Around the Saint-Venant system

Course 2

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LJLL - January 30, 2024

Course materials available at
https://team.inria.fr/ange/course-materials/

From Navier-Stokes to Saint-Venant

- The Navier-Stokes or Euler system

$$
\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\
&\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z} \\
&\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z}
\end{aligned}
$$

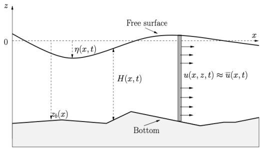
$$
\begin{aligned}
&\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\
&\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0 \\
&\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g
\end{aligned}
$$

+ Kinematic (and dynamic) boundary conditions

$$
\frac{\partial z_b}{\partial x} u_b - w_b = 0, \qquad \frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} u_s - w_s = 0 \quad \dots
$$

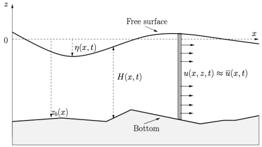
- Asymptotic expansion for $\varepsilon = \frac{h}{\lambda}$
- The Saint-Venant system

$$
\begin{aligned}
&\frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0 \\
&\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x} \left(H\bar{u}^2 + \frac{g}{2} H^2 \right) = -gH \frac{\partial z_b}{\partial x}
\end{aligned}
$$



The Saint-Venant system

- Formulation (up to $\mathcal{O}(\varepsilon^2)$) terms



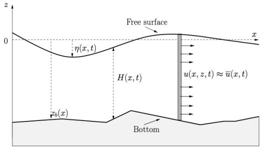
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\overline{u}) = 0,$$

$$\frac{\partial (H\overline{u})}{\partial t} + \frac{\partial (H\overline{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H \frac{\partial \overline{u}}{\partial x}\right) - \frac{\kappa \overline{u}}{1 + \frac{\kappa}{3\nu}H},$$

- Friction laws
  - Navier $S_f = \kappa\overline{u}$, Manning-Strickler $S_f = C_f \frac{\overline{u}|\overline{u}|}{H^{\frac{4}{3}}}$,
  - Darcy-Weisbach $S_f = C_f \frac{\overline{u}|\overline{u}|}{H}$
- Energy balance, vertical velocity, passive tracer
- In 2d
- Why the Saint-Venant system is ony valid for $\frac{\partial z_b}{\partial x} \ll 1$?

Do not forget the validity domain !

Energy balance for the Saint-Venant system



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\Big(4\nu H\frac{\partial \bar{u}}{\partial x}\Big) - \frac{\kappa \bar{u}}{1+\frac{\kappa}{3\nu}H} \qquad (*)$$

- Energy balance $= (*) \times \bar{u}$
- Advection terms

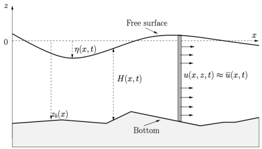
$$\bar{u} \times \left(\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x}\right) = \frac{\partial (H\bar{u}^2)}{\partial t} + \frac{\partial (\bar{u}H\bar{u}^2)}{\partial x} - H\bar{u}\frac{\partial \bar{u}}{\partial t} - H\bar{u}^2\frac{\partial \bar{u}}{\partial x}$$

$$= \frac{\partial (H\bar{u}^2)}{\partial t} + \frac{\partial (\bar{u}H\bar{u}^2)}{\partial x} - \frac{H}{2}\frac{\partial \bar{u}^2}{\partial t} - \frac{H\bar{u}}{2}\frac{\partial \bar{u}^2}{\partial x}$$

$$= \frac{\partial (H\bar{u}^2)}{\partial t} + \frac{\partial (\bar{u}H\bar{u}^2)}{\partial x} - \frac{\partial}{\partial t}\Big(\frac{H}{2}\bar{u}^2\Big) - \frac{\partial}{\partial x}\left(\frac{\bar{u}H\bar{u}^2}{2}\right)$$

$$+ \frac{\bar{u}^2}{2}\frac{\partial H}{\partial t} + \frac{\bar{u}^2}{2}\frac{\partial (H\bar{u})}{\partial x}$$

Energy balance for Saint-Venant (cont'd)



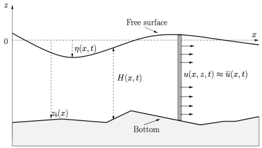
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1+\frac{\kappa}{3\nu}H} \qquad (*)$$

- Pressure terms

$$\begin{aligned}
\bar{u} \times \left(gH\frac{\partial (H+z_b)}{\partial x}\right) &= \frac{\partial}{\partial x}\left(gH\bar{u}(H+z_b)\right) - g(H+z_b)\frac{\partial (H\bar{u})}{\partial x} \\
&= \frac{\partial}{\partial x}\left(gH\bar{u}(H+z_b)\right) + g(H+z_b)\frac{\partial H}{\partial t} \\
&= \frac{\partial}{\partial x}\left(\bar{u}\left(gH\left(\frac{H}{2}+z_b\right) + \frac{g}{2}H^2\right)\right) + g\eta\frac{\partial \eta}{\partial t} \\
&= \frac{\partial}{\partial x}\left(\bar{u}\left(\frac{g}{2}(\eta^2 - z_b^2) + \frac{g}{2}H^2\right)\right) + \frac{\partial}{\partial t}\left(\frac{g}{2}(\eta^2 - z_b^2)\right)
\end{aligned}$$

Energy balance for Saint-Venant (cont'd)



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$

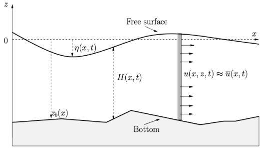
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H \frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1 + \frac{\kappa}{3\nu}H} \qquad (*)$$

- Energy balance $= (*) \times \bar{u}$
- Advection + pressure terms

$$\bar{u} \times \left(\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + gH\frac{\partial (H + z_b)}{\partial x}\right) =$$

$$\frac{\partial}{\partial t}\left(\frac{H\bar{u}^2}{2} + \frac{g}{2}(\eta^2 - z_b^2)\right) + \frac{\partial}{\partial x}\left(\bar{u}\left(\frac{H\bar{u}^2}{2} + \frac{g}{2}(\eta^2 - z_b^2) + \frac{g}{2}H^2\right)\right)$$

Energy balance for Saint-Venant (cont'd)



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1+\frac{\kappa}{3\nu}H} \qquad (*)$$

- Rheology $(*) \times \bar{u}$

$$\bar{u} \times \left(\frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1+\frac{\kappa}{3\nu}H}\right) =$$

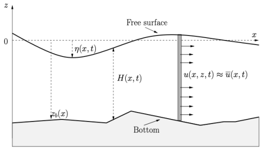
$$\frac{\partial}{\partial x}\left(4\nu H\bar{u}\frac{\partial \bar{u}}{\partial x}\right) - 4\nu H\left(\frac{\partial \bar{u}}{\partial x}\right)^2 - \frac{\kappa \bar{u}^2}{1+\frac{\kappa}{3\nu}H}$$

- Sum

$$\frac{\partial (H\bar{E})}{\partial t} + \frac{\partial}{\partial x}\left(\bar{u}\left(H\bar{E} + \frac{g}{2}H^2 - 4\nu H\frac{\partial \bar{u}}{\partial x}\right)\right) = -4\nu H\left(\frac{\partial \bar{u}}{\partial x}\right)^2 - \frac{\kappa \bar{u}^2}{1+\frac{\kappa}{3\nu}H}$$

with $H\bar{E} = \frac{H\bar{u}^2}{2} + \frac{g}{2}\left(\eta^2 - z_b^2\right)$

Energy balance for Saint-Venant (cont'd)



- What if

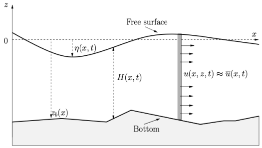
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + H\frac{\partial}{\partial x}\left(4\nu\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa\bar{u}}{1+\frac{\kappa}{3\nu}H}$$

- Rheology terms

$$\bar{u} \times H\frac{\partial}{\partial x}\left(4\nu\frac{\partial \bar{u}}{\partial x}\right) = \frac{\partial}{\partial x}\left(4\nu H\bar{u}\frac{\partial \bar{u}}{\partial x}\right) - 4\nu\frac{\partial (H\bar{u})}{\partial x}\frac{\partial \bar{u}}{\partial x}$$
$$= \frac{\partial}{\partial x}\left(4\nu H\bar{u}\frac{\partial \bar{u}}{\partial x}\right) - 4\nu H\left(\frac{\partial \bar{u}}{\partial x}\right)^2 - 4\nu\bar{u}\frac{\partial H}{\partial x}\frac{\partial \bar{u}}{\partial x}$$

- What if any coefficient is modified !

A passive tracer



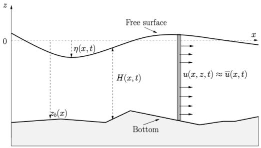
- A sort of "ink" whose concentration is $T(x, z, t)$
  `run:anim/upwelling_aeres1.avi`
- Governing equation

$$\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + w\frac{\partial T}{\partial z} = \mu_T \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

or equivalently $(+T\ \nabla \cdot \mathrm{u})$

$$\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (wT)}{\partial z} = \mu_T \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

A passive tracer (cont'd)



- Governing equation

$$\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (wT)}{\partial z} = \mu_T \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

- A vertical integration from z_b to η

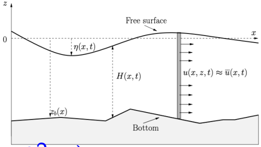
$$0 = \int_{z_b}^{\eta} \left(\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (wT)}{\partial z} \right) dz =$$

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} T dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} (uT) dz - T_s \left(\frac{\partial \eta}{\partial t} + u_s \frac{\partial \eta}{\partial x} - w_s \right) + T_b \left(u_b \frac{\partial z_b}{\partial x} - w_b \right)$$

- The shallow water assumption ($\mu_T = 0$)

$$\frac{\partial (H\bar{T})}{\partial t} + \frac{\partial (H\bar{u}\bar{T})}{\partial x} = 0 \qquad \text{with } H\bar{T} = \int_{z_b}^{\eta} T dz$$

A passive tracer (cont'd)



- Governing equation

$$\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (wT)}{\partial z} = \mu_T \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

- Boundary conditions

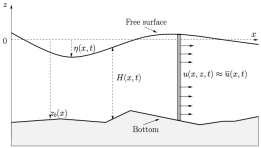
$$\left.\frac{\partial T}{\partial \underline{n}}\right|_s = \left.\frac{\partial T}{\partial \underline{n}}\right|_b = 0 \qquad (\alpha(T_0 - T))$$

or equivalently $\nabla T|_s \cdot \underline{n}_s = \nabla T|_b \cdot \underline{n}_b = 0$

- A vertical integration from z_b to z

$$\mu_T \int_{z_b}^{\eta} \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) dz = \mu_T \frac{\partial}{\partial x} \int_{z_b}^{\eta} \left(\frac{\partial T}{\partial x} \right) dz$$
$$- \mu_T \frac{\partial \eta}{\partial x} \left.\frac{\partial T}{\partial x}\right|_s + \mu_T \left.\frac{\partial T}{\partial z}\right|_s + \mu_T \frac{\partial z_b}{\partial x} \left.\frac{\partial T}{\partial x}\right|_b - \mu_T \left.\frac{\partial T}{\partial z}\right|_b$$

A passive tracer (cont'd)



- Governing equation

$$\frac{\partial T}{\partial t} + \frac{\partial (uT)}{\partial x} + \frac{\partial (wT)}{\partial z} = \mu_T \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right)$$

- Finally

$$\frac{\partial (H\bar{T})}{\partial t} + \frac{\partial (H\bar{u}\bar{T})}{\partial x} = \frac{\partial}{\partial x} \left(\mu_T H \frac{\partial \bar{T}}{\partial x} \right)$$

or equivalently

$$\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} = \frac{1}{H}\frac{\partial}{\partial x} \left(\mu_T H \frac{\partial \bar{T}}{\partial x} \right)$$

The vertical velocity

- The divergence free condition

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

- After an integration in z over $[z_b, z]$

$$\int_{z_b}^{z} \frac{\partial u}{\partial x} dx + w - w_b = 0$$

and using the Leibniz rule

$$w = w_b - \int_{z_b}^{z} \frac{\partial u}{\partial x} dx = w_b - u_b \frac{\partial z_b}{\partial x} - \frac{\partial}{\partial x} \int_{z_b}^{z} u dx = -\frac{\partial}{\partial x} \int_{z_b}^{z} u dx$$

The vertical velocity (cont'd)

- The divergence free condition is equivalent to

$$w = -\frac{\partial}{\partial x}\int_{z_b}^{z} u dx$$

- The shallow water regime $u = \bar{u} + \mathcal{O}(\varepsilon)$

$$w = -\frac{\partial}{\partial x}\int_{z_b}^{z} \bar{u} dx + \mathcal{O}(\varepsilon) = -\frac{\partial}{\partial x}\big((z - z_b)\bar{u}\big) + \mathcal{O}(\varepsilon)$$
$$= -(z - z_b)\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial z_b}{\partial x} + \mathcal{O}(\varepsilon)$$

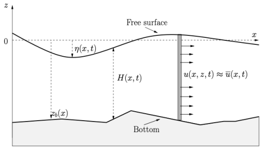
- An integration from z_b to η gives

$$\int_{z_b}^{\eta} w dz = -\frac{H^2}{2}\frac{\partial \bar{u}}{\partial x} + H\bar{u}\frac{\partial z_b}{\partial x} + \mathcal{O}(\varepsilon)$$

- A choice

$$H\bar{w} = -\frac{H^2}{2}\frac{\partial \bar{u}}{\partial x} + H\bar{u}\frac{\partial z_b}{\partial x} \quad \text{i.e.} \quad \bar{w} = -\frac{H}{2}\frac{\partial \bar{u}}{\partial x} + \bar{u}\frac{\partial z_b}{\partial x}$$

A physical limitation



$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}\right) = 0$$

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x}$$

- The slope of the topography

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z}_b = \frac{z_b}{h}$$

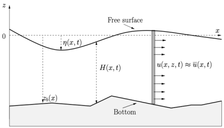
leading to

$$\frac{\partial z_b}{\partial x} = \varepsilon \frac{\partial \tilde{z}_b}{\partial \tilde{x}} = \mathcal{O}(\varepsilon)$$

Analytical solutions

- Stationary analytical solutions (anim)
  - continuous
  - with shocks
- Time-dependent analytical solutions
  - dam-break
  - double rarefaction (anim)
  - parabolic bowl (Thacker) (anim)

A nonlinear system of cons. laws



- The Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$

$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial (H\bar{u}^2)}{\partial x} + \frac{g}{2}\frac{\partial H^2}{\partial x} = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1+\frac{\kappa}{3\nu}H}$$

- A rewritting $\frac{\partial X}{\partial t} + \frac{\partial}{\partial x}F(X) = S_b(X) + S_{f,v}(X)$

  where $F(X) = \begin{pmatrix} H\bar{u} \\ H\bar{u}^2 + \frac{g}{2}H^2 \end{pmatrix}$, $X = \begin{pmatrix} H \\ H\bar{u} \end{pmatrix}$

- Non conservtive form

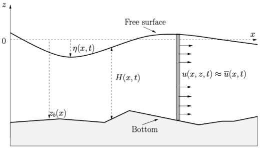
$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + g\frac{\partial H}{\partial x} = -g\frac{\partial z_b}{\partial x} + \frac{1}{H}\frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{H\left(1+\frac{\kappa}{3\nu}H\right)}$$

- Quasi-linear form $\frac{\partial Y}{\partial t} + A(Y)\frac{\partial Y}{\partial x} = \tilde{S}_b(Y) + \tilde{S}_f(Y)$

  with $A(Y) = \begin{pmatrix} \bar{u} & H \\ g & \bar{u} \end{pmatrix}$, $\quad Y = \begin{pmatrix} H \\ \bar{u} \end{pmatrix}$

A nonlinear system of cons. laws (cont'd)



- Non conservtive form

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$

$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + g\frac{\partial H}{\partial x} = -g\frac{\partial z_b}{\partial x} + \frac{1}{H}\frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{H\left(1 + \frac{\kappa}{3\nu}H\right)}$$

- Quasi-linear form

$$\frac{\partial Y}{\partial t} + A(Y)\frac{\partial Y}{\partial x} = \cancel{\tilde{S}_b(Y)} + \cancel{\tilde{S}_f(Y)}$$

with $A(Y) = \begin{pmatrix} \bar{u} & H \\ g & \bar{u} \end{pmatrix}$

- Eigenvalues/eigenvectors

$$\begin{vmatrix} \bar{u} - \lambda & H \\ g & \bar{u} - \lambda \end{vmatrix} = 0 \qquad \rightarrow \qquad \lambda_{\pm} = \bar{u} \pm \sqrt{gH}$$

Riemann invariants

- A very simple case

$$\frac{\partial Y}{\partial t} + A_0 \frac{\partial Y}{\partial x} = 0 \qquad (^*)$$

- If A_0 is diagonalizable over $\mathbb{R}$ then

$$P^{-1} A_0 P = A_0' = \mathrm{d}iag(\lambda_1, \ldots, \lambda_N)$$

- First step $P^{-1} \times (^*)$ gives

$$\frac{\partial (P^{-1} Y)}{\partial t} + P^{-1} A_0 \frac{\partial Y}{\partial x} = 0$$

- Second step $P^{-1} P = I_N$ hence

$$\frac{\partial (P^{-1} Y)}{\partial t} + P^{-1} A_0 P \frac{\partial (P^{-1} Y)}{\partial x} = 0 \quad \Leftrightarrow \quad \frac{\partial \tilde{Y}}{\partial t} + A_0' \frac{\partial \tilde{Y}}{\partial x} = 0$$

- Finally

$$\frac{\partial \tilde{Y}_i}{\partial t} + \lambda_i \frac{\partial \tilde{Y}_i}{\partial x} = 0 \quad \Leftrightarrow \quad \tilde{Y}_i = f_i(x - \lambda_i t)$$

Riemann invariants (cont'd)

- In practice

$$\frac{\partial Y}{\partial t} + \begin{pmatrix} \bar{u} & H \\ g & \bar{u} \end{pmatrix} \frac{\partial Y}{\partial x} = 0 \qquad (^*)$$

- Let us assume $\frac{\partial R_+}{\partial t} + \lambda_+ \frac{\partial R_+}{\partial x} = 0$ with $R_+ = R_+(Y)$ then

$$\nabla_Y R_+ \cdot \frac{\partial Y}{\partial t} + \lambda_+ \nabla_Y R_+ \cdot \frac{\partial Y}{\partial x} = 0$$

- Using $(^*)$, it comes

$$\nabla_Y R_+ \cdot (\lambda_+ I_N - A) \frac{\partial Y}{\partial x} = 0$$

- Necessarily $\nabla_Y R_+$ is a left eigenvector of A

$$\lambda_\pm = u \pm \sqrt{gH}, \qquad \nabla R_\pm = \begin{pmatrix} \pm\sqrt{g} \\ \sqrt{H} \end{pmatrix}.$$

Riemann invariants (cont'd)

- Quasilinear form for SV (without topography)

$$\frac{\partial H}{\partial t} + \bar{u}\frac{\partial H}{\partial x} + H\frac{\partial \bar{u}}{\partial x} = 0$$
$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + g\frac{\partial H}{\partial x} = 0$$

- Equivalently ($c = \sqrt{gH}$)

$$\frac{\partial (2c)}{\partial t} + \bar{u}\frac{\partial (2c)}{\partial x} + c\frac{\partial \bar{u}}{\partial x} = 0$$
$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + c\frac{\partial (2c)}{\partial x} = 0$$

- The sum/difference of the two above equaions gives

$$\frac{\partial (\bar{u}+2c)}{\partial t} + (\bar{u}+c)\frac{\partial (\bar{u}+2c)}{\partial x} = 0$$
$$\frac{\partial (\bar{u}-2c)}{\partial t} + (\bar{u}-c)\frac{\partial (\bar{u}-2c)}{\partial x} = 0$$

Riemann invariants (cont'd)

- The Saint-Venant system

$$\frac{\partial Y}{\partial t} + A(Y)\frac{\partial Y}{\partial x} = 0$$

  with eigenvalues $\lambda_\pm$, (right) eigenvectors $r_\pm$

- Definition GNL (resp. LD) fields

$$\nabla\lambda_\pm . r_\pm \neq 0 \qquad (\text{resp.} \nabla\lambda_\pm . r_\pm = 0) \quad \forall Y$$

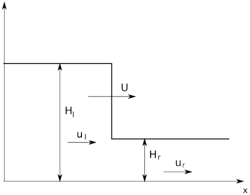
- The Saint-Venant system with a tracer

$$\frac{\partial H}{\partial t} + \bar{u}\frac{\partial H}{\partial x} + H\frac{\partial \bar{u}}{\partial x} = 0$$
$$\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + g\frac{\partial H}{\partial x} = 0$$
$$\frac{\partial T}{\partial t} + \bar{u}\frac{\partial T}{\partial x} = 0$$

- Eigenvalues $\bar{u} - \sqrt{gH}, \bar{u}, \bar{u} + \sqrt{gH}$ and $\nabla_Y \bar{u} \cdot r_{\bar{u}} = 0$ since

$$\nabla_Y \bar{u} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \qquad r_{\bar{u}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Particular solutions



- Discontinuous solutions
  - shocks

$$X(x,t) = \begin{cases} X_l & \text{if } \frac{x}{t} < U \\ X_r & \text{if } \frac{x}{t} > U \end{cases}$$

  - the Rankine-Hugoniot condition

$$-U(X_r - X_l) = F(X_r) - F(X_l)$$

  - dissipation
- Analytical solutions
  - "lake at rest"
  - stationary solutions
  - time dependent (flat/non flat bottom) (anim)
  - non unique analytical solutions

The topography

- Hydrodynamic regimes

$$|\bar{u}| < \sqrt{gH}, \qquad |\bar{u}| > \sqrt{gH} \quad \text{and} \quad |\bar{u}| = \sqrt{gH}$$

- The topography as an unknow

$$\begin{aligned} &\frac{\partial z_b}{\partial t} = 0, \\ &\frac{\partial H}{\partial t} + \frac{\partial H\bar{u}}{\partial x} = 0, \\ &\frac{\partial H\bar{u}}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + gH^2/2\right) + gH\frac{\partial z_b}{\partial x} = 0. \end{aligned}$$

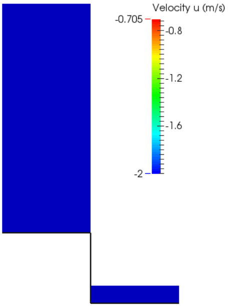
- Eigenvalues associated with $\lambda_0 = 0$ and $\lambda_\pm = \overline{u} \pm \sqrt{gH}$
- Strictly hyperbolic if $|\overline{u}| \neq \sqrt{gH}$ else *resonant* ...very complex

Multiple solutions : HR vs. modified HR

- Eigenelements (eigenvalues $\lambda_{\pm} = \overline{u} \pm \sqrt{gH}$, $\lambda_0 = 0$)

$$l_{\pm} = \begin{pmatrix} 0 \\ 1 \\ \overline{u} \pm \sqrt{gH} \end{pmatrix} \text{ and } l_0 = \begin{pmatrix} gH - \overline{u}^2 \\ -gH \\ 0 \end{pmatrix},$$

- Analytical solutions (Andrianov - Seguin)



Asymptotic expansion

- A model (typically a PDE)

$$\mathcal{P}(\underline{u}, \varepsilon) = 0$$

$\underline{u}$ is the unknown and $\varepsilon \ll 1$ a small parameter

- An expansion $\underline{u} = \underline{u}_0 + \varepsilon \underline{u}_1 + \varepsilon^2 \underline{u}_2 + \dots$
- A cascade of models

$$\mathcal{P}(\underline{u}_0, 0) = 0, \qquad \mathcal{P}_1(\underline{u}_1, \underline{u}_0) = 0, \qquad \mathcal{P}_2(\underline{u}_2, \underline{u}_1, \underline{u}_0) = 0 \quad \dots$$

- In our domains : several small parameters
  - shallowness of the fluid domain $\frac{H_0}{L} \ll 1$
  - the Froude number $Fr = \frac{\bar{u}}{\sqrt{gH}} \ll 1$
  - the gravity waves & the acoustic waves $\frac{\sqrt{gH}}{c_{sound}} \ll 1$
  - ...

Weaknesses of asymptotic expansions

- Existence of solutions for the cascade of models ?
  - analysis results required for the sequence of sub-problems

$$\mathcal{P}_i(\underline{u}_i, \{\underline{u}_j\}_{0\leq j<i}) = 0$$

  - loss of some properties (conservation, symmetry...) for the sub-problems
- Rate of convergence of the expansion ?

$$\left\| \underline{u} - \sum_{i=0}^{N} \varepsilon^i \underline{u}_i \right\| = \mathcal{O}\left(\frac{1}{N^p}\right)$$

A possible way : weak/variational asymptotic expansion

- An expansion $\underline{u}^w = \underline{u}_0^w + \underline{u}_1^w + \underline{u}_2^w + \ldots$
- Weak form

$$\int_\Omega \varphi_i \mathcal{P}(\underline{u}^w, \varepsilon) d\omega = 0 \qquad i \in \{0, 1, 2, \ldots\}$$

Asymptotic expansion : 1^{st} example

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} + \kappa u = 0$$

with $\kappa = \varepsilon\kappa_1$ and $\varepsilon \ll 1$

- General solution of the PDE (for any f_0 and $\tilde{f}_0$)

$$u(x,t) = f_0(x-ct)e^{-\frac{\kappa}{c}x} + \tilde{f}_0(x-ct)e^{-\kappa t}$$

- Using asymptotic expansion $u_0 + \varepsilon u_1$, the solutions have to satisfy

$$\begin{cases} \frac{\partial u_0}{\partial t} + c\frac{\partial u_0}{\partial x} = 0 \\ \frac{\partial u_1}{\partial t} + c\frac{\partial u_1}{\partial x} + \kappa_1 u_0 = 0 \end{cases}$$

- The solutions are

$$\begin{cases} u_0(x,t) = F_0(x-ct) \\ u_1(x,t) = -\frac{\kappa_1}{c}xF_0(x-ct) + \tilde{F}_0(x-ct) \end{cases}$$

- Only choice $\begin{cases} F_0(x-ct) = f_0(x-ct) + \tilde{f}_0(x-ct) \\ \tilde{F}_0(x-ct) = -\frac{\kappa_1}{c}(x-ct)\tilde{f}_0(x-ct) \end{cases}$

BUT $u \approx u_0 + \varepsilon u_1$ only for $\varepsilon \ll 1$, x, t small

Asymptotic expansion : 1^{st} example (cont'd)

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = -\kappa u, \qquad \kappa = \varepsilon\kappa_1,\ \varepsilon \ll 1$$

- A simple solution

$$u(x,t) = \sin(x-ct)e^{-\frac{\kappa}{c}x}$$

- Asymptotic expansion $u \approx u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots$
- Cascade of models

$$\frac{\partial u_i}{\partial t} + c\frac{\partial u_i}{\partial x} = -\kappa_1 u_{i-1}$$

- The solutions are

$$u_i = (-1)^i \frac{\kappa_1^i x^i}{i!c^i}\sin(x-ct) \quad (+f_i(x-ct))$$

- Leading to

$$\lim_{N\to+\infty}\sum_{i=0}^{N}\varepsilon^i u_i(x,t) = \lim_{N\to+\infty}\sin(x-ct)\sum_{i=0}^{N}(-1)^i\frac{\varepsilon^i\kappa_1^i x^i}{i!c^i} = u(x,t)$$

- BUT: the energy balances ! $\quad (-\kappa_1 u_{i-1}u_i \geq 0)$

Asymptotic expansion : 2^{nd} example

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \mu\frac{\partial^2 u}{\partial x^2} = 0 \qquad (*)$$

with $\mu = \varepsilon\mu_1$ and $\varepsilon \ll 1$

- Using asymptotic expansion $u_0 + \varepsilon u_1$, the solutions have to satisfy

$$\begin{cases} \frac{\partial u_0}{\partial t} + u_0\frac{\partial u_0}{\partial x} = 0, \\ \frac{\partial u_1}{\partial t} + \frac{\partial(u_0 u_1)}{\partial x} = \mu_1\frac{\partial^2 u_0}{\partial x^2}, \end{cases}$$

- Some solutions are

$$\begin{cases} u_0(x,t) = \frac{x-x_0}{t-t_0} \\ u_1(x,t) = \frac{1}{x-x_0} f_1\left(\frac{t-t_0}{x-x_0}\right) \end{cases}$$

for any function f_1

Conclusion

- u_0 is solution of the general equation (*)
- $u \not\approx u_0 + \varepsilon u_1$ because u_1 is not necessarily bounded when $t \to t_0$, $t \ll 1$, $x \to x_0$, $x \ll 1$

Asymptotic expansion : 3^{rd} example

$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = 0$$
$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{H}\right) + gH\frac{\partial H}{\partial x} = 0$$

- Using asymp. expansion $q_0 + \varepsilon q_1 + \varepsilon^2 q_2$, $H_0 + \varepsilon H_1 + \varepsilon^2 H_2$, we get
  - $H_0 = cst$, $q_0 = 0$
  - in ε
$$\frac{\partial h_1}{\partial t} + \frac{\partial q_1}{\partial x} = 0$$
$$\frac{\partial q_1}{\partial t} + gH_0\frac{\partial h_1}{\partial x} = 0$$
  - in ε^2
$$\frac{\partial h_2}{\partial t} + \frac{\partial q_2}{\partial x} = 0$$
$$\frac{\partial q_2}{\partial t} + gH_0\frac{\partial h_2}{\partial x} = -\frac{1}{H_0}\frac{\partial q_1^2}{\partial x} - \frac{g}{2}\frac{\partial h_1^2}{\partial x}$$

Asymptotic expansion : 3^{rd} example (cont'd)

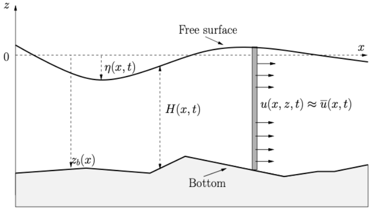
$$\frac{\partial H}{\partial t} + \frac{\partial q}{\partial x} = 0$$
$$\frac{\partial q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{q^2}{H}\right) + gH\frac{\partial H}{\partial x} = 0$$

- Using asymp. expansion $u_0 + \varepsilon u_1 + \varepsilon^2 u_2$, $H_0 + \varepsilon H_1 + \varepsilon^2 H_2$, we get (with $c_0 = \sqrt{gH_0}$)
  - $H_0 = cst$, $q_0 = 0$
  - in ε
  $$h_1(x,t) = f_1(x - c_0 t) + f_2(x + c_0 t)$$
  $$q_1(x,t) = c_0\left(f_1(x - c_0 t) - f_2(x + c_0 t)\right)$$
  - in ε^2 with $f_1(x) = \sin(x)$, $f_2(x) = 0$
  $$h_2 = f_3(x - c_0 t) + f_4(x + c_0 t) + \frac{3x}{4H_0}\sin(2(x - c_0 t))$$
  $$q_2 = c_0\left(f_3(x - c_0 t) - f_4(x + c_0 t)\right) + \frac{3g}{8c_0}\left(\cos(2(x - c_0 t))\right.$$
  $$\left. + 2x\sin(2(x - c_0 t))\right) + f_5(t)$$

Models simpler than Saint-Venant

- The Saint-Venant system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0,$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x}$$



- The wave equation

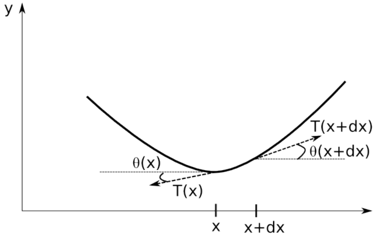
$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gH_0\frac{\partial \eta}{\partial x}\right) = 0$$

- 3 simulations

The wave equation

- FPD

$$\rho dx \frac{\partial^2 y}{\partial t^2} = T(x+dx)\sin(\theta(x+dx,t)) - T(x)\sin(\theta(x,t))$$



- $T \approx T_0 = cste$ and $dx \to 0$

$$\rho dx \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial \sin(\theta(x,t))}{\partial x} dx$$

- But $\theta \ll 1$

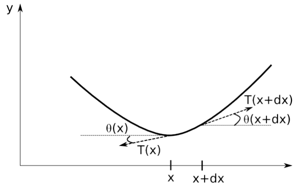
$$\theta(x,t) \approx \frac{\delta y}{\delta x} = \frac{\partial y}{\partial x} \quad \text{and} \quad \sin\theta \approx \theta$$

- The obtained equation

$$\rho \frac{\partial^2 y}{\partial t^2} = T_0 \frac{\partial^2 y}{\partial x^2}$$

or equivalently $\frac{\partial^2 y}{\partial t^2} = c_0^2 \frac{\partial^2 y}{\partial x^2}$ with $c_0 = \sqrt{\frac{T_0}{\rho}}$

The wave equation (cont'd)



- Governing equation

$$\frac{\partial^2 y}{\partial t^2} = c_0^2 \frac{\partial^2 y}{\partial x^2}$$

- Fondamental solution (d'Alembert)

$$y(x,t) = f_1(x - c_0 t) + f_2(x + c_0 t)$$

- Dispersion relation / dispersive media

$$f(x,t) = Ae^{i(kx - \omega t)}$$

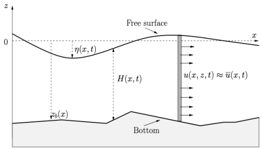
leading to the dispersion relation $\frac{\omega}{k} = c_0$ $\quad ((-\omega)^2 i^2 A - c_0^2 i^2 k^2)\, e^{i(kx-\omega t)}$

- Damped waves

$$\frac{\partial^2 y}{\partial t^2} + \lambda \frac{\partial y}{\partial t} - c_0^2 \frac{\partial^2 y}{\partial x^2} = 0 \qquad (1)$$

dispersive or not ? What are the solutions of Eq. (1) ?

Models simpler than Saint-Venant



- The Saint-Venant system

$$(i) \quad \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$

$$(ii) \quad \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x}$$

- Froude number

$$Fr = \frac{\bar{u}}{\sqrt{gH}} \qquad \left(\ll 1 \text{ for wave regime i.e. } \bar{u}^2 \ll gH\right)$$

- Approximate momentum equation

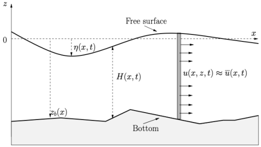
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(\frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x}$$

or equivalently

$$\frac{\partial (H\bar{u})}{\partial t} + gH\frac{\partial (H + z_b)}{\partial x} = 0$$

Models simpler than Saint-Venant (cont'd)

- The modified Saint-Venant system



$$(i) \quad \frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$

$$(ii) \quad \frac{\partial (H\bar{u})}{\partial t} + gH\frac{\partial (H + z_b)}{\partial x} = 0$$

- $\frac{\partial (i)}{\partial t} - \frac{\partial (ii)}{\partial x}$ gives

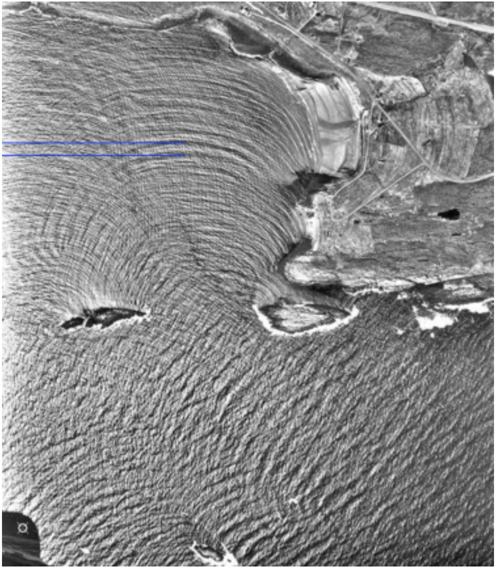
$$\frac{\partial^2 H}{\partial t^2} - \frac{\partial}{\partial x}\left(gH\frac{\partial (H + z_b)}{\partial x}\right) = 0$$

- $H \approx H_0(x)$ and $\eta = H + z_b$

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gH_0\frac{\partial \eta}{\partial x}\right) = 0 \qquad \left(\frac{\partial^2 \eta}{\partial t^2} - g\bar{H}_0\frac{\partial^2 \eta}{\partial x^2} = 0\right)$$

- 3 simulations

Gravity waves



Stationary waves

- For the wave equation (stationary_wave1.avi)

$$\frac{\partial^2 \eta}{\partial t^2} - c_0^2 \frac{\partial^2 \eta}{\partial x^2} = 0$$

- Explanation

$$\eta_0 \sin(x - c_0 t) + \eta_0 \sin(x + c_0 t) = 2\eta_0 \sin(x) \sin(c_0 t)$$

- For the wave equation (with topography) (stationary_wave2.avi)

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left(gH_0 \frac{\partial \eta}{\partial x}\right) = 0$$

- For the Saint-Venant system (stationary_wave3.avi)

Beyond shallow water type models

The Saint-Venant system is

- widely used in 1d and 2d
- a good approximation of NS for long wave phenomena
- a good model for river flows,. . .

It can be

- enriched with several terms (curvature, stiff topography,. . .)
- used in many context
  - blood flows
  - traffic flows

■ Some remaining difficulties arround the Shallow Water syst. but scientific challenges and real-life applications often concern more complex models.

■ Applications to social sciences, humanities

Non-hydrostatic models

$$\left\{\begin{array}{l} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \rho_0 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z}\right) + \frac{\partial p}{\partial x} = 0 \\ \rho_0 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z}\right) + \frac{\partial p}{\partial z} = -\rho_0 g \end{array}\right.$$

- Illustrations
  - (Hydro vs. non-hydro), (Wave over a beach), (Mascaret)
- An extensive literature
  - many models/results (Bona, Boussinesq, Green-Nagdhi, Peregrine / Lannes, Saut, Duchêne, Kazerani)
  - no more hyperbolic systems
  - often based on irrotational flows
- Derivation of SW non-hydrostatic models
  - a classical tool : asymptotic expansion
  - but strong limitations
- No more compressible fluid mechanics
  - p : lagrange multiplier

Averaged Euler (M3AS 2011, DCDS 2015)

- The Euler system
$$\left\{\begin{array}{l}\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0\\ \frac{\partial u}{\partial t}+\frac{\partial u^2}{\partial x}+\frac{\partial uw}{\partial z}+\frac{\partial p}{\partial x}=0\\ \frac{\partial w}{\partial t}+\frac{\partial uw}{\partial x}+\frac{\partial w^2}{\partial z}+\frac{\partial p}{\partial z}=-g\end{array}\right.$$
- Boundary conditions
  - kinematic (bottom + free surface), dynamical ($p_s = p^a$)
- Energy equality: a constraint

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E\ dz+\frac{\partial}{\partial x}\int_{z_b}^{\eta} u\left(E+p\right)dz=0,\qquad E=\frac{u^2+w^2}{2}+gz$$

☐ A dispersive shallow water system

$$\left\{\begin{array}{l}\frac{\partial H}{\partial t}+\frac{\partial (H\bar{u})}{\partial x}=0\\ \frac{\partial (H\bar{u})}{\partial t}+\frac{\partial}{\partial x}\left(H\bar{u}^2+\frac{g}{2}H^2+H\overline{p}_{nh}\right)=-\left(gH+2\overline{p}_{nh}\right)\frac{\partial z_b}{\partial x}\\ \frac{\partial (H\bar{w})}{\partial t}+\frac{\partial (H\bar{w}\bar{u})}{\partial x}=2\overline{p}_{nh}\\ \bar{w}=-\frac{H}{2}\frac{\partial \bar{u}}{\partial x}+\frac{\partial z_b}{\partial x}\bar{u}\end{array}\right.$$

The rotating shallow water system

- Saint-Venant + Coriolis

$$\frac{\partial h}{\partial t} + \nabla_{x,y} \cdot (h\mathrm{u}) = 0,$$
$$\frac{\partial (h\mathrm{u})}{\partial t} + \nabla_{x,y} \cdot (h\mathrm{u} \otimes \mathrm{u}) + \nabla_{x,y}\left(\frac{g}{2}h^2\right) = -gh\nabla_{x,y} z_b - \Omega h\mathrm{u}^{\perp}$$

- A source term without any influence over the energy balance
- Some characteristic / analytical solutions

The rotating shallow water system (cont'd)

Let f be any real value function, h_0 a non negative constant and z_b^0 a constant. Then the variables h, u, v, w, p defined by

$$
\begin{aligned}
h(t,x,y) &= h_0 + \frac{1}{2g}\int_0^{x^2+y^2} f(z)\big(f(z)+\Omega\big)dz \\
u(t,x,y,z) &= yf(x^2+yr) \\
v(t,x,y,z) &= -xf(x^2+y^2) \\
w(t,x,y,z) &= -\partial_x((z-z_b)u) - \partial_y((z-z_b)v) \\
p(t,x,y,z) &= \rho_0 g(h+z_b-z) \\
z_b(x,y) &= z_b^0 - \frac{\Omega}{g}\int_0^{x^2+y^2} f(z)dz
\end{aligned}
$$

are analytical solutions of the rotating hydrostatic Euler system. Since the expressions for u and v do not depend on the variable z, the proposed analytical solution is also an analytical solution for the Saint-Venant system with Coriolis.

The rotating shallow water system (cont'd)