# Non-hydrostatic shallow water model and Gradient Discretization Method

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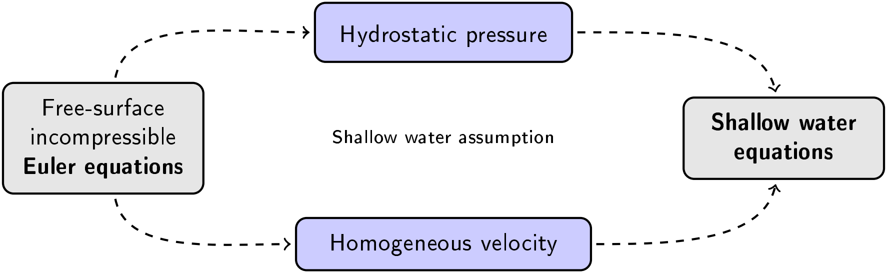
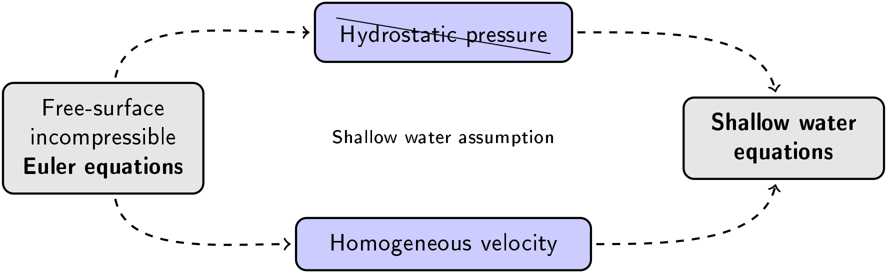
# Outline

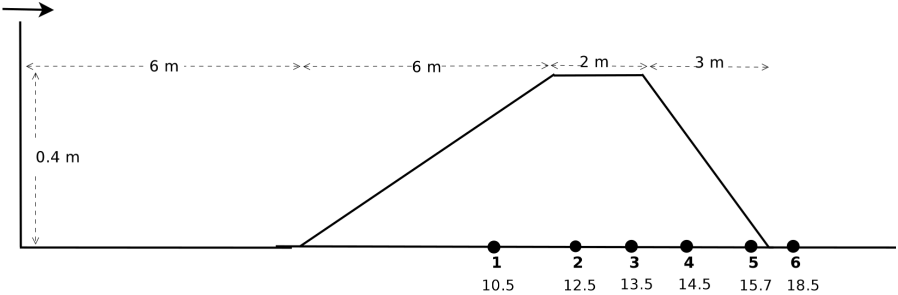
1. Dispersive effects modelling
   - Overview of free-surface flows
   - A depth-averaged Euler model
   - Numerical analysis
2. Gradient Discretization Method
   - Problematic and objective
   - Example: linear stationary diffusion problem
   - The conforming $\mathbb{P}_1$ Finite Elements case
3. Application of GDM to the elliptic problem
   - Gradient scheme for the elliptic problem
   - GDM for mixed BCs (homo. Dirichlet)

# Free-surface flow modelling



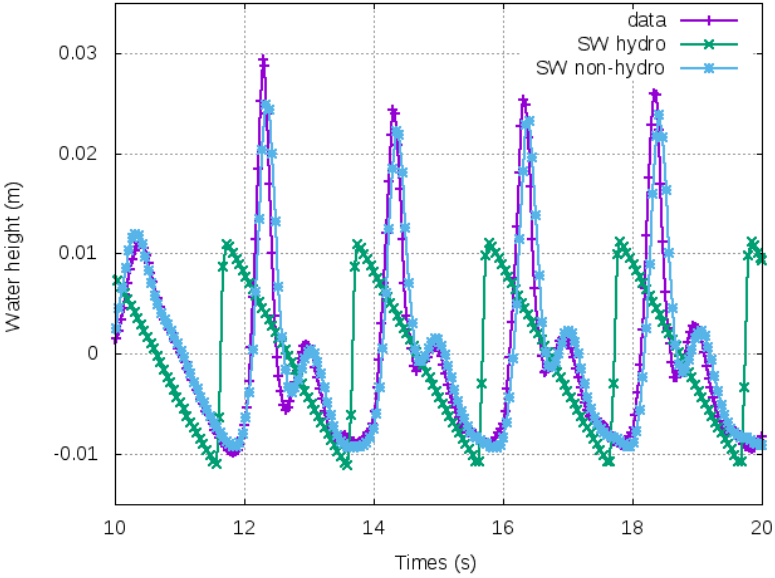

Free-surface
incompressible
**Euler equations**
Hydrostatic pressure
Shallow water assumption
Homogeneous velocity
**Shallow water equations**

# Emphasis of non-hydrostatic effects




M.-W. Dingemans, *Wave propagation over uneven bottoms* (**Adv. Ser. Ocean Eng.**, 1997)

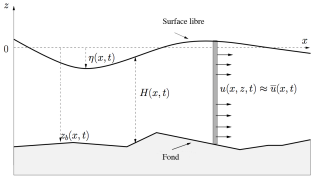
# Emphasis of non-hydrostatic effects



# A depth-averaged Euler model

PhD of Nora Aïssiouene
(LJLL, 2016) :

- water height $H$
- velocity $\overline{\mathbf{u}} = (\overline{\mathbf{v}}, \overline{w})$
- (non-hydrostatic) pressure $p_{nh}$



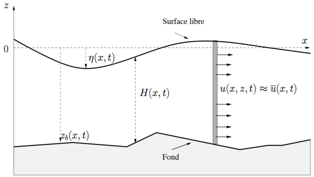


N. Aïssiouene, M.-O. Bristeau, E. Godlewski, J. Sainte-Marie, *A combined finite volume – finite element scheme for a dispersive shallow water system* (**Netw. Heterog. Media** 11(1), 2016)

N. Aïssiouene, M.-O. Bristeau, E. Godlewski, A. Mangeney, C. Parés, J. Sainte-Marie, *A two-dimensional method for a dispersive shallow water model* (submitted)

PhD of Nora Aïssiouene (LJLL, 2016) :

- water height $H$
- velocity $\overline{\mathbf{u}} = (\overline{\mathbf{v}}, \overline{w})$
- (non-hydrostatic) pressure $p_{nh}$
- $\alpha = 2$ ($\alpha = \frac{3}{2}$ : Serre-Green-Naghdi)




$$\frac{\partial H}{\partial t} + \bar{\nabla} \cdot (H\overline{\mathbf{u}}) = 0$$

$$\frac{\partial (H\overline{\mathbf{u}})}{\partial t} + \bar{\nabla} \cdot (H\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) + \bar{\nabla}\left(g\frac{H^2}{2}\right) + \nabla_{sw}^{\alpha}\, p_{nh} + gH\bar{\nabla} z_b = 0$$

$$\mathrm{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} = 0$$

**Notations**

$$\overline{\mathbf{u}} = \begin{pmatrix} \overline{\mathbf{v}} \\ \overline{w} \end{pmatrix} \qquad \nabla_{sw}^{\alpha}\, p = \begin{pmatrix} H\nabla_{\mathbf{x}} p + p\nabla_{\mathbf{x}}(H + 2z_b) \\ -\alpha p \end{pmatrix}$$

$$\bar{\nabla} = \begin{pmatrix} \nabla_{\mathbf{x}} \\ 0 \end{pmatrix} \qquad \mathrm{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} = \nabla_{\mathbf{x}} \cdot (H\overline{\mathbf{v}}) - \overline{\mathbf{v}} \cdot \nabla_{\mathbf{x}}(H + 2z_b) + \alpha\overline{w}$$

PhD of Nora Aïssiouene (LJLL, 2016) :

- water height $H$
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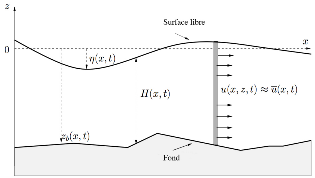


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**Numerical strategy**: Time splitting (projection/correction) Hyperbolic solver / Dispersive solver

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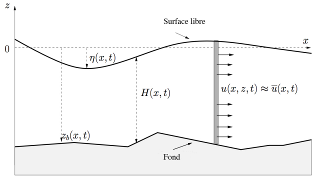


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$$\mathrm{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} = 0$$

**Numerical strategy**: Time splitting (projection/correction) Hyperbolic solver / Dispersive solver

# Correction step

## Focus on the elliptic part - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

Find $p_{nh} : \Omega \to \mathbb{R}$ and $\overline{\mathbf{u}} : \Omega \to \mathbb{R}^{d+1}$ s.t.

$$
\begin{aligned}
H\overline{\mathbf{u}} + \nabla_{sw}^{\alpha} p_{nh} &= g \quad \text{on } \Omega \\
\operatorname{div}_{sw}^{\alpha} \overline{\mathbf{u}} &= f \quad \text{on } \Omega \\
H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s &= \phi \quad \text{on } \Gamma_n \\
p_{nh} &= 0 \quad \text{on } \Gamma_d
\end{aligned}
$$

with

$$
\begin{aligned}
\zeta &:= H + 2z_b \\
\partial\Omega &= \Gamma = \Gamma_d \cup \Gamma_n \\
\boldsymbol{n}_s &= (\boldsymbol{n}_{\Gamma_n}, 0)
\end{aligned}
$$

$$
\begin{aligned}
\nabla_{sw}^{\alpha} p_{nh} &= (H\nabla p_{nh} + p_{nh}\nabla\zeta, -\alpha p_{nh}) \\
\operatorname{div}_{sw}^{\alpha} \overline{\mathbf{u}} &= \operatorname{div}(H\overline{\boldsymbol{v}}) - \overline{\boldsymbol{v}} \cdot \nabla\zeta + \alpha\overline{w}
\end{aligned}
$$

## Focus on the elliptic part - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)

Find $p_{nh} : \Omega \to \mathbb{R}$ and $\overline{\mathbf{u}} : \Omega \to \mathbb{R}^{d+1}$ s.t.

$$\begin{aligned} H\overline{\mathbf{u}} + \nabla^{\alpha}_{sw}\, p_{nh} &= g && \text{on } \Omega \\ \operatorname{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} &= f && \text{on } \Omega \\ H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s &= \phi && \text{on } \Gamma_n \\ p_{nh} &= 0 && \text{on } \Gamma_d \end{aligned}$$

with

$$\zeta := H + 2z_b \qquad \nabla^{\alpha}_{sw}\, p_{nh} = (H\nabla p_{nh} + p_{nh}\nabla\zeta, -\alpha p_{nh})$$

$$\partial\Omega = \Gamma = \Gamma_d \cup \Gamma_n \qquad \operatorname{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} = \operatorname{div}(H\overline{\boldsymbol{v}}) - \overline{\boldsymbol{v}} \cdot \nabla\zeta + \alpha\overline{w}$$

$$\boldsymbol{n}_s = (\boldsymbol{n}_{\Gamma_n}, 0)$$

**Stokes-type formula :**

$$\int_{\Omega} \nabla^{\alpha}_{sw}\, p_{nh} \cdot \overline{\mathbf{u}} = -\int_{\Omega} p_{nh}\, \operatorname{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} + \int_{\partial\Omega} \gamma p_{nh} H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s$$

# Elliptic part: change of formulation

From a mixed formulation on ($p_{nh}$ , $\overline{\mathbf{u}}$ ) ...

$$
\begin{aligned}
H\overline{\mathbf{u}} + \nabla_{sw}^{\alpha}\, p_{nh} &= g && \text{on } \Omega \\
\operatorname{div}_{sw}^{\alpha}\, \overline{\mathbf{u}} &= f && \text{on } \Omega
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$$

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From a mixed formulation on $(p_{nh}, \overline{\mathbf{u}})$ ...

$$H\overline{\mathbf{u}} + \nabla^{\alpha}_{sw} p_{nh} = g \qquad \text{on } \Omega$$
$$\operatorname{div}^{\alpha}_{sw} \overline{\mathbf{u}} = f \qquad \text{on } \Omega$$

... to a conform formulation on $p_{nh}$

$$-\operatorname{div}^{\alpha}_{sw}\left(\frac{1}{H}\nabla^{\alpha}_{sw} p_{nh}\right) = f - \operatorname{div}^{\alpha}_{sw}\left(\frac{1}{H}g\right) \qquad \text{on } \Omega$$
$$\overline{\mathbf{u}} = \frac{1}{H}\left(g - \nabla^{\alpha}_{sw} p_{nh}\right) \qquad \text{on } \Omega$$

*under assumption* $0 < \underline{H} \leq H(\mathbf{x}) \leq \overline{H}$

# Elliptic part: change of formulation

From a mixed formulation on ( $p_{nh}$ , $\overline{\mathbf{u}}$ ) . . .

$$\begin{aligned} H\overline{\mathbf{u}} + \nabla^{\alpha}_{sw}\, p_{nh} &= g && \text{on } \Omega \\ \operatorname{div}^{\alpha}_{sw}\, \overline{\mathbf{u}} &= f && \text{on } \Omega \end{aligned}$$

. . . to a conform formulation on $p_{nh}$

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*under assumption* $0 < \underline{H} \leq H(\mathbf{x}) \leq \overline{H}$

Ani Miraçi's Master internship (summer 2017):

- conforming method: easier to implement & smaller linear system
- on simple 1D tests: similar accuracy as mixed formulation

# How are you with combinatorics ?

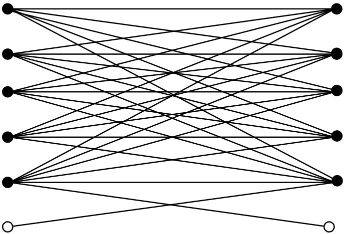
If we have to analyse the convergence of each numerical method for each model
. . .

| METHODS | PROBLEMS |
| --- | --- |
| FE | Heat equation |
| mixed FE | Stefan problem |
| MPFA | Porous media flow |
| DDFV | Richards equation |
| dG | Incompressible Navier-Stokes |
| . . . | . . . |

If we have to analyse the convergence of each numerical method for each model …

METHODS

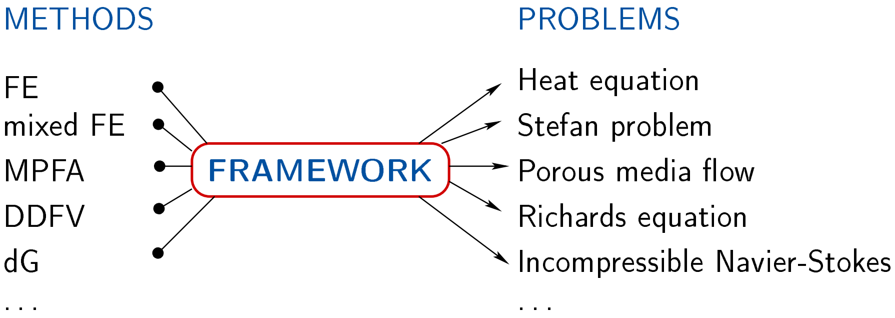
FE
mixed FE
MPFA
DDFV
dG
…



PROBLEMS

Heat equation
Stefan problem
Porous media flow
Richards equation
Incompressible Navier-Stokes
…

*each line = one analysis to perform*




**Objective**

The framework identifies a few key properties that all methods satisfy, and that are sufficient for all convergence analyses

$$\begin{aligned} -\mathrm{div}(\Lambda \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

- $\Omega$ open bounded in $\mathbb{R}^d$
- $\Lambda : \Omega \to M_d(\mathbb{R})$ bounded uniformly coercive
- $f \in L^2(\Omega)$

Idea : in the weak formulation of the PDE
replace the space and operators by the discrete ones

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**Weak formulation**:

Find $u \in H_0^1(\Omega)$ such that, $\forall\, v \in H_0^1(\Omega)$,

$$\int_\Omega \Lambda\,\nabla u \cdot \nabla v \;=\; \int_\Omega f\, v$$

$$\begin{aligned} -\mathrm{div}(\Lambda \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

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**Weak formulation**:

Find $u \in H_0^1(\Omega)$ such that, $\forall\, v \in H_0^1(\Omega)$,

$$\int_\Omega \Lambda \nabla u \cdot \nabla v = \int_\Omega f\, v$$

**Gradient scheme**:

Find $u_\mathcal{D} \in X_{\mathcal{D},0}$ such that, $\forall v_\mathcal{D} \in X_{\mathcal{D},0}$,

$$\int_\Omega \Lambda\, \nabla_\mathcal{D} u_\mathcal{D} \cdot \nabla_\mathcal{D} v_\mathcal{D} = \int_\Omega f\, \Pi_\mathcal{D} v_\mathcal{D}$$

# Gradient Discretisation - (GD) for homogeneous Dirichlet BC

$$\mathcal{D} = (X_{\mathcal{D},0}\ ,\ \Pi_{\mathcal{D}}\ ,\ \nabla_{\mathcal{D}})$$

- discrete space $X_{\mathcal{D},0} = \mathbb{R}^{\{d.o.f.\}}$ ($X_{\mathcal{D},0}$ suited to boundary conditions)
- reconstruction of function $\Pi_{\mathcal{D}}\ :\ X_{\mathcal{D},0} \to L^2(\Omega)$ linear mapping
- reconstruction of gradient $\nabla_{\mathcal{D}}\ :\ X_{\mathcal{D},0} \to L^2(\Omega)^d$ linear mapping, such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$

- **Coercivity** (*discrete Poincaré inequality*)

$$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D},\mathbf{0}}\setminus\{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^2}}{\|\nabla_{\mathcal{D}} v\|_{L^2}}$$

$C_{\mathcal{D}_m}$ **remains bounded**

- **GD-Consistency** (*FE interpolation error*)

$$\forall \varphi \in H^1_0(\Omega)\,, \quad S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},\mathbf{0}}} \left( \|\Pi_{\mathcal{D}} v - \varphi\|_{L^2} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2} \right)$$

$S_{\mathcal{D}_m} \to 0$

- **Limit-conformity** (*FE consistency*)

$$\forall \boldsymbol{\varphi} \in H_{\mathrm{div}}(\Omega)\,, \; W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{u\in X_{\mathcal{D},\mathbf{0}}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} u\|_{L^2}} \left| \int_\Omega (\nabla_{\mathcal{D}} u \cdot \boldsymbol{\varphi} + \Pi_{\mathcal{D}} u \,\mathrm{div}\boldsymbol{\varphi}) \right|$$

$W_{\mathcal{D}_m} \to 0$

**Weak formulation :**

$$\text{Find } u \in H_0^1(\Omega) \text{ such that, } \forall\, v \in H_0^1(\Omega),$$
$$\int_\Omega \Lambda \nabla u \cdot \nabla v \; = \int_\Omega f\, v$$

**Gradient scheme :**

$$\text{Find } u_\mathcal{D} \in X_{\mathcal{D},0} \text{ such that, } \forall v_\mathcal{D} \in X_{\mathcal{D},0},$$
$$\int_\Omega \Lambda \nabla_\mathcal{D} u_\mathcal{D} \cdot \nabla_\mathcal{D} v_\mathcal{D} \; = \int_\Omega f\, \Pi_\mathcal{D} v_\mathcal{D}$$

**Error estimate**

$$\|\Pi_\mathcal{D} u_\mathcal{D} - u\|_{L^2} + \|\nabla_\mathcal{D} u_\mathcal{D} - \nabla u\|_{L^2} \leq C(1 + C_\mathcal{D})\,[S_\mathcal{D}(u) + W_\mathcal{D}(\Lambda \nabla u)]$$

On a triangular/tetrahedral mesh, $\mathcal{V}$ = set of vertices of the mesh

Gradient discretisation :

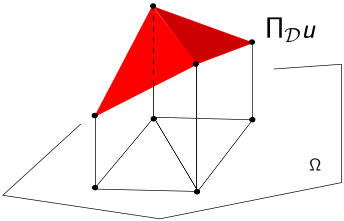
- $X_{\mathcal{D},0} := \{ (u_s)_{s\in\mathcal{V}} \ : \ u_s = 0 \text{ if } s \in \partial\Omega \}$
- $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \to C(\Omega) \ ; \ u \mapsto u_h = \sum_{s\in\mathcal{V}} u_s\varphi_s$

  with $\varphi_s$ $\mathbb{P}_1$ FE shape function associated to vertex $s$
- $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \to L^2(\Omega)^d \ ; \ u \mapsto \nabla_{\mathcal{D}} u = \nabla u_h$ (piecewise constant function)
  - $(\nabla_{\mathcal{D}} u)_{|K} = \nabla(\Pi_{\mathcal{D}} u)_{|K}$

Poincaré inequality $\Longrightarrow$ **coercivity**

$S_{\mathcal{D}}(\varphi) \leq C\,h \Longrightarrow$ **GD-consistency**

$W_{\mathcal{D}}(\varphi) = 0 \Longrightarrow$ **limit conformity**



# Summarize

## GDMs in few words

- a framework to study convergence analyses
- replace the space and operators by the discrete ones
- choose $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ and ensure **coercivity**, **consistency** and **limit-conformity** property


J. Droniou, R. Eymard, T. Gallouët, C. Guichard, R. Herbin, *The gradient discretisation method* Springer International Publishing AG, 82, 2018, Mathématiques et Applications

J. Droniou, R. Eymard, R. Herbin, *Gradient schemes: generic tools for the numerical analysis of diffusion equations* (**M2AN** 50(3), 2016)

# Summarize

## GDMs in few words

- a framework to study convergence analyses
- replace the space and operators by the discrete ones
- choose $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ and ensure **coercivity**, **consistency** and **limit-conformity** property

## Other methods known to be GDMs

- mass-lumped conforming $\mathbb{P}_1$ Finite Elements
- some Finite Volume schemes : VAG, DDFV, SUSHI, ...
- non-conforming FE method, including non-conforming $\mathbb{P}_k$
- Hybrid Mimetic Mixed method
- Hybrid high-order methods
- . . .

# Elliptic part: GDMs

**Elliptic problem - $\Omega \subset \mathbb{R}^d$ ($d = 1$ or $d = 2$)**

Find $p_{nh} : \Omega \to \mathbb{R}$ and $\overline{\mathbf{u}} : \Omega \to \mathbb{R}^{d+1}$ s.t.

$$-\operatorname{div}_{sw}^{\alpha}\left(\frac{1}{H}\nabla_{sw}^{\alpha} p_{nh}\right) = f - \operatorname{div}_{sw}^{\alpha}\left(\frac{1}{H}\mathbf{g}\right) \quad \text{on } \Omega$$

$$H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s = \phi \quad \text{on } \Gamma_n$$

$$p_{nh} = 0 \quad \text{on } \Gamma_d$$

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$$H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s = \phi \quad \text{on } \Gamma_n$$

$$p_{nh} = 0 \quad \text{on } \Gamma_d$$

**Weak formulation :**

Find $\hat{p} \in H_0^1(\Omega)$ such that $\forall \hat{q} \in H_0^1(\Omega)$,

$$\int_\Omega \frac{(H\nabla\hat{p} + \hat{p}\nabla\zeta)\cdot(H\nabla\hat{q} + \hat{q}\nabla\zeta) + \alpha^2\hat{p}\hat{q}}{H}\, dx$$

$$= \int_\Omega f\hat{q} + \frac{\mathbf{g}_1\cdot\nabla\zeta - \alpha g_2}{H}\hat{q}\, dx + \int_\Omega \mathbf{g}_1\cdot\nabla\hat{q}\, dx - \int_{\Gamma_n}\phi\hat{q}\, ds$$

**Elliptic problem - $\Omega \subset \mathbb{R}^d$ $(d = 1 \text{ or } d = 2)$**

Find $p_{nh} : \Omega \to \mathbb{R}$ and $\overline{\mathbf{u}} : \Omega \to \mathbb{R}^{d+1}$ s.t.

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$$H\overline{\mathbf{u}} \cdot \boldsymbol{n}_s = \phi \qquad \text{on } \Gamma_n$$

$$p_{nh} = 0 \qquad \text{on } \Gamma_d$$

**Gradient scheme :**

Find $p \in X_{\mathcal{D},0}$ such that $\forall q \in X_{\mathcal{D},0}$,

$$\int_{\Omega} \frac{(H\nabla_{\mathcal{D}} p + \Pi_{\mathcal{D}} p \nabla\zeta) \cdot (H\nabla_{\mathcal{D}} q + \Pi_{\mathcal{D}} q \nabla\zeta) + \alpha^2 \Pi_{\mathcal{D}} p \Pi_{\mathcal{D}} q}{H} \, dx$$

$$= \int_{\Omega} f \Pi_{\mathcal{D}} q + \frac{\mathbf{g}_1 \cdot \nabla\zeta - \alpha g_2}{H} \Pi_{\mathcal{D}} q \, dx + \int_{\Omega} \mathbf{g}_1 \cdot \nabla_{\mathcal{D}} q \, dx - \int_{\Gamma_n} \phi \mathbb{T}_{\mathcal{D},\Gamma_n} q \, ds$$

# Gradient Discretisation - (GD) for mixed BCs (homo. Dirichlet)

$$\mathcal{D} = (X_{\mathcal{D}} \ , \ \Pi_{\mathcal{D}} \ , \ \nabla_{\mathcal{D}} \ , \ \mathbb{T}_{\mathcal{D},\Gamma_n})$$

- discrete space $X_{\mathcal{D}} = X_{\mathcal{D},\Gamma_d} \oplus X_{\mathcal{D},\Omega,\Gamma_n}$ direct sum of two finite dimensional vector spaces on $\mathbb{R}$
- reconstruction of function $\Pi_{\mathcal{D}} \ : \ X_{\mathcal{D}} \to L^2(\Omega)$ linear mapping
- reconstruction of gradient $\nabla_{\mathcal{D}} \ : \ X_{\mathcal{D}} \to L^2(\Omega)^d$ linear mapping, such that $\|\nabla_{\mathcal{D}} \cdot \|_{L^2(\Omega)^d}$ is a norm on $X_{\mathcal{D},\Omega,\Gamma_n}$
- reconstruction of trace $\mathbb{T}_{\mathcal{D},\Gamma_n} \ : \ X_{\mathcal{D}} \to L^2(\Gamma_n)$ linear mapping

- **Coercivity**

$$C_{\mathcal{D}} = \max_{v\in X_{\mathcal{D},\Omega,\Gamma_n}\setminus\{0\}} \left( \max\left\{ \frac{\|\Pi_{\mathcal{D}} v\|_{L^2(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}}, \frac{\|\mathbb{T}_{\mathcal{D},\Gamma_n} v\|_{L^2(\Gamma_n)}}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}} \right\}\right)$$

$C_{\mathcal{D}_m}$ **remains bounded**

- **GD-Consistency**

$$\forall \varphi \in H^1(\Omega)\,,\ S_{\mathcal{D}}(\varphi) = \min_{v\in X_{\mathcal{D},\Omega,\Gamma_n}} \left( \|\Pi_{\mathcal{D}} v - \varphi\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla\varphi\|_{L^2(\Omega)^d} \right)$$

$S_{\mathcal{D}_m} \to 0$

- **Limit-conformity**

$$\forall \boldsymbol{\varphi} \in H_{div,\Gamma_n}(\Omega)\,,\ W_{\mathcal{D}}(\boldsymbol{\varphi}) = \max_{v\in X_{\mathcal{D},\Omega,\Gamma_n}\setminus\{0\}} \frac{1}{\|\nabla_{\mathcal{D}} v\|_{L^2(\Omega)^d}} \left| \int_\Omega (\nabla_{\mathcal{D}} v\cdot\boldsymbol{\varphi} + \Pi_{\mathcal{D}} v \,\mathrm{div}\boldsymbol{\varphi}) - \int_{\Gamma_n} \mathbb{T}_{\mathcal{D},\Gamma_n} v \gamma_{\mathbf{n}}\phi \right|$$

$W_{\mathcal{D}_m} \to 0$

# To conclude

## GDMs on the elliptic part

- Gradient scheme framework includes: Conforming and Nonconforming FE, some FV schemes, etc...
- Deals with several BCs case-by-case
- Error estimate on $\|\Pi_{\mathcal{D}} p - \hat{p}\|_{L^2(\Omega)} + \|\nabla_{\mathcal{D}} p - \nabla \hat{p}\|_{L^2(\Omega)^d}$

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## Perspectives: Abstract GDM

- Same processing of different BCs
- Non-classic operators $\nabla_{sw}^{\alpha}$ and $\mathrm{div}_{sw}^{\alpha}$ related by a Stokes-type formula
- Can use a quadrature formula for $-\alpha p$ in $\nabla_{sw}^{\alpha} p$ ($\neq -\alpha \Pi_{\mathcal{D}} p$)
- Comparison between error estimate given by GDM and A-GDM ?