From Navier-Stokes to Saint-Venant

Course 1

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https://team.inria.fr/ange/course-materials/

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Eulerian vs. Lagrangian description

- Euler

$$\underline{u}(x, y, z, t)$$

- Lagrange

$$M(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad \frac{dM}{dt} = \underline{u}(M(t), t)$$

flow of the trajectories

$$\begin{aligned} \varphi_t \ : \ \mathbb{R}^3 &\longrightarrow \mathbb{R}^3 \\ M(t_0) &\mapsto M(t) \end{aligned}$$

- Relation Lagrangian/Eulerian description

$$\frac{Df}{Dt} = \frac{d}{dt} f(M(t), t) = \frac{\partial f}{\partial t} + \underline{u}.\nabla f.$$

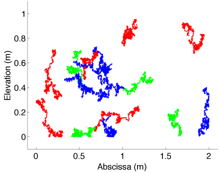
- Fluid mechanics vs. solid mechanics
  - "particle position"
  - time constant

Illustrations

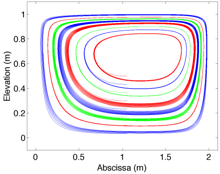
- The wind-driven cavity (Eulerian description)

(anim/advect_wind_tracer.avi) (anim/advect_diff_wind_tracer.avi)

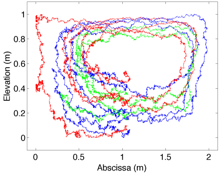
- Lagrange



$$dM = d\omega_t$$



$$dM = \underline{u}dt$$



$$dM = \underline{u}dt + d\omega_t$$

Origins of the Euler/NS system

- Mass within a volume V

$$m = \iiint_V \rho dv$$

- Mass conservation

$$0 = \frac{dm}{dt} = \iiint_V \frac{\partial \rho}{\partial t} dv + \oiint_S \rho \underline{u}.\underline{ds}$$

- Green-Ostrogradsky formula

$$\oiint_S \rho \underline{u}.\underline{ds} = \iiint_V \operatorname{div}(\rho \underline{u}) dv$$

- local mass conservation equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$$

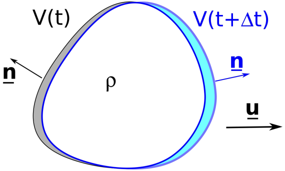
- When $\rho = cst$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Origins of the Euler/NS system (cont'd)

- An "intuitive" explanation

$$\lim_{\Delta t \to 0} \frac{\iiint_{V(t+\Delta t)} \rho dv - \iiint_{V(t)} \rho dv}{\Delta t} = \oiint_S \rho \underline{u} \cdot \underline{ds}$$



- Green-Ostrogradsky formula

$$\iiint_V \operatorname{div}(\rho \underline{u}) dv = \oiint_S \rho \underline{u} . \underline{ds}$$

nothing else than (in higher dimension)

$$\int_a^b \frac{df}{dx} dx = [f(x)]_a^b = f(b) - f(a)$$

Origins of the Euler/NS system (cont'd)

- Divergence free condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Variation of velocity

$$d\underline{u} = \underline{u}(x + udt, y + vdt, z + wdt, t + dt) - \underline{u}(x, y, z, t)$$

i.e.

$$d\underline{u} = \frac{\partial \underline{u}}{\partial x} udt + \frac{\partial \underline{u}}{\partial y} vdt + \frac{\partial \underline{u}}{\partial z} wdt + \frac{\partial \underline{u}}{\partial t} dt$$

- Acceleration $\underline{a}$ defined by $d\underline{u} = \underline{a}dt$

$$\underline{a} = \frac{\partial \underline{u}}{\partial t} + u\frac{\partial \underline{u}}{\partial x} + v\frac{\partial \underline{u}}{\partial y} + w\frac{\partial \underline{u}}{\partial z}$$

- Fundamental law of dynamics

$$\rho \underline{a} - \mathrm{div}\,(\sigma_T) = \rho \underline{g}$$

with $\sigma_T = -pI_d + \sigma$

Origins of the Euler/NS system (cont'd)

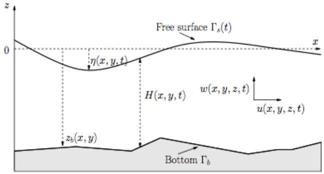
- Small displacement (with $\underline{u} = (u, v, w)^T$)

$$dx = udt, \quad dy = vdt, \quad dz = wdt$$

- Variation of velocity

$$\begin{aligned}
d\underline{u} =& \underline{u}(x + dx, y + dy, z + dz, t + dt) - \underline{u}(x, y, z, t) \\
=& \underline{u}(x + udt, y + vdt, z + wdt, t + dt) - \underline{u}(x, y, z, t) \\
=& \underline{u}(x + udt, y + vdt, z + wdt, t + dt) - \underline{u}(x, y + vdt, z + wdt, t + dt) \\
& + \underline{u}(x, y + vdt, z + wdt, t + dt) - \underline{u}(x, y, z, t) \\
=& udt\frac{\partial \underline{u}}{\partial x} + \underline{u}(x, y + vdt, z + wdt, t + dt) - \underline{u}(x, y, z, t) \\
=& udt\frac{\partial \underline{u}}{\partial x} + \underline{u}(x, y + vdt, z + wdt, t + dt) - \underline{u}(x, y, z + wdt, t + dt) \\
& + \underline{u}(x, y, z + wdt, t + dt) - \underline{u}(x, y, z, t) \\
=& udt\frac{\partial \underline{u}}{\partial x} + vdt\frac{\partial \underline{u}}{\partial y} + \underline{u}(x, y, z + wdt, t + dt) - \underline{u}(x, y, z, t) \\
& = \dots
\end{aligned}$$

The Navier-Stokes equations



- Equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

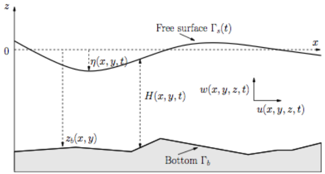
$$\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z},$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z},$$

with $\sigma_{xx} = 2\mu \frac{\partial u}{\partial x}$, $\sigma_{xz} = \sigma_{zx} = \mu \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right)$, $\sigma_{zz} = 2\mu \frac{\partial w}{\partial z}$

- Role of the pressure
- Completed with an energy equality
- Boundary conditions ?

The Navier-Stokes equations (cont'd)



- Kinematic boundary conditions
  - at the bottom

$$\frac{\partial z_b}{\partial x} u_b - w_b = 0 \qquad \left(\frac{\partial z_b}{\partial x} u_b + \frac{\partial z_b}{\partial y} v_b - w_b = 0 \right)$$

  - at the free surface

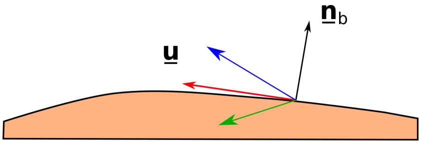
$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} u_s - w_s = 0$$

- Dynamic boundary conditions
  - $(\sigma - pI_d)n_s = -p^a(x,t)n_s + Wt_s$
  - $(\sigma - pI_d)n_b - \big(n_b.(\sigma - pI_d)n_b\big)n_b = \kappa \underline{u}_b$

    or equivalently $t_b.(\sigma - pI_d)n_b = \kappa \sqrt{1 + \left(\frac{\partial z_b}{\partial x}\right)^2} u_b$

Kinematic boundary conditions

- At the bottom $\underline{u}_b \cdot \underline{n}_b = 0$ with $\underline{u}_b = \begin{pmatrix} u(t,x,z_b(x)) \\ w(t,x,z_b(x)) \end{pmatrix}$



Normal to the bottom $x \mapsto z_b(x)$ given by $\underline{n}_b = \frac{1}{\sqrt{1+\left(\frac{\partial z_b}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial z_b}{\partial x} \\ 1 \end{pmatrix}$

- At the free surface $\frac{\partial \eta}{\partial t} + u_b \frac{\partial \eta}{\partial x} = w_s$ with $\eta = H + z_b$

$$\frac{D(z-\eta)}{Dt} = 0$$

but $\frac{D(z-\eta)}{Dt} = \frac{\partial(z-\eta)}{\partial t} + \underline{u}_s \cdot \nabla(z-\eta) = -\frac{\partial \eta}{\partial t} + \begin{vmatrix} u_s \\ w_s \end{vmatrix} \cdot \begin{vmatrix} -\frac{\partial \eta}{\partial x} \\ 1 \end{vmatrix}$

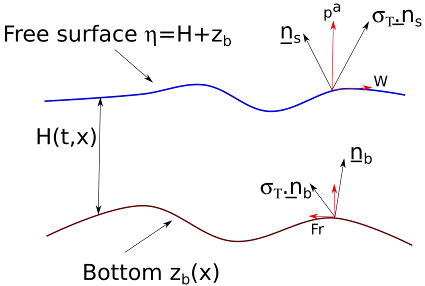
Dynamic boundary conditions

- The incompressible Navier-Stokes equations

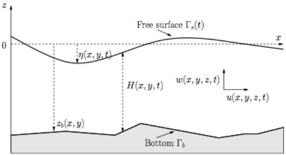
$$\begin{cases} \nabla.\underline{u} = 0 \\ \rho_0 \left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u} \right) + \nabla p = \nabla \cdot \sigma + \rho_0 \underline{g} \end{cases}$$

with

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{x,z} \\ \sigma_{zx} & \sigma_{zz} \end{pmatrix}, \qquad \sigma_T = \sigma - p1$$



The viscosity tensor : two equivalent expressions ?



- First version

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z},$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z},$$

with $\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}$, $\sigma_{xz} = \sigma_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$, $\sigma_{zz} = 2\mu\frac{\partial w}{\partial z}$

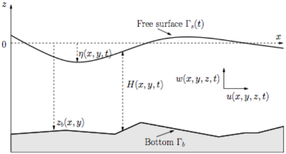
- Second version

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right),$$

The viscosity tensor : two equivalent expressions ?



- Second version

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2} \right),$$

$$\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial z^2} \right),$$

with $\tilde{\sigma}_{xx} = \mu \frac{\partial u}{\partial x}$, $\tilde{\sigma}_{xz} = \mu \frac{\partial u}{\partial z}$, $\tilde{\sigma}_{zx} = \mu \frac{\partial w}{\partial x}$, $\tilde{\sigma}_{zz} = \mu \frac{\partial w}{\partial z}$

- Or equivalently

$$\begin{cases} \nabla . \underline{u} = 0 \\ \rho_0 \left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla) \underline{u} \right) + \nabla p = \nabla \cdot \tilde{\sigma} + \rho_0 \underline{g} \end{cases}$$

and $\tilde{\sigma} = \begin{pmatrix} \tilde{\sigma}_{xx} & \tilde{\sigma}_{xz} \\ \tilde{\sigma}_{zx} & \tilde{\sigma}_{zz} \end{pmatrix}$ that is not symmetric !

- Moreover $\sigma \cdot \underline{n}_s \neq \tilde{\sigma} \cdot \underline{n}_s$

The Euler system

- Equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$
$$\rho_0 \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + w \frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0,$$
$$\rho_0 \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + w \frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g,$$

- Boundary conditions
  - kinematic (bottom + free surface), dynamical ($p_s = p^a$)
- Energy equality: a constraint

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} E \; dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u \left(E + p \right) dz = 0$$

with

$$E = \rho_0 \frac{u^2 + w^2}{2} + \rho_0 g z$$

- The Euler system and physical solutions ?

An exercise

- The incompressible Euler system with free surface

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$

$$(i) \quad \rho_0 \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} \right) + \frac{\partial p}{\partial x} = 0,$$

$$(ii) \quad \rho_0 \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} \right) + \frac{\partial p}{\partial z} = -\rho_0 g,$$

- Boundary conditions
  - $\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x} u_s - w_s = 0, \quad \frac{\partial z_b}{\partial x} u_b - w_b = 0, \quad p_s = p^a$
- Energy equality

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} E \; dz + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u\,(E+p)\, dz = 0$$

with $E = \rho_0 \frac{u^2+w^2}{2} + \rho_0 g z$

- Obtained from : $(i) \times u + (ii) \times w$

Correction

- Energy = momentum × velocity

$$\rho_0\left(\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}\right)+\frac{\partial p}{\partial x}\right)\times u=0$$

  or equivalently

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}u^2\right)+u\frac{\partial}{\partial x}\left(\frac{\rho_0}{2}u^2\right)+w\frac{\partial}{\partial z}\left(\frac{\rho_0}{2}u^2\right)+\frac{\partial(pu)}{\partial x}-p\frac{\partial u}{\partial x}=0$$

  and using the divergence free condition, it comes

$$(i)\quad \frac{\partial}{\partial t}\left(\frac{\rho_0}{2}u^2\right)+\frac{\partial}{\partial x}\left(u\frac{\rho_0}{2}u^2\right)+\frac{\partial}{\partial z}\left(w\frac{\rho_0}{2}u^2\right)+\frac{\partial(pu)}{\partial x}-p\frac{\partial u}{\partial x}=0$$

- Likewise for the momentum equation along z, we get

$$(ii)\quad \frac{\partial}{\partial t}\left(\frac{\rho_0}{2}w^2\right)+\frac{\partial}{\partial x}\left(u\frac{\rho_0}{2}w^2\right)+\frac{\partial}{\partial z}\left(w\frac{\rho_0}{2}w^2\right)+\frac{\partial(pw)}{\partial z}-p\frac{\partial w}{\partial z}=-\rho_0 g w$$

- The sum of *(i)* and *(ii)* gives

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}(u^2+w^2)\right)+\frac{\partial}{\partial x}\left(u\frac{\rho_0}{2}(u^2+w^2)\right)+\frac{\partial}{\partial z}\left(w\frac{\rho_0}{2}(u^2+w^2)\right)$$
$$+\frac{\partial(pu)}{\partial x}+\frac{\partial(pw)}{\partial z}=-\rho_0 g w$$

Correction (cont'd)

- But from the divergence free condition, we get

$$\frac{\partial(zu)}{\partial x} + \frac{\partial(zw)}{\partial z} = w \qquad \left(z\frac{\partial u}{\partial x} + z\frac{\partial w}{\partial z} = 0\right)$$

- Leading to

$$\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}(u^2+w^2) + \rho_0 g z\right) + \frac{\partial}{\partial x}\left(u\left(\frac{\rho_0}{2}(u^2+w^2) + \rho_0 g z + p\right)\right)$$
$$+ \frac{\partial}{\partial z}\left(w\left(\frac{\rho_0}{2}(u^2+w^2) + \rho_0 g z + p\right)\right) = 0$$

or equivalently

$$\frac{\partial E}{\partial t} + \frac{\partial\Big(u(E+p)\Big)}{\partial x} + \frac{\partial\Big(w(E+p)\Big)}{\partial z} = 0$$

- Energy equality (for smooth solutions)

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E \; dz + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u\,(E+p)\,dz = 0$$

with $E = \rho_0 \frac{u^2+w^2}{2} + \rho_0 g z$

The Leibniz's rule

$$\frac{\partial}{\partial x}\int_{a(x)}^{b(x)} f(x,z)dz = \int_{a(x)}^{b(x)} \frac{\partial f}{\partial x}dz + \frac{\partial b}{\partial x} f\left(x, b(x)\right) - \frac{\partial a}{\partial x} f\left(x, a(x)\right)$$

Starting from the vertical mean of the local energy balance

$$\int_{z_b}^{\eta}\left(\frac{\partial}{\partial t}\left(\frac{\rho_0}{2}(u^2+w^2)+\rho_0 gz\right)+\frac{\partial}{\partial x}\left(u\left(\frac{\rho_0}{2}(u^2+w^2)+\rho_0 gz+p\right)\right)\right)dz$$
$$+\int_{z_b}^{\eta}\frac{\partial}{\partial z}\left(w\left(\frac{\rho_0}{2}(u^2+w^2)+\rho_0 gz+p\right)\right)dz = 0$$

and applying the Leibniz's rule gives

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} E\ dz + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u\left(E+p\right)dz - \frac{\partial \eta}{\partial t}E_s - \frac{\partial \eta}{\partial x}u_S(E_s+p_s)$$
$$+\frac{\partial z_b}{\partial x}u_b(E_b+p_b) + w_s(E_s+p_s) - w_b(E_b+p_b) = 0$$

with $E = \rho_0\frac{u^2+w^2}{2} + \rho_0 gz$

Incompressible versus compressible fluids

- Compressible Navier-Stokes equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$
$$\frac{\partial (\rho \underline{u})}{\partial t} + \nabla \cdot (\rho \underline{u} \otimes \underline{u}) + \nabla p = \rho \mathrm{g} + \nabla \cdot \sigma$$
$$\frac{\partial E}{\partial t} + \nabla \cdot \Big(\underline{u}(E + p - \sigma) \Big) = -\nabla \cdot Q_T$$
$$f(\rho, T, p) = 0 \quad de = \frac{p}{\rho^2} d\rho + T ds$$

with $E = \rho \frac{|\underline{u}|^2}{2} + \rho e + \rho g z$

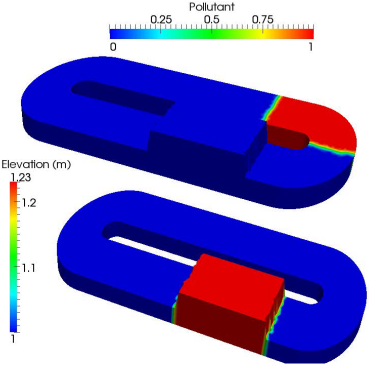
- Incompressible Navier-Stokes equations

$$\nabla \cdot \underline{u} = 0$$
$$\rho_0 \left(\frac{\partial \underline{u}}{\partial t} + \nabla \cdot (\underline{u} \otimes \underline{u}) \right) + \nabla p = \rho_0 \underline{g} + \nabla \cdot \sigma$$
$$\frac{\partial E}{\partial t} + \nabla \cdot \Big(\underline{u}(E + p - \sigma) \Big) = -\sigma : D(\underline{\varepsilon}) \qquad \text{(redundant)}$$

- Compressible $\leftrightarrow$ incompressible : singular limit

Free surface and compressible models

- We (often) consider incompressible fluids but because of the free surface, the models have compressible features
- Several velocities



Fluids with complex rheology

- Newtonian fluids

$$\sigma_{v,xx} = 2\mu\frac{\partial u}{\partial x}, \qquad \sigma_{v,xz} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}),$$
$$\sigma_{v,zz} = 2\mu\frac{\partial w}{\partial z}, \qquad \sigma_{v,zx} = \mu(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}),$$

- The Mohr-Coulomb criterion (anim/compar_rheol.avi)

$$\sigma_T = \sigma_N \tan(\phi) + c$$

  c: cohesion, ϕ: internal friction angle

- The Drucker-Prager criterion

$$\begin{cases} \sigma = \sigma_v + \kappa\frac{\sigma_v}{\|\sigma_v\|} & \text{if } \|\sigma_v\| \neq 0, \\ \|\sigma\| \leq \kappa & \text{else} \end{cases}$$

  with $\kappa = \sqrt{2}\lambda[p]_+$

- Also Herschel-Bulkley fluid,...

The Navier-Stokes equations

- Equations

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0,$$
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xz}}{\partial z},$$
$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} + \frac{\partial p}{\partial z} = -g + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zz}}{\partial z},$$

$$\sigma_{xx} = 2\mu\frac{\partial u}{\partial x}, \quad \sigma_{xz} = \sigma_{zx} = \mu\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big), \quad \sigma_{zz} = 2\mu\frac{\partial w}{\partial z},$$

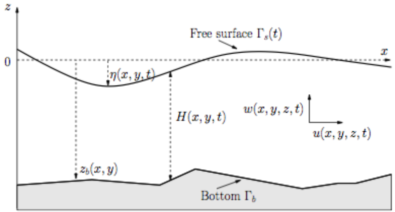
- Kinematic boundary conditions
  - at the bottom

$$\frac{\partial z_b}{\partial x}u_b - w_b = 0 \qquad \left(\frac{\partial z_b}{\partial x}u_b + \frac{\partial z_b}{\partial y}v_b - w_b = 0\right)$$

  - at the free surface

$$\frac{\partial \eta}{\partial t} + \frac{\partial \eta}{\partial x}u_s - w_s = 0$$

Boundary conditions for Navier-Stokes



- Normals

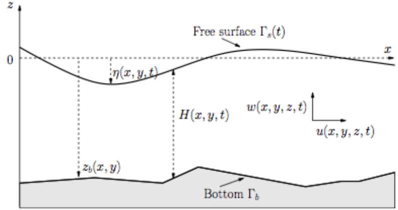
$$\mathbf{n}_s = \frac{1}{\sqrt{1+\left(\frac{\partial \eta}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial \eta}{\partial x} \\ 1 \end{pmatrix}, \quad \mathbf{n}_b = \frac{1}{\sqrt{1+\left(\frac{\partial z_b}{\partial x}\right)^2}} \begin{pmatrix} -\frac{\partial z_b}{\partial x} \\ 1 \end{pmatrix}.$$

- Free surface

$$\mu\left(\left.\frac{\partial u}{\partial z}\right|_s + \left.\frac{\partial w}{\partial x}\right|_s\right) - \frac{\partial \eta}{\partial x}\left(2\mu \left.\frac{\partial u}{\partial x}\right|_s - p_s\right) = p^a \frac{\partial \eta}{\partial x},$$

$$2\mu \left.\frac{\partial w}{\partial z}\right|_s - p_s - \mu \frac{\partial \eta}{\partial x}\left(\left.\frac{\partial u}{\partial z}\right|_s + \left.\frac{\partial w}{\partial x}\right|_s\right) = -p^a,$$

Boundary conditions for Navier-Stokes (cont'd)



- At the bottom

$$\mu \left(\left.\frac{\partial w}{\partial x}\right|_b + \left.\frac{\partial u}{\partial z}\right|_b \right) - \frac{\partial z_b}{\partial x} \left(2\mu \left.\frac{\partial u}{\partial x}\right|_b - p_b \right)$$

$$+ \frac{\partial z_b}{\partial x} \left(2\mu \left.\frac{\partial w}{\partial z}\right|_b - p_b - \mu \frac{\partial z_b}{\partial x} \left(\left.\frac{\partial u}{\partial z}\right|_b + \left.\frac{\partial w}{\partial x}\right|_b \right) \right)$$

$$= \kappa \left(1 + \left(\frac{\partial z_b}{\partial x} \right)^2 \right)^{1/2} u_b,$$

- Mainly

$$\mu \left.\frac{\partial u}{\partial z}\right|_b = \kappa u_b + \dots$$

Shallow water approximation

- Rescaling $\varepsilon = h/\lambda$
- Rescaling
  - Velocities : $U = \lambda/T = C$, $W = h/T = \varepsilon C$
  - Time : $T = \lambda/C$
  - Pressure $P = C^2$
- Variables without dimension

$$\tilde{x} = \frac{x}{\lambda}, \quad \tilde{z} = \frac{z}{h}, \quad \tilde{\eta} = \frac{\eta}{h}, \quad \tilde{t} = \frac{t}{T},$$

$$\tilde{p} = \frac{p}{P}, \quad \tilde{u} = \frac{u}{U}, \quad \text{and} \quad \tilde{w} = \frac{w}{W}.$$

- Reynolds number, Froude number, bottom friction

$$\tilde{\nu} = \frac{\mu}{U\lambda} = \frac{1}{Re}, \quad \tilde{g} = \frac{gh}{U^2} = \frac{1}{Fr^2}, \quad \tilde{\kappa} = \frac{\kappa}{U},$$

Shallow water approximation (cont'd)

- Dimensionless 2D Navier-Stokes equations

$$
\begin{aligned}
&\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0 \\
&\frac{\partial \tilde{u}}{\partial \tilde{t}} + \frac{\partial \tilde{u}^2}{\partial \tilde{x}} + \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{z}} + \frac{\partial \tilde{p}}{\partial \tilde{x}} = \frac{\partial}{\partial \tilde{x}}\left(2\tilde{\nu}\frac{\partial \tilde{u}}{\partial \tilde{x}}\right) \\
&\qquad\qquad + \frac{\partial}{\partial \tilde{z}}\left(\frac{\tilde{\nu}}{\varepsilon^2}\frac{\partial \tilde{u}}{\partial \tilde{z}} + \tilde{\nu}\frac{\partial \tilde{w}}{\partial \tilde{x}}\right) \\
&\varepsilon^2\left(\frac{\partial \tilde{w}}{\partial \tilde{t}} + \frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{x}} + \frac{\partial \tilde{w}^2}{\partial \tilde{z}}\right) + \frac{\partial \tilde{p}}{\partial \tilde{z}} = -1 + \frac{\partial}{\partial \tilde{x}}\left(\tilde{\nu}\frac{\partial \tilde{u}}{\partial \tilde{z}} + \varepsilon^2\tilde{\nu}\frac{\partial \tilde{w}}{\partial \tilde{x}}\right) \\
&\qquad\qquad + \frac{\partial}{\partial \tilde{z}}\left(2\tilde{\nu}\frac{\partial \tilde{w}}{\partial \tilde{z}}\right)
\end{aligned}
$$

- Boundary conditions
  - kinematic (not modified)

Shallow water approximation (cont'd)

- Boundary conditions
  - at the free surface

$$\frac{\tilde{\nu}}{\varepsilon}\left(\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s + \varepsilon^2 \left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_s\right) - \varepsilon\frac{\partial \tilde{\eta}}{\partial \tilde{x}}\left(2\tilde{\nu}\left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_s - \tilde{p}_s\right) = \varepsilon \tilde{p}^a \frac{\partial \tilde{\eta}}{\partial \tilde{x}},$$

$$2\tilde{\nu}\left.\frac{\partial \tilde{w}}{\partial \tilde{z}}\right|_s - \tilde{p}_s - \varepsilon\tilde{\nu}\frac{\partial \tilde{\eta}}{\partial \tilde{x}}\left(\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s + \varepsilon^2 \left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_s\right) = \tilde{p}^a,$$

  - at the bottom

$$\frac{\tilde{\nu}}{\varepsilon}\left(\varepsilon^2\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_b + \left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b\right) - \varepsilon\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(2\tilde{\nu}\left.\frac{\partial \tilde{u}}{\partial \tilde{x}}\right|_b - \tilde{p}_b\right)$$
$$+\varepsilon\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(2\tilde{\nu}\left.\frac{\partial \tilde{w}}{\partial \tilde{z}}\right|_b - \tilde{p}_b - \tilde{\nu}\frac{\partial \tilde{z}_b}{\partial \tilde{x}}\left(\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b + \varepsilon^2\left.\frac{\partial \tilde{w}}{\partial \tilde{x}}\right|_b\right)\right)$$
$$= \tilde{\kappa}\left(1 + \varepsilon^2\left(\frac{\partial z_b}{\partial x}\right)^2\right)^{1/2} \tilde{u}_b,$$

Hydrostatic Navier-Stokes system

- With initial variables

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

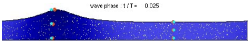
$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\left(2\nu\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(\nu\frac{\partial u}{\partial z} + \nu\frac{\partial w}{\partial x}\right)$$

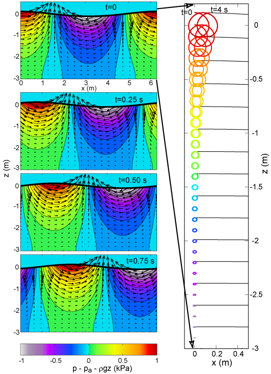
$$\frac{\partial p}{\partial z} = -g + \frac{\partial}{\partial x}\left(\nu\frac{\partial u}{\partial z} + \nu\frac{\partial w}{\partial x}\right) + \frac{\partial}{\partial z}\left(2\nu\frac{\partial w}{\partial z}\right)$$

- "A good model" (Brenier, Grenier, Masmoudi, Bresch. . .)
- Simplified role of the pressure
- Rather complex to analyse and solve

Validity of the hydrostatic assumption

- OK for river flows, tsunami,...
- Questionable for short waves





Validity of the hydrostatic assumption (cont'd)

- OK for river flows, tsunami,. . .
- Questionable for short waves
  Indeed, the hydrostatic Euler system writes

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} \right) + \frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial z} = -\rho_0 g$$

with $p_s = p^a$, leading to $p = p^a + \rho_0 g(\eta - z)$ hence

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

$$\rho_0 \left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} \right) + \rho_0 g \frac{\partial \eta}{\partial x} = 0$$

Vertically averaged hydrostatic Euler system

- Still with initial variables and $\rho_0 = 1$
- Hydrostatic Euler system

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$
$$\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0$$
$$\frac{\partial p}{\partial z} = -g$$

- Averaged version

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u dz = 0,$$
$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} u \; dz + \frac{\partial}{\partial x}\left(\int_{z_b}^{\eta} u^2 dz + \int_{z_b}^{\eta} p \; dz\right) = -p_b \frac{\partial z_b}{\partial x} + p^a \frac{\partial \eta}{\partial x}$$
$$p = p^a + g(\eta - z)$$

- A closure relation needed

The Leibniz's rule (act II)

- The vertical mean of the divergence free condition

$$\int_{z_b}^{\eta} \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) dz = 0$$

gives

$$0 = \frac{\partial}{\partial x} \int_{z_b}^{\eta} u dz - \frac{\partial \eta}{\partial x} u_s + \frac{\partial z_b}{\partial x} u_b + w_s - w_b$$

and using the kinematic boundary conditions, we get ($\eta = H + z_b$)

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u dz = 0 \qquad \frac{\partial H}{\partial t} + \frac{\partial}{\partial x} \int_{z_b}^{\eta} u dz = 0$$

- An exercise

$$\int_{z_b}^{\eta} \left(\frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} \right) dz = 0$$

gives (?)

$$\frac{\partial}{\partial t} \int_{z_b}^{\eta} u \; dz + \frac{\partial}{\partial x} \left(\int_{z_b}^{\eta} u^2 dz + \int_{z_b}^{\eta} p \; dz \right) = -p_b \frac{\partial z_b}{\partial x} + p^a \frac{\partial \eta}{\partial x}$$

Closure relations

- Rescaled viscosity & friction

$$\tilde{\nu} = \varepsilon \nu_0, \qquad \tilde{\kappa} = \varepsilon \kappa_0$$

Closure relations (cont'd)

- Rescaled boundary conditions give

$$\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s = \mathcal{O}(\varepsilon) \qquad \left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b = \mathcal{O}(\varepsilon)$$

couple with

$$\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} = \mathcal{O}(\varepsilon)$$

gives

$$\frac{\partial \tilde{u}}{\partial \tilde{z}} - \left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_b = \int_{\tilde{z}_b}^{\tilde{z}} \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} dz = \mathcal{O}(\varepsilon)$$

or

$$\left.\frac{\partial \tilde{u}}{\partial \tilde{z}}\right|_s - \frac{\partial \tilde{u}}{\partial \tilde{z}} = \int_{\tilde{z}}^{\tilde{\eta}} \frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2} dz = \mathcal{O}(\varepsilon)$$

- Hence

$$\frac{\partial \tilde{u}}{\partial \tilde{z}} = \mathcal{O}(\varepsilon)$$

Closure relations (cont'd)

- We have obtained

$$\frac{\partial \tilde{u}}{\partial \tilde{z}} = \mathcal{O}(\varepsilon)$$

giving

$$\tilde{u} - \tilde{u}_b = \int_{\tilde{z}_b}^{\tilde{z}} \frac{\partial \tilde{u}}{\partial \tilde{z}} dz = \mathcal{O}(\varepsilon)$$

or

$$\tilde{u}_s - \tilde{u} = \int_{\tilde{z}}^{\tilde{\eta}} \frac{\partial \tilde{u}}{\partial \tilde{z}} dz = \mathcal{O}(\varepsilon)$$

i.e.

$$\tilde{u} = \alpha(t, x) + \mathcal{O}(\varepsilon)$$

- Hence

$$\tilde{u} = \overline{\tilde{u}} + \mathcal{O}(\varepsilon)$$

or equivalently $u = \overline{u} + \mathcal{O}(\varepsilon)$ with $\overline{u} = \frac{1}{H} \int_{z_b}^{\eta} u dz$

The Saint-Venant system

- From $u = \overline{u} + \mathcal{O}(\varepsilon)$ we get

$$u^2 = \overline{u}^2 + \mathcal{O}(\varepsilon)$$

- The averaged hydrostatic Euler system

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}\int_{z_b}^{\eta} u dz = 0$$

$$\frac{\partial}{\partial t}\int_{z_b}^{\eta} u \ dz + \frac{\partial}{\partial x}\left(\int_{z_b}^{\eta} u^2 dz + \int_{z_b}^{\eta} p \ dz\right) = -p_b \frac{\partial z_b}{\partial x} + p^a \frac{\partial \eta}{\partial x}$$

$$p = p^a + g(\eta - z)$$

writes, up to $\mathcal{O}(\varepsilon)$ terms

$$\frac{\partial H}{\partial t} + \frac{\partial (H\overline{u})}{\partial x} = 0$$

$$\frac{\partial (H\overline{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\overline{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + p^a \frac{\partial \eta}{\partial x}$$

with $H\overline{u} = \int_{z_b}^{\eta} u dz$, $p^a = 0$, $\int_{z_b}^{\eta} p \ dz = \frac{g}{2}H^2$

The Saint-Venant system (cont'd)

- Extended formulation

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -H\frac{\partial p^a}{\partial x} - gH\frac{\partial z_b}{\partial x} - \kappa\bar{u}$$

- Up to $\mathcal{O}(\varepsilon^2)$ terms

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -H\frac{\partial p^a}{\partial x} - gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa\bar{u}}{1 + \frac{\kappa}{3\nu}H}$$

- Energy balance, vertical velocity, passive tracer
- Friction laws
  - Navier $S_f = \kappa\bar{u}$, Manning-Strickler $S_f = C_f\frac{\bar{u}|\bar{u}|}{H^{\frac{4}{3}}}$,
  - Darcy-Weisbach $S_f = C_f\frac{\bar{u}|\bar{u}|}{H}$
- The Saint-Venant system in 2d

The Saint-Venant system (cont'd)

- Up to $\mathcal{O}(\varepsilon^2)$ terms

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) = 0$$
$$\frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\nu H\frac{\partial \bar{u}}{\partial x}\right) - \frac{\kappa \bar{u}}{1 + \frac{\kappa}{3\nu}H}$$

- Exercise : the energy balance ?

Typical simulations

- A tsunami (occured versus realistic) (chili0.avi) (mediter0.avi)
- A landslide (landslide_tsunami1.avi) (landslide_tsunami3.avi)
- A channel with a bump (bump.avi)
- Time dependent analytical solutions (ritter.avi)
- Outside the domain of validity (dam_with_step.avi)
- Wind effects (wind_sv.avi) (wind_step.avi)