(Not so) Hyperbolic models for complex flows
Applications to environnemental flows
Modeling, numerical analysis & simulations

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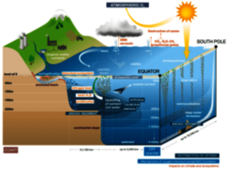
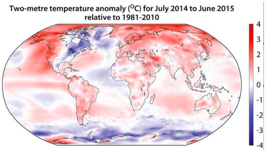
https://team.inria.fr/ange/course-materials/

Lab. J.-L. Lions - January 2023

From climate changes to natural hazards

- Long-term / large scale evolutions
  - climate change, bio-geo-chemistry
  - acification, CO_2 storage, bio-diversity. . .
- Necessary adaptations
  - landslides, flooding
  - pollutions

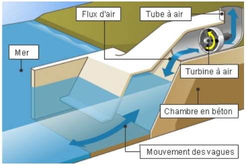


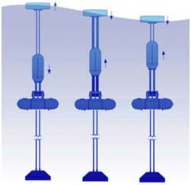


Marine energies

Capture the waves energy

- Various tested systems
- Objective be able to simulate the fluid/structure interaction in order to optimize the system





Scientific program

- 3 aspects : modelling, analysis & simulation
- From Navier-Stokes to simpler models e.g. the Saint-Venant system

$$\left\{ \begin{array}{l} \nabla.\underline{u} = 0, \\ \dot{\underline{u}} + (\underline{u}.\nabla)\underline{u} + \nabla p = G, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} \end{array} \right.$$

- Hyperbolic conservation laws

$$\frac{\partial X}{\partial t} + \frac{\partial F(X)}{\partial x} = S$$

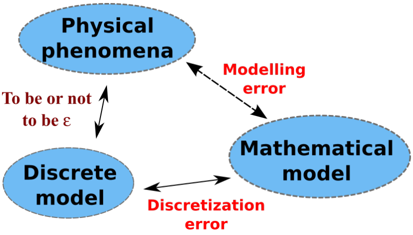
  - represent various phenomena
  - difficult to analyse and simulate, (not so) hyperbolic systems
- Academic & complex models
  - numerical analysis of academic models
  - beyond HCL, what to do in front of "complex models" ?

A non-exhaustive list of open questions

- Modelling of complex flows
- Well-posedness of some models
  - simple models versus useful models
  - practical interest ?
- Accurate & long term simulations
  - numerical techniques (consistency, accuracy. . .) requires $\Delta t, \Delta x \ll 1$
  - but in ocean global circulation models $\Delta t, \Delta x \gg 1$
- Multi-scale & multi-physics models
  - uncertainties, stochasticity
- Control & optimization
- Coupling between models & data

3 complementary aspects

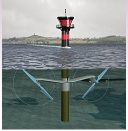
- Modelling
- Analysis
- Numerical methods / simulation



$$\|E_\phi - E_{i,j}\| \leq \|E_\phi - E_{C^0}\| + \|E_{C^0} - E_{i,j}\|$$

Context





- Risk assessment
- Sustainable development, sustainable environment

Novelty

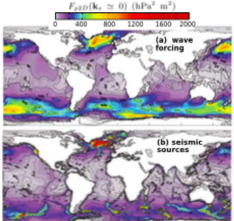
- Many available measurements (satellites, accelerometers,. . .)
- Complex/new phenomena are extracted from these data
- Significant advances made by IA and ML (GraphCast, Pangu-Weather,. . .)

Sophisticated models are needed

- But simpler models are useful
- Rigourously derived, accurate & efficient numerical methods
- An example: the seismic noise
  - interaction of gravity waves
  - bottom pressure variations

    generate seismic waves

    ANR project MIMOSA with IPGP, Ifremer, Barcelona Univ.



L. Euler in *Principes généraux du mouvement des fluides (1757)*

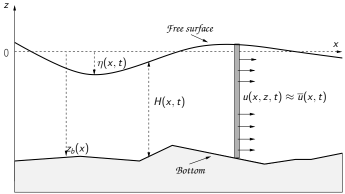
$$\left\{ \begin{array}{l} \nabla . \underline{u} = 0, \\ \dot{\underline{u}} + (\underline{u}.\nabla)\underline{u} + \nabla p = G, \end{array} \right.$$

Cependant tout ce que la Théorie des fluides renferme, eſt contenu dans les deux équations rapportées cy-deſſus (§. XXXIV.), de ſorte que ce ne ſont pas les principes de Méchanique qui nous manquent dans la pourſuite de ces recherches, mais uniquement l'Analyſe, qui n'eſt pas encore aſſés cultivée, pour ce deſſein : & partant on voit clairement, quelles découvertes nous reſtent encore à faire dans cette Science, avant que nous puiſſions arriver à une Théorie plus parfaite du mouvement des fluides.

The cornerstone : the Saint-Venant system

$$\left\{ \begin{array}{l} \nabla.\underline{u} = 0, \\ \dot{\underline{u}} + (\underline{u}.\nabla)\underline{u} + \nabla p = G, \end{array} \right. \Rightarrow \left\{ \begin{array}{l} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} \end{array} \right.$$

- $\varepsilon = H_0/L_0 \ll 1$, approx. $\mathcal{O}(\varepsilon^2)$, Saint-Venant 1872
- Reduced complexity, various applications, HCL
- Robust numerical tools are required



Water depth $H = \eta - z_b$,
velocity $\underline{u} = (u, w)$,
$\overline{u}(x,t) = \frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$

Some remaining difficulties arround the Shallow Water syst. but scientific challenges and real-life applications concern more complex models.

Outline of the course

- Modelling, numerical methods (JSM)
- Numerical analysis of hyperbolic PDEs (NA)
- `https://team.inria.fr/ange/course-materials/`

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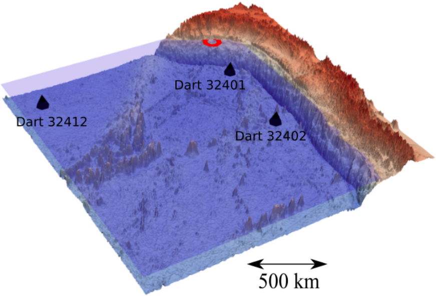
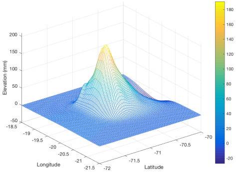
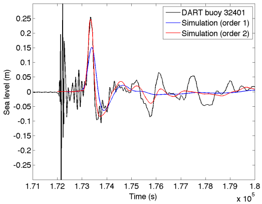
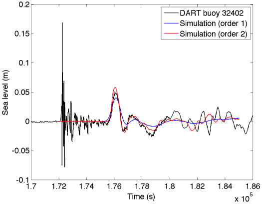
- From Navier-Stokes to Saint-Venant
    - asymptotic expansion of a complex model
    - interest & difficulties
- The Saint-Venant system
    - properties of the Saint-Venant
- Numerical analysis of hyperbolic conservation laws
    - finite volumes discretization
    - enforcement of physical properties (positivity, equilibrium, energy)
- Introduction to numerical oceanography (J. Deshayes)

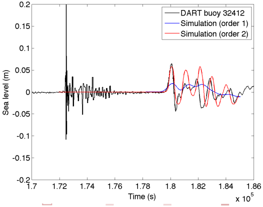
Tsunami modelling/simulation (Chile - 2014)





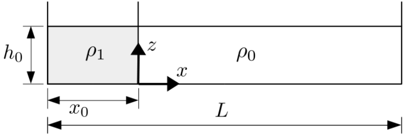




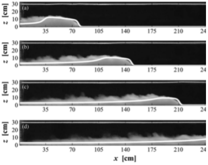


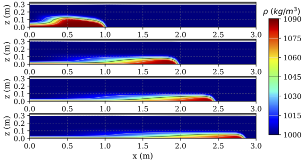
A typical example

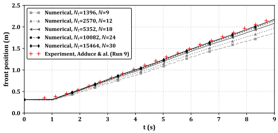
- The experimental device [Adduce *et al.*, J. Hydraul. Eng. 2012] (anim) (Gibraltar strait)



- Measurements versus simulation results

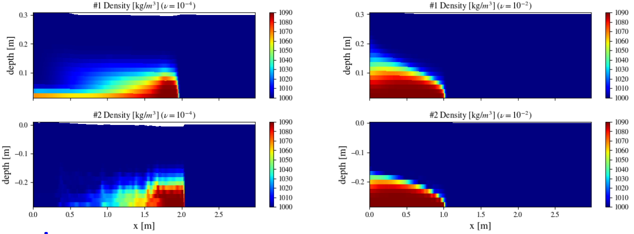




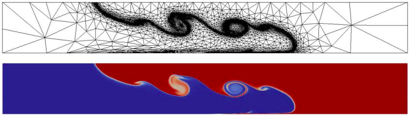


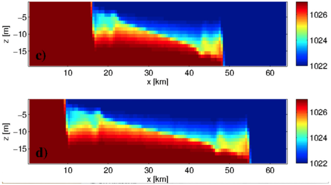
A typical example (cont'd)

Comparison of 2 different codes (experiment proposed by Adduce)



Other codes





See also Wroniszewski et al. Benchmarking of Navier-Stokes codes for free surface simulations, Coastal Eng., 91:1-17,2014

Complex rheology flows

1d Saint-Venant with Coulomb

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0 \\ \partial_t(h\bar{u}) + \partial_x(h\bar{u}^2 + \frac{g}{2}h^2) = -gh\partial_x z_b - \mu h \text{sign}(\bar{u}), \end{cases}$$

Saint-Venant with Bingham

$$\begin{cases} \partial_t h + \nabla_{x,y} \cdot (h\mathrm{v}) = 0 \\ \partial_t(h\mathrm{v}) + \nabla_{x,y} \cdot (h\mathrm{v} \otimes \mathrm{v} + \frac{g}{2}h^2) - \nabla_{x,y} \cdot \sigma = -gh\nabla_{x,y} z_b, \end{cases}$$

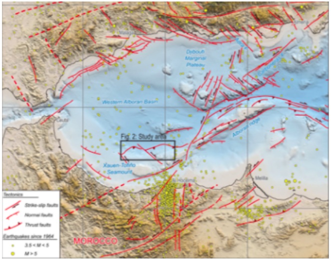
with

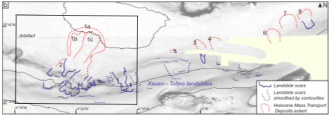
$$\sigma = \begin{cases} 2\mu h D(\mathrm{v}) + \lambda h \nabla_{x,y} \cdot v\, 1 + gh\dfrac{D(v)}{|D(v)|} & \textit{if } D(v) \neq 0, \\ |\sigma| < gb & \textit{if } D(v) = 0, \end{cases}$$

and $D(v) = \frac{1}{2}(\nabla_{x,y} v + \nabla_{x,y}^T v)$, $\mathrm{v} = (u_1, u_2)^T$

Complex rheology flows - illustrations

The Alboran sea





(Coulomb6) (Coulomb3) (Bingham)

(Tsunami 1) (Tsunami 2) (Tsunami 3)

Incompressible Euler system & dispersive models

Non-hydrostatic terms (dispersive terms)

$$\frac{\partial w}{\partial t} + \frac{\partial uw}{\partial x} + \frac{\partial w^2}{\partial z} + \frac{\partial p}{\partial z} = -g \quad \Rightarrow \quad \frac{\partial p}{\partial z} = -g$$

- Open problems (Hydro vs. non-hydro)(Wave over a beach)(Flow over a bump) (Mascaret)
- Collaborations with EDF, IPGP, LOCEAN...
- Many difficulties
  - only hyperbolic features
  - dispersive terms
  - numerical treatement of p
  - often costly algorithms