Low Froude regime and dispersive effects in kinetic formulations

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1 General description

The shallow water system describes the evolution of the water height h and the mean horizontal speed u when the spatial domain is much larger in the horizontal direction than in the vertical one, which is the case in oceans and rivers. In 1D for simplicity, the system writes

$$\begin{cases} \partial_t h(t,x) + \partial_x (hu)(t,x) = 0 & t \in \mathbb{R}^+,\ x \in \mathbb{R} \\ \partial_t (hu)(t,x) + \partial_x \left(hu^2 + \dfrac{g}{2} h^2 \right)(t,x) = gh\partial_x b(x) & \end{cases} \tag{1}$$

where g is the gravitationnal constant and b accounts for the (slowly varying) bottom topography.

System (1) focuses on the macroscopic quantities h and u. Its kinetic formulation describes the same phenomenon at the mesoscopic scale. The evolution of $f(t,x,\xi)$ the proportion of "particles" moving at speed ξ at time t and point x is given by

$$\partial_t f(t,x,\xi) + \xi \partial_x f(t,x,\xi) = Q(f,t,x,\xi), \qquad t \in \mathbb{R}^+,\ x \in \mathbb{R},\ \xi \in \mathbb{R} \tag{2}$$

where Q is a collision operator.

The kinetic description (2) is of larger dimension, but is linear, while the macroscopic description (1) does not depend on ξ, but is nonlinear. The two are linked by the relations, see [1] :

$$\rho(t,x) = \int f(t,x,\xi)\, d\xi \text{ and } (\rho u)(t,x) = \int \xi f(t,x,\xi)\, d\xi \tag{3}$$

and reciprocally, for some Maxwellian function χ,

$$f(t,x,\xi) = \frac{h(t,x)}{\sqrt{gh}} \chi \left(\frac{\xi - u(t,x)}{\sqrt{gh}} \right) \tag{4}$$

In other words, depending on the interest, the study can be about the Boltzmann equation instead of the Euler system for instance.

Kinetic schemes rely on the same approach : instead of solving (1) directly, an additionnal variable is introduced and a linear scheme for (2) is proposed. Since they follow "physical" paths, those schemes naturally conserve crucial properties of the model (conservation, dissipation ...), at the price of dealing with numerical integrations (3) to recover the macroscopic quantities.

In this PhD we first propose to keep a trace of the Froude number in (2), see Section 2. Second, we will study a diffusive-dispersive conservation law. This is a first step for understanding dispersive model as Green-Naghdi, see Section 3. All along the PhD, the understanding of the continuous PDEs, which is interesting by itself, will also be the base for new kinetic schemes.

2 Low Froude number in the kinetic formulation

In many oceanic phenomena modelled by the shallow water equation, there is presence of multiscale behaviours. More precisely, it can be observed that $\sqrt{gh}$ the speed of the gravitational waves on the ocean surface is much higher than the one of the water u. In other words, the Froude number $\mathbf{Fr} = \frac{u}{\sqrt{gh}}$ may be very small and we are interesting in the following system (where α is a dimensionless parameter)

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{h^2}{2\mathbf{Fr}^2}\right) = -\dfrac{\alpha}{\mathbf{Fr}^2}\partial_z b \end{cases} \tag{5}$$

Classical explicit schemes are constrained by a CFL condition of the type $\Delta t \leq \frac{\Delta x}{|u|(1+1/\mathbf{Fr})}$ and thus use ridiculously small time steps when $\mathbf{Fr}$ is small ; the reasonable restriction is $\Delta t \leq \frac{\Delta x}{|u|}$.

The existing solutions [2] consist in impliciting a "smart" part of the equation (5), so that the CFL condition relaxes to $\Delta x \leq \frac{\Delta t}{|u|}$ while avoiding the resolution of large nonlinear systems. To our knowledge, they are all based on the macroscopic equation (1).

We would like to investigate the kinetic approach, which hasn't be studied yet. With a carefull analysis based on dimensionless parameters, we should be able to propose a correct expansion of (4) in terms of the Froude number $\mathbf{Fr}$. Then, taking advantage of the linearity of (2), a linear implicit-explicit scheme should be quite obvious to write. Eventually, at the continuous level, we will explore the links between the Froude number and the relaxation time ε, in the case of the BGK collision operator.

3 Kinetic formulation for nonclassical shocks

In the simplest setting [3], nonclassical shocks are discontinous solutions of the scalar law

$$\partial_t u + \partial_x u^3 = 0$$

that are limits, when ε tends to 0, of the following diffusive dispersive

$$\partial_t u_\varepsilon + \partial_x u^3 = \varepsilon \partial_{xx} u_\varepsilon + \alpha \varepsilon^2 \partial_{xxx} u_\varepsilon. \tag{6}$$

The classical case corresponds to $\alpha = 0$ and yields the well known Krushkov theory, where discontinuous solutions verify the entropy inequality $\partial_t \eta(u) + \partial_x g(u) \leq 0$ for <u>any</u> convex function η (and its associated entropy flux g). When $\alpha \neq 0$, one can obtain only <u>one</u> entropy inequality, and a so-called kinetic condition is necessary to select shocks and define a unique weak solution. We plan to propose a nonclassical kinetic formulation that selects the limit of (6), in the spirit of [5]. At the kinetic level, the fact that only one entropy inequality holds prevent us from applying existing methods, and the key point will be the interpretation of the kinetic relation. The numerical approximation of such problems is still largely open [4], since it is very difficult to maintain, at the discrete level, the correct balance between diffusive and dispersive effects. A reliable kinetic interpretation would be a first step towards a new class of schemes.

Références

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