frac{c}{2}(u^n)^2\right) = -\frac{1}{2}(u^{n+1} - u^n)^2$$

Numerical scheme (in time)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$$

- Explicit Euler scheme

$$u^{n+1} = u^n - \Delta t^n c \frac{\partial u^n}{\partial x}$$

  - First order scheme
  - Stable under a CFL condition
  - Easy to implement
  - Energy added

- Implicit Euler scheme

$$u^{n+1} = u^n - \Delta t^n c \frac{\partial u^{n+1}}{\partial x}$$

  - First order scheme
  - "No restriction" for Δt^n
  - More difficult to implement
  - Energy dissipated

Stability versus accuracy

Numerical scheme (cont'd)

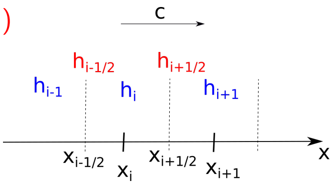
$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \qquad c > 0$$

- Finite volume approximation

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_{i+1/2}^n - h_{i-1/2}^n)$$

  - A natural choice $h_{i+1/2}^n = \frac{h_{i+1}^n + h_i^n}{2}$

$$h_i^{n+1} = h_i^n - \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - h_{i-1}^n)$$



  - Upwinding $h_{i+1/2}^n = h_i^n$, $h_{i-1/2}^n = h_{i-1}^n$

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_i^n - h_{i-1}^n)$$

Numerical scheme (cont'd)

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \qquad c > 0$$

- Finite volume approximation

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_{i+1/2}^n - h_{i-1/2}^n)$$

- Explanation

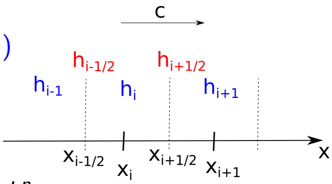
$$\begin{aligned}
h_i^{n+1} &= h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_i^n - h_{i-1}^n) \\
&= h_i^n - \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - h_{i-1}^n) + \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - 2h_i^n + h_{i-1}^n) \\
&= h_i^n - \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - h_{i-1}^n) + \frac{c}{2}\Delta t^n \Delta x_i \frac{h_{i+1}^n - 2h_i^n + h_{i-1}^n}{\Delta x_i^2} \\
&= h_i^n - \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - h_{i-1}^n) + \frac{c}{2}\Delta t^n \Delta x_i \Delta h_i^n
\end{aligned}$$

Numerical scheme (cont'd)

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \qquad c > 0$$

- Stability of the schemes (invariant domain $h(t,x) \geq 0$)
  - A natural choice $h^n_{i+1/2} = \frac{h^n_{i+1}+h^n_i}{2}$

$$h_i^{n+1} = h_i^n - \frac{c}{2}\frac{\Delta t^n}{\Delta x_i}(h^n_{i+1} - h^n_{i-1})$$



  - Upwinding $h^n_{i+1/2} = h^n_i$, $h^n_{i-1/2} = h^n_{i-1}$

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h^n_i - h^n_{i-1})$$

or

$$h_i^{n+1} = \left(1 - c\frac{\Delta t^n}{\Delta x_i}\right) h_i^n + c\frac{\Delta t^n}{\Delta x_i} h^n_{i-1}$$

with the CFL condition $\Delta t^n \leq \max_i \frac{\Delta x_i}{c}$

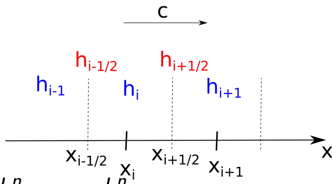
Numerical scheme (cont'd)

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0$$

- Stability of the schemes (invariant domain $h(t,x) \geq 0$)
  - When $c > 0$, upwinding $h^n_{i+1/2} = h^n_i$, $h^n_{i-1/2} = h^n_{i-1}$

    $$h_i^{n+1} = \left(1 - c\frac{\Delta t^n}{\Delta x_i}\right) h_i^n + c\frac{\Delta t^n}{\Delta x_i} h_{i-1}^n$$

    with the CFL condition $\Delta t^n \leq \max_i \frac{\Delta x_i}{c}$



  - When $c < 0$, upwinding $h^n_{i+1/2} = h^n_{i+1}$, $h^n_{i-1/2} = h^n_i$

    $$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_{i+1}^n - h_i^n)$$

    or

    $$h_i^{n+1} = \left(1 + c\frac{\Delta t^n}{\Delta x_i}\right) h_i^n - c\frac{\Delta t^n}{\Delta x_i} h_{i+1}^n$$

    with the CFL condition $\Delta t^n \leq \max_i \frac{\Delta x_i}{-c}$

Discrete entropy

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0 \quad (*)$$

- Continuous level $\quad (^*) \times h$ or $(^*) \times F'(h)$

$$\frac{\partial}{\partial t}\left(\frac{h^2}{2}\right) + \frac{\partial}{\partial x}\left(c\frac{h^2}{2}\right) = 0 \qquad \Bigg| \qquad \frac{\partial F(h)}{\partial t} + \frac{\partial (cF(h))}{\partial x} = 0$$

- Finite volume approximation (centered $h^n_{i+1/2} = (h^n_{i+1} + h^n_i)/2$)

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h^n_{i+1/2} - h^n_{i-1/2})$$

- Let us assume

$$\frac{(h_i^{n+1})^2}{2} \leq \frac{(h_i^n)^2}{2} - \frac{\Delta t^n}{\Delta x_i}\left(\frac{c(h^n_{i+1/2})^2}{2} - \frac{c(h^n_{i-1/2})^2}{2}\right)$$

leading to

$$\sum_i \frac{(h_i^{n+1})^2}{2} \leq \sum_i \frac{(h_i^n)^2}{2} - \frac{\Delta t^n}{\Delta x_i}\left(\frac{c(h^n_{N+1/2})^2}{2} - \frac{c(h^n_{1/2})^2}{2}\right)$$

Discrete entropy (cont'd)

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \quad c > 0$$

- Continuous level

$$\frac{\partial}{\partial t}\left(\frac{h^2}{2}\right) + \frac{\partial}{\partial x}\left(c\frac{h^2}{2}\right) = 0 \qquad \Bigg| \qquad \frac{\partial F(h)}{\partial t} + \frac{\partial (cF(h))}{\partial x} = 0$$

- Finite volume approximation (upwind scheme)

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_i^n - h_{i-1}^n) = \left(1 - c\frac{\Delta t^n}{\Delta x_i}\right) h_i^n + c\frac{\Delta t^n}{\Delta x_i} h_{i-1}^n \qquad (^*)$$

- Leading to $(^*) \times h_i^n$

$$\frac{(h_i^{n+1})^2}{2} = \frac{(h_i^n)^2}{2} - \frac{\Delta t^n}{\Delta x_i}\left(\frac{c(h_i^n)^2}{2} - \frac{c(h_{i-1}^n)^2}{2}\right) + \frac{1}{2}(h_i^{n+1} - h_i^n)^2 - \frac{c\Delta t^n}{2\Delta x_i}(h_i^n - h_{i-1}^n)^2$$

i.e.

$$\frac{(h_i^{n+1})^2}{2} = \frac{(h_i^n)^2}{2} - \frac{\Delta t^n}{\Delta x_i}\left(\frac{c(h_i^n)^2}{2} - \frac{c(h_{i-1}^n)^2}{2}\right) - \frac{c\Delta t^n}{2\Delta x_i}\left(1 - \frac{c\Delta t^n}{\Delta x_i}\right)(h_i^n - h_{i-1}^n)^2$$

Discrete entropy (cont'd)

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = 0, \quad c > 0$$

- Continuous level

$$\frac{\partial}{\partial t}\left(\frac{h^2}{2}\right) + \frac{\partial}{\partial x}\left(c\frac{h^2}{2}\right) = 0 \qquad \Bigg| \qquad \frac{\partial F(h)}{\partial t} + \frac{\partial (cF(h))}{\partial x} = 0$$

- Finite volume approximation (upwind scheme)

$$h_i^{n+1} = h_i^n - c\frac{\Delta t^n}{\Delta x_i}(h_i^n - h_{i-1}^n) = \left(1 - c\frac{\Delta t^n}{\Delta x_i}\right) h_i^n + c\frac{\Delta t^n}{\Delta x_i} h_{i-1}^n \qquad (*)$$

- $(*)$ evaluated with any convex function $F(.)$ i.e.
  $F\Big((1-\lambda)x + \lambda y\Big) \leq (1-\lambda)F(x) + \lambda F(y)$

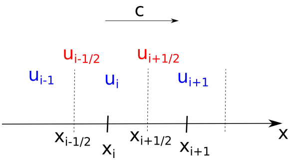
$$F\left(h_i^{n+1}\right) \leq \left(1 - c\frac{\Delta t^n}{\Delta x_i}\right) F\left(h_i^n\right) + c\frac{\Delta t^n}{\Delta x_i} F\left(h_{i-1}^n\right)$$

i.e.

$$F\left(h_i^{n+1}\right) \leq F\left(h_i^n\right) - c\frac{\Delta t^n}{\Delta x_i}\left(F\left(h_i^n\right) - F\left(h_{i-1}^n\right)\right)$$

With a source term

$$\frac{\partial h}{\partial t} + c\frac{\partial h}{\partial x} = f, \qquad c > 0$$



- Finite volume approximation

$$\frac{h_i^{n+1} - h_i^n}{\Delta t^n} = -\frac{c}{\Delta x_i}(h_{i+1/2}^n - h_{i-1/2}^n) + f_i^n = -\frac{c}{\Delta x_i}(h_i^n - h_{i-1}^n) + f_i^n$$

- Discrete entropy

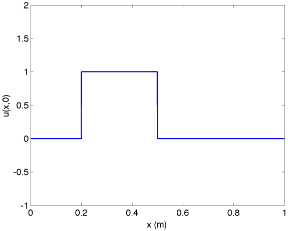
$$\frac{(h_i^{n+1})^2 - (h_i^n)^2}{2\Delta t^n} + c\frac{(h_i^n)^2 - (h_{i-1}^n)^2}{2\Delta x_i} - f_i^n h_i^n = \frac{\Delta t^n}{2}\left(\frac{h_i^{n+1} - h_i^n}{\Delta t^n}\right)^2 - \frac{c}{2\Delta x_i}(h_i^n - h_{i-1}^n)^2$$

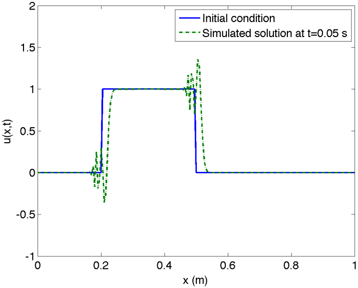
i.e.

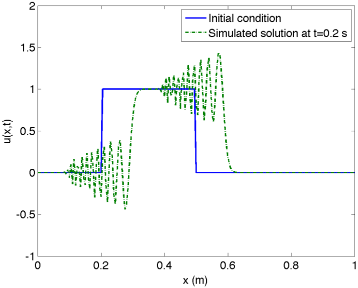
$$\frac{(h_i^{n+1})^2 - (h_i^n)^2}{2\Delta t^n} + \frac{c}{\Delta x_i}\left(\frac{(h_i^n)^2}{2} - \frac{(h_{i-1}^n)^2}{2}\right) - f_i^n h_i^n =$$

$$-\frac{c\Delta x_i}{2}\left(1 - \frac{c\Delta t^n}{\Delta x_i}\right)\left(\frac{h_i^n - h_{i-1}^n}{\Delta x_i}\right)^2$$

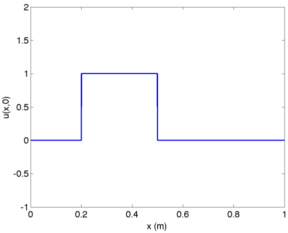
Centered scheme

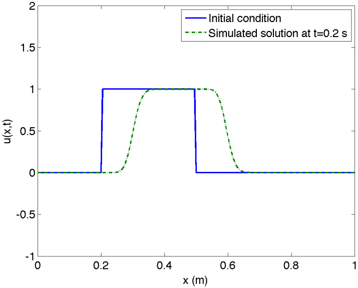


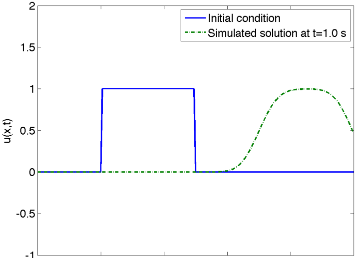


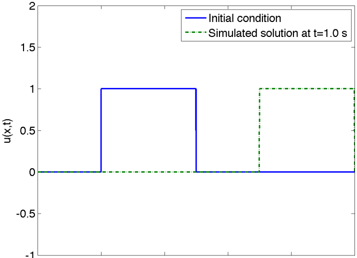


Upwind scheme





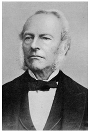




Important dates

- Leonhard Euler (1707-1783)
  - $$\begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0 \\ \rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g \end{cases}$$

- Navier (1785-1836) / Stokes (1819-1903)
  - $$\begin{cases} \mathrm{div}\, \underline{u} = 0 \\ \rho_0 \left(\dot{\underline{u}} + (\underline{u}.\nabla)\underline{u} \right) + \nabla p = \rho_0 \mathrm{G} + \mathrm{div}\, \underline{\underline{\sigma}} \\ \underline{u} = (u, w) \end{cases}$$

- A. J.-C. Barré de Saint-Venant (1797-1886)
  - $$\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x} \left(H\bar{u}^2 + \frac{g}{2} H^2 \right) = -gH \frac{\partial z_b}{\partial x} \end{cases}$$

Kinetic interpretation of the Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + gH^2/2\right) = -gH\frac{\partial z_b}{\partial x}$$

Proposition (Lions, Perthame, Tadmor, Souganidis)

The functions $(H, \overline{u})$ are weak solutions of the SV system if and only if the equilibrium $M(x,t,\xi)$ is solution of the kinetic equation

$$(\mathcal{B}) \qquad \frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi),$$

with $M = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right)$, $c = \sqrt{\frac{gH}{2}}$ *and* $\int_{\mathbb{R}} Q\ d\xi = \int_{\mathbb{R}} \xi Q\ d\xi = 0.$
The solution is an entropy solution if additionally

$$\chi(z) = \frac{1}{\pi}\sqrt{1-\frac{z^2}{4}}.$$

Kinetic interpretation of the SV system (cont'd)

- Let a function χ satisfying

$$\chi(z) = \chi(-z), \quad \chi(z) \geq 0, \quad \chi(z) = 0 \text{ for } |z| \geq \omega, \quad \int_{\mathbb{R}} \chi(z) dz = \int_{\mathbb{R}} z^2 \chi(z) dz = 1$$

- The Gibbs equilibrium (density) $M = M(t, x, \xi)$ is defined by

$$M = \frac{H}{c} \chi \left(\frac{\xi - \bar{u}}{c} \right)$$

with $c = \sqrt{\frac{gH}{2}}$

- Simple computations gives

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} M d\xi = \begin{pmatrix} H \\ H\bar{u} \\ H\bar{u}^2 + \frac{g}{2} H^2 \end{pmatrix} \quad \text{and} \quad \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \\ \xi^2 \end{pmatrix} \frac{\partial M}{\partial \xi} d\xi = \begin{pmatrix} 0 \\ -H \\ -2H\bar{u} \end{pmatrix}$$

Kinetic interpretation of the SV system (cont'd)

Proof (based on the change of variable $\xi = cz + \bar{u}$, $d\xi = cdz$)

$$\int_{\mathbb{R}} M d\xi = \int_{\mathbb{R}} \frac{H}{c}\chi\left(\frac{\xi - \bar{u}}{c}\right) d\xi = H \int_{\mathbb{R}} \chi(z) dz = H$$

$$\int_{\mathbb{R}} \xi M d\xi = \int_{\mathbb{R}} \xi \frac{H}{c}\chi\left(\frac{\xi - \bar{u}}{c}\right) d\xi = H \int_{\mathbb{R}} (cz + \bar{u})\chi(z) dz = H\bar{u}$$

$$\begin{aligned}\int_{\mathbb{R}} \xi^2 M d\xi &= \int_{\mathbb{R}} \xi^2 \frac{H}{c}\chi\left(\frac{\xi - \bar{u}}{c}\right) d\xi = H \int_{\mathbb{R}} (cz + \bar{u})^2 \chi(z) dz \\ &= H \int_{\mathbb{R}} (c^2 z^2 + 2c\bar{u}z + \bar{u}^2)\chi(z) dz = H(\bar{u}^2 + c^2) = H\bar{u}^2 + \frac{g}{2}H^2\end{aligned}$$

Kinetic interpretation of the SV system (cont'd)

Using the previous computations

$$\int_{\mathbb{R}} M d\xi = H, \quad \int_{\mathbb{R}} \xi M d\xi = H\bar{u}, \quad \int_{\mathbb{R}} \xi^2 M d\xi = H\bar{u}^2 + \frac{g}{2}H^2 \ \ldots$$

Proof $\int_{\mathbb{R}} (\mathcal{B}) d\xi$

$$\int_{\mathbb{R}} \left(\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} \right) d\xi = \int_{\mathbb{R}} Q d\xi$$

leading to

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} M d\xi + \frac{\partial}{\partial x} \int_{\mathbb{R}} \xi M d\xi - g \frac{\partial z_b}{\partial x} \int_{\mathbb{R}} \frac{\partial M}{\partial \xi} d\xi = \int_{\mathbb{R}} Q d\xi$$

that is

$$\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0$$

Kinetic interpretation of the SV system (cont'd)

Using the previous computations

$$\int_{\mathbb{R}} M d\xi = H, \quad \int_{\mathbb{R}} \xi M d\xi = H\bar{u}, \quad \int_{\mathbb{R}} \xi^2 M d\xi = H\bar{u}^2 + \frac{g}{2}H^2 \ \ldots$$

Proof $\int_{\mathbb{R}} \xi \times (\mathcal{B}) d\xi$

$$\int_{\mathbb{R}} \xi \left(\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} \right) d\xi = \int_{\mathbb{R}} \xi Q d\xi$$

leading to

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} \xi M d\xi + \frac{\partial}{\partial x} \int_{\mathbb{R}} \xi^2 M d\xi - g \frac{\partial z_b}{\partial x} \int_{\mathbb{R}} \xi \frac{\partial M}{\partial \xi} d\xi = \int_{\mathbb{R}} \xi Q d\xi$$

that is

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x} \left(H\bar{u}^2 + gH^2/2 \right) = -gH \frac{\partial z_b}{\partial x}$$

The energy balance

see Audusse, Bouchut, Bristeau, JSM *Math. of Comp.* 2016

- With the particular choice
$$\chi(z) = \frac{1}{\pi}\sqrt{\left|1 - \frac{z^2}{4}\right|_+}$$
we get
$$\int_{\mathbb{R}} \mathcal{H}(M) d\xi = \frac{H\bar{u}^2}{2} + \frac{g}{2}(\eta^2 - z_b^2) = H\bar{E}$$
with $\mathcal{H}(M) = \frac{\xi^2}{2} M + \frac{g^2\pi^2}{6} M^3 + g z_b M$
- Therefore
$$\partial_M \mathcal{H}(M) \left(\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} \right) = \partial_M \mathcal{H}(M) Q(x,t,\xi)$$
leads to
$$\frac{\partial \mathcal{H}(M)}{\partial t} + \xi \frac{\partial \mathcal{H}(M)}{\partial x} - g \frac{\partial z_b}{\partial x}\frac{\partial \mathcal{H}(M)}{\partial \xi} = \partial_M \mathcal{H}(M) Q(x,t,\xi)$$

The energy balance (cont'd)

see Audusse, Bouchut, Bristeau, JSM *Math. of Comp.* 2016

- With the particular choice

$$\chi(z) = \frac{1}{\pi}\sqrt{\left|1 - \frac{z^2}{4}\right|_+}$$

  we get

$$\int_{\mathbb{R}} \mathcal{H}(M) d\xi = \frac{H\bar{u}^2}{2} + \frac{g}{2}(\eta^2 - z_b^2) = H\bar{E}$$

  with $\mathcal{H}(M) = \frac{\xi^2}{2}M + \frac{g^2\pi^2}{6}M^3 + gz_bM$

- Therefore

$$\int_{\mathbb{R}} \left(\frac{\partial \mathcal{H}(M)}{\partial t} + \xi \frac{\partial \mathcal{H}(M)}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial \mathcal{H}(M)}{\partial \xi} \right) d\xi = \int_{\mathbb{R}} \partial_M \mathcal{H}(M) Q(x,t,\xi) d\xi$$

  with

$$\partial_M \mathcal{H}(M) = \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}M^2 + gz_b = \frac{\xi^2}{2} + gH - \frac{(\xi - \bar{u})^2}{2} + gz_b$$

$$= g(H + z_b) - \frac{\bar{u}^2}{2} + \bar{u}\xi$$

The energy balance (cont'd)

see Audusse, Bouchut, Bristeau, JSM *Math. of Comp.* 2016

- With the particular choice

$$\chi(z) = \frac{1}{\pi}\sqrt{\left|1 - \frac{z^2}{4}\right|_+}$$

  we get

$$\int_{\mathbb{R}} \mathcal{H}(M)d\xi = \frac{H\bar{u}^2}{2} + \frac{g}{2}(\eta^2 - z_b^2) = H\bar{E}$$

  with $\mathcal{H}(M) = \frac{\xi^2}{2}M + \frac{g^2\pi^2}{6}M^3 + gz_bM$
- Therefore

$$\int_{\mathbb{R}} \left(\frac{\partial\mathcal{H}(M)}{\partial t} + \xi\frac{\partial\mathcal{H}(M)}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial\mathcal{H}(M)}{\partial\xi}\right) d\xi = \int_{\mathbb{R}} \partial_M\mathcal{H}(M)Q(x,t,\xi)d\xi = 0$$

  leads to

$$\frac{\partial(H\bar{E})}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}\left(H\bar{E} + \frac{g}{2}H^2\right)\right) = 0$$

  with $H\bar{E} = \frac{H\bar{u}^2}{2} + \frac{g}{2}\left(\eta^2 - z_b^2\right)$

Interest of the kinetic description

$$\left\{ \begin{array}{l} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0 \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + gH^2/2\right) = -gH\frac{\partial z_b}{\partial x} \\ \frac{\partial (H\bar{E})}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}\left(H\bar{E} + \frac{g}{2}H^2\right)\right) = 0 \end{array} \right.$$

$$\Updownarrow$$

$$\frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi) = \text{"}0\text{"}$$

- Equivalence between a system of nonlinear conservation laws and a linear (almost scalar) equation
- Strong analysis results & stable/accurate numerical schemes
- Kinetic representation $\neq$ kinetic interpretation

Relaxation & kinetic description

- The Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = 0$$

- The Boltzmann equation $\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} = Q$
- BGK relaxation (Bouchut, Perthame)

$$\frac{\partial f}{\partial t} + \xi \frac{\partial f}{\partial x} = \frac{M - f}{\varepsilon}$$

with

$$\int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} f d\xi = \int_{\mathbb{R}} \begin{pmatrix} 1 \\ \xi \end{pmatrix} M d\xi$$

- BGK allows to recover a single entropy (enough to prove the convergence)

Kinetic scheme

- The Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = 0$$

- The Boltzmann equation

$$\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} = Q$$

- Kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n\right)$$
$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

- Definition of $M_{i\pm 1/2}^n$? Upwinding !

Kinetic scheme (cont'd)

- Kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n\right)$$
$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

- Upwinding

$$M_{i+1/2}^n = \begin{cases} M_i^n & \text{if } \xi \geq 0 \\ M_{i+1}^n & \text{if } \xi < 0 \end{cases}$$

- Leading to

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n - |\xi|_- M_i^n - |\xi|_+ M_{i-1}^n\right)$$

Kinetic scheme (cont'd)

- Kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n\right)$$

$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

- Leading to

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n - |\xi|_- M_i^n - |\xi|_+ M_{i-1}^n\right)$$

- Positivity of the density : $f_i^{n+1-} \geq 0 \;\; \forall i, n$?

$$f_i^{n+1-} = \left(1 - |\xi|\frac{\Delta t^n}{\Delta x_i}\right) M_i^n - \frac{\Delta t^n}{\Delta x_i}|\xi|_- M_{i+1}^n + \frac{\Delta t^n}{\Delta x_i}|\xi|_+ M_{i-1}^n$$

that is a convex combination

Macroscopic scheme

- Kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\Big(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n - |\xi|_- M_i^n - |\xi|_+ M_{i-1}^n\Big) \qquad (^*)$$

- $(^*)$ multiplied by 1 and integrated in ξ

$$\int_{\mathbb{R}} f_i^{n+1-} d\xi = \int_{\mathbb{R}} M_i^n d\xi - \frac{\Delta t^n}{\Delta x_i}\Big(\int_{\mathbb{R}} \big(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n\big) d\xi - \int_{\mathbb{R}} \big(|\xi|_- M_i^n + |\xi|_+ M_{i-1}^n\big) d\xi\Big)$$

that is

$$H_i^{n+1} = H_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{F}_{H,i+1/2}^n - \mathcal{F}_{H,i-1/2}^n\right)$$

with $\mathcal{F}_{H,i+1/2}^n = \int_{\xi \leq 0} \xi M_{i+1}^n d\xi + \int_{\xi \geq 0} \xi M_i^n d\xi$

Macroscopic scheme (cont'd)

- Kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\Big(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n - |\xi|_- M_i^n - |\xi|_+ M_{i-1}^n\Big) \qquad (^*)$$

- Mass conservation equation

$$H_i^{n+1} = H_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{F}_{H,i+1/2}^n - \mathcal{F}_{H,i-1/2}^n\right)$$

- $(^*)$ multiplied by ξ and integrated in ξ

$$\int_{\mathbb{R}} \xi f_i^{n+1-} d\xi = \int_{\mathbb{R}} \xi M_i^n d\xi - \frac{\Delta t^n}{\Delta x_i}\Big(\int_{\mathbb{R}} \xi\big(|\xi|_- M_{i+1}^n + |\xi|_+ M_i^n\big) d\xi - \int_{\mathbb{R}} \xi\big(|\xi|_- M_i^n + |\xi|_+ M_{i-1}^n\big) d\xi\Big)$$

that is

$$(H\bar{u})_i^{n+1} = (H\bar{u})_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{F}_{H\bar{u},i+1/2}^n - \mathcal{F}_{H\bar{u},i-1/2}^n\right)$$

with $\mathcal{F}_{H\bar{u},i+1/2}^n = \int_{\xi\leq 0} \xi^2 M_{i+1}^n d\xi + \int_{\xi\geq 0} \xi^2 M_i^n d\xi$

A discrete entropy inequality ?

- Using the kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n\right)$$
$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

- We seek for a relation having the form

$$\mathcal{H}(f_i^{n+1-}) = \mathcal{H}(M_i^n) - \frac{\Delta t^n}{\Delta x_i}\left(\xi \mathcal{H}(M_{i+1/2}^n) - \xi \mathcal{H}(M_{i-1/2}^n)\right) + \frac{\Delta t^n}{\Delta x_i}\mathcal{D}_i^n$$

with $\mathcal{H}(M) = \frac{\xi^2}{2}M + \frac{g^2\pi^2}{6}M^3$, $M_{i+1/2}^n = M_i^n 1_{\xi \geq 0} + M_{i+1}^n 1_{\xi < 0}$
and $\mathcal{D}_i^n \leq 0$

- This will give us

$$\int_{\mathbb{R}} \mathcal{H}(f^{n+1-})_i d\xi \leq \int_{\mathbb{R}} \mathcal{H}(M_i^n) d\xi - \frac{\Delta t^n}{\Delta x_i}\left(\int_{\mathbb{R}} \xi \mathcal{H}(M_{i+1/2}^n) d\xi - \int_{\mathbb{R}} \xi \mathcal{H}(M_{i-1/2}^n) d\xi\right)$$

in other words $E_i^{n+1} \leq E_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{G}_{i+1/2}^n - \mathcal{G}_{i-1/2}^n\right)$

A discrete entropy inequality (flat bottom)

- Using the kinetic scheme

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n\right) \quad (^*)$$

$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

- First we focus on $(^*)$ for $\xi \le 0$ corresponding to

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1}^n - \xi M_i^n\right)$$

multiplied by $\partial_M \mathcal{H}(M_i^n)$, it gives

$$\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n) = -\frac{\Delta t^n}{\Delta x_i}\xi \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)$$

with $\partial_M \mathcal{H}(M_i^n) = \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}\left(M_i^n\right)^2$

A discrete entropy inequality (cont'd)

- We have for $\xi \leq 0$

$$\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n) = -\frac{\Delta t^n}{\Delta x_i}\xi \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)$$

with $\partial_M \mathcal{H}(M_i^n) = \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}\left(M_i^n\right)^2$

- But using a Taylor expansion formula

$$\begin{aligned}
\mathcal{H}(M_{i+1}^n) &= \mathcal{H}(M_i^n) + \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right) \\
&+ \frac{1}{2}\partial_M^2 \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)^2 + \frac{1}{6}\partial_M^3 \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)^3 \\
&= \mathcal{H}(M_i^n) + \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right) \\
&+ \frac{1}{2}\left(\partial_M^2 \mathcal{H}(M_i^n) + \frac{1}{3}\partial_M^3 \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)\right)\left(M_{i+1}^n - M_i^n\right)^2 \\
&= \mathcal{H}(M_i^n) + \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right) + \frac{g^2\pi^2}{6}\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2
\end{aligned}$$

with $\partial_M^2 \mathcal{H}(M_i^n) = g^2\pi^2 M_i^n$, $\partial_M^3 \mathcal{H}(M_i^n) = g^2\pi^2$

A discrete entropy inequality (cont'd)

- We have for $\xi \leq 0$

$$\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n) = -\frac{\Delta t^n}{\Delta x_i}\xi \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right)$$

with $\partial_M \mathcal{H}(M_i^n) = \frac{\xi^2}{2} + \frac{g^2\pi^2}{2}\left(M_i^n\right)^2$

- Using the Taylor expansion formula

$$\mathcal{H}(M_{i+1}^n) = \mathcal{H}(M_i^n) + \partial_M \mathcal{H}(M_i^n)\left(M_{i+1}^n - M_i^n\right) + \frac{g^2\pi^2}{6}\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2$$

- Hence we get (for $\xi \leq 0$)

$$\begin{aligned}\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n) = &-\frac{\Delta t^n}{\Delta x_i}\xi\left(\mathcal{H}(M_{i+1}^n) - \mathcal{H}(M_i^n)\right)\\ &+\frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2\end{aligned}$$

A discrete entropy inequality (cont'd)

- We have obtained (for $\xi \le 0$)

$$\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n) = -\frac{\Delta t^n}{\Delta x_i}\xi\left(\mathcal{H}(M_{i+1}^n) - \mathcal{H}(M_i^n)\right) + \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2$$

- But for the quantity $\partial_M \mathcal{H}(M_i^n)(f_i^{n+1-} - M_i^n)$, we also have

$$\mathcal{H}(f_i^{n+1-}) = \mathcal{H}(M_i^n) + \partial_M \mathcal{H}(M_i^n)\left(f_{i+1}^{n+1-} - M_i^n\right) + \frac{g^2\pi^2}{6}\left(2M_i^n + f_i^{n+1-}\right)\left(f_i^{n+1-} - M_i^n\right)^2$$

leading to

$$\mathcal{H}(f_i^{n+1-}) = \mathcal{H}(M_i^n) - \frac{\Delta t^n}{\Delta x_i}\xi\left(\mathcal{H}(M_{i+1}^n) - \mathcal{H}(M_i^n)\right) + \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2 + \frac{g^2\pi^2}{6}\left(\frac{\Delta t^n}{\Delta x_i}\right)^2\xi^2\left(2M_i^n + f_i^{n+1-}\right)\left(M_{i+1}^n - M_i^n\right)^2$$

A discrete entropy inequality (cont'd)

- We have obtained (for $\xi \leq 0$)

$$\begin{aligned}\mathcal{H}(f_i^{n+1-}) =& \mathcal{H}(M_i^n) - \frac{\Delta t^n}{\Delta x_i}\xi\left(\mathcal{H}(M_{i+1}^n) - \mathcal{H}(M_i^n)\right)\\ &+ \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi\left(2M_i^n + M_{i+1}^n\right)\left(M_{i+1}^n - M_i^n\right)^2\\ &+ \frac{g^2\pi^2}{6}\left(\frac{\Delta t^n}{\Delta x_i}\right)^2\xi^2\left(2M_i^n + f_i^{n+1-}\right)\left(M_i^n - M_{i-1}^n\right)^2\end{aligned}$$

- Likewise for $\xi \geq 0$, we get

$$\begin{aligned}\mathcal{H}(f_i^{n+1-}) =& \mathcal{H}(M_i^n) - \frac{\Delta t^n}{\Delta x_i}\xi\left(\mathcal{H}(M_i^n) - \mathcal{H}(M_{i-1}^n)\right)\\ &- \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi\left(2M_i^n + M_{i-1}^n\right)\left(M_i^n - M_{i-1}^n\right)^2\\ &+ \frac{g^2\pi^2}{6}\left(\frac{\Delta t^n}{\Delta x_i}\right)^2\xi^2\left(2M_i^n + f_i^{n+1-}\right)\left(M_i^n - M_{i-1}^n\right)^2\end{aligned}$$

A discrete entropy inequality (cont'd)

- Finally we have obtained

$$\begin{aligned}\mathcal{H}(f_i^{n+1-}) =& \mathcal{H}(M_i^n) - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{H}(\xi M_{i+1/2}^n) - \mathcal{H}(\xi M_{i-1/2}^n)\right) \\ &+ \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi 1_{\xi\le 0}\left(2M_i^n + M_{i+1}^n + \frac{\Delta t^n}{\Delta x_i}N_{i+}^n\right)\left(M_{i+1}^n - M_i^n\right)^2 \\ &- \frac{\Delta t^n}{\Delta x_i}\frac{g^2\pi^2}{6}\xi 1_{\xi\ge 0}\left(2M_i^n + M_{i-1}^n + \frac{\Delta t^n}{\Delta x_i}N_{i-}^n\right)\left(M_i^n - M_{i-1}^n\right)^2\end{aligned}$$

- This gives us

$$\int_{\mathbb{R}}\mathcal{H}(f^{n+1-})_i d\xi \le \int_{\mathbb{R}}\mathcal{H}(M_i^n)d\xi - \frac{\Delta t^n}{\Delta x_i}\left(\int_{\mathbb{R}}\xi\mathcal{H}(M_{i+1/2}^n)d\xi - \int_{\mathbb{R}}\xi\mathcal{H}(M_{i-1/2}^n)d\xi\right)$$

in other words $E_i^{n+1} \le E_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\mathcal{G}_{i+1/2}^n - \mathcal{G}_{i-1/2}^n\right)$

Num. treatment of sources terms

Summary

- The homogeneous Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = 0 \qquad \left(-gH\frac{\partial z_b}{\partial x}\right)$$

- Finite volume discretization / upwinding (run:st_venant_center.avi)

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right)$$

Source terms

- Naive approaches fail (run:anim/not_well_balanced.avi)

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right) + \sigma_i^n S_i^n, \qquad S_i^n = \begin{pmatrix} 0 \\ -\frac{gH_i^n}{2}(z_{b,i+1} - z_{b,i-1}) \end{pmatrix}$$

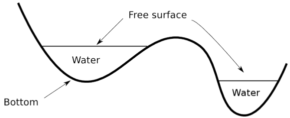
Num. treatment of sources terms (cont'd)

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x}$$

At rest $\bar{u} = 0$, $\eta = H + z_b = cste$

- The flux $F(X)$ with $X = (H, 0)^T$

$$F(X) = \begin{pmatrix} 0 \\ \frac{g}{2} H^2 \end{pmatrix}$$



- Finite volume scheme for $n = 0$ (Rusanov fluxes)

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n\right) + \sigma_i^n S_i^n \qquad S_i^n = \begin{pmatrix} 0 \\ -\frac{gH_i^n}{2}(z_{b,i+1} - z_{b,i-1}) \end{pmatrix}$$

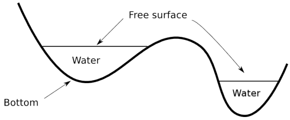
with $\mathcal{F}_{i+1/2}^n = \begin{pmatrix} 0 \\ \frac{g}{4}\left((H_{i+1}^n)^2 + (H_i^n)^2\right) \end{pmatrix} - \max_{k,j}\left(|\lambda_k^n(X_{i+j}^n)|\right) \frac{X_{i+1}^n - X_i^n}{2}$

Num. treatment of sources terms (cont'd)

At rest $\bar{u} = 0$, $\eta = H + z_b = cste$

- The flux $F(X)$ with $X = (H, 0)^T$

$$F(X) = \begin{pmatrix} 0 \\ \frac{g}{2} H^2 \end{pmatrix}$$



- Finite volume scheme for $n = 0$ (Rusanov fluxes)

$$X_i^{n+1} = X_i^n - \sigma_i^n \left(\mathcal{F}_{i+1/2}^n - \mathcal{F}_{i-1/2}^n \right) + \sigma_i^n S_i^n \qquad S_i^n = \begin{pmatrix} 0 \\ -\frac{gH_i^n}{2}(z_{b,i+1} - z_{b,i-1}) \end{pmatrix}$$

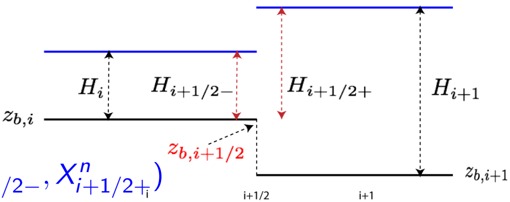
with $\mathcal{F}_{H,i+1/2}^n = -\max_{k,j} \left(|\lambda_k^n(X_{i+j}^n)| \right) \frac{H_{i+1}^n - H_i^n}{2}$

- Hence

$$H_i^{n+1} = H_i^n + \sigma_i^n \left(\lambda_{i+1/2}^{max} \frac{H_{i+1}^n - H_i^n}{2} - \lambda_{i-1/2}^{max} \frac{H_i^n - H_{i-1}^n}{2} \right)$$

Well-balanced schemes (Hyd. rec. [ABBKP 04])

- Main ideas



$$\mathcal{F}^n_{i+1/2} = \mathcal{F}(X^n_{i+1/2-}, X^n_{i+1/2+})$$

and

$$\mathcal{S}_{b,i} = \begin{pmatrix} 0 \\ \frac{g}{2}(H^n_{i+1/2-})^2 - \frac{g}{2}(H^n_{i-1/2+})^2 \end{pmatrix}$$

- Reconstruction ($\bar{u}^n_{i+1/2-} = \bar{u}^n_i$ and $\bar{u}^n_{i+1/2+} = \bar{u}^n_{i+1}$)

$$\hat{H}^n_{i+1/2-} = H^n_i + z_{b,i} - z_{b,i+1/2}, \quad \hat{H}^n_{i+1/2+} = H^n_{i+1} + z_{b,i+1} - z_{b,i+1/2}$$

with $z_{b,i+1/2} = \max(z_{b,i}, z_{b,i+1})$ and
$H^n_{i+1/2\pm} = \max(0, \hat{H}^n_{i+1/2\pm})$

- Properties $\hat{H}^n_{i+1/2+} = \hat{H}^n_{i+1/2-} \quad \Rightarrow \quad H^n_{i+1} + z_{b,i+1} = H^n_i + z_{b,i}$

Saint-Venant with topography

From

$$\frac{\partial H}{\partial t} + \frac{\partial H\bar{u}}{\partial x} = 0$$
$$\frac{\partial H\bar{u}}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + gH^2/2\right) = -gH\frac{\partial z_b}{\partial x}$$

to

$$\frac{\partial M}{\partial t} + \xi\frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi),$$

Discrete form

$$f_i^{n+1-} = M_i^n - \frac{\Delta t^n}{\Delta x_i}\left(\xi M_{i+1/2}^n - \xi M_{i-1/2}^n + \delta M_{i+1/2-}^n - \delta M_{i-1/2+}^n\right)$$
$$M_i^{n+1} = f_i^{n+1-} + \Delta t^n Q_i^n$$

with $\delta M_{i+1/2-}^n = M_i^n - M_{i+1/2-}^n$ and $M_{i+1/2-}^n = \frac{H_{i+1/2-}^n}{c_{i+1/2-}^n}\chi\left(\frac{\xi - u_i^n}{c_{i+1/2-}^n}\right)$
where $H_{i+1/2-}^n$ is defined using the hyd. rec. technique

Other aspects

- ...
- Second order schemes
  - Space - MUSCL type techniques
  - Time - Runge-Kutta type techniques such as the Heun method given by:

    In order to solve $\frac{\partial X}{\partial t} = F(X, t)$, we define

$$\begin{aligned}\tilde{X}^{n+1} &= X^n + \Delta t^n F(X^n, t^n)\\ \tilde{X}^{n+2} &= \tilde{X}^{n+1} + \Delta t^n F(\tilde{X}^{n+1}, t^n)\\ X^{n+1} &= \frac{X^n + \tilde{X}^{n+2}}{2}\end{aligned}$$

$$\begin{aligned}\tilde{X}^{n+1} &= X^n + \tilde{\Delta} t^n F(X^n, t^n)\\ \tilde{X}^{n+2} &= \tilde{X}^{n+1} + \tilde{\Delta} t^{n+1} F(\tilde{X}^{n+1}, t^n)\\ X^{n+1} &= (1-\gamma) X^n + \gamma \tilde{X}^{n+2}\\ \Delta t^n &= \frac{2\tilde{\Delta} t^n \tilde{\Delta} t^{n+1}}{\tilde{\Delta} t^n + \tilde{\Delta} t^{n+1}}, \quad \gamma = \frac{(\Delta t^n)^2}{2\tilde{\Delta} t^n \tilde{\Delta} t^{n+1}}\end{aligned}$$

- Prove that the Heun scheme is a second order scheme ?