# Wave-structure interaction for long wave models

## in the presence of a freely moving body on the bottom

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# Motivation

*Mathematical motivation* : a better understanding of the water waves problem

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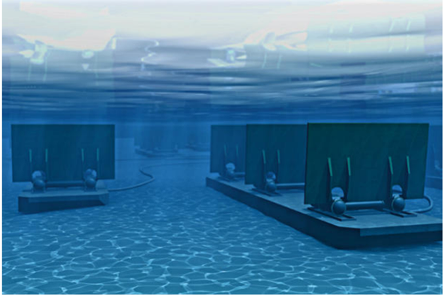
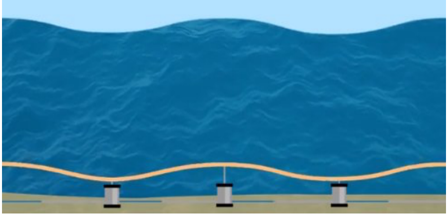
*Mathematical motivation* : a better understanding of the water waves problem
*Real life applications* : Coastal engineering and wave energy converters

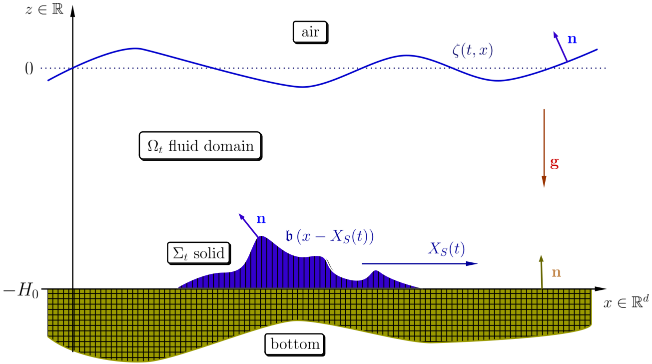


(a) Wave Roller



(b) Wave Carpet

# The physical domain for the wave-structure interaction problem




$$\Omega_t = \left\{(x,z) \in \mathbb{R}^2 \,:\, -H_0 + b(x - X_S(t)) < z < \zeta(t,x)\right\}.$$

# References

- In the case of a predefined evolution of the bottom topography :
  - T. Alazard, N. Burq, and C. Zuily, On the Cauchy Problem for the water waves with surface tension (*2011*),
  - F. Hiroyasu, andT. Iguchi, A shallow water approximation for water waves over a moving bottom (*2015*),
  - B. Melinand, A mathematical study of meteo and landslide tsunamis (*2015*) ;
- Fluid - submerged solid interaction :
  - G-H. Cottet, and E. Maitre, A level set method for fluid-structure interactions with immersed surfaces (*2006*),
  - P. Guyenne, and D. P. Nicholls, A high-order spectral method for nonlinear water waves over a moving bottom (*2007*),
  - S. Abadie et al., A fictious domain approach based on a viscosity penalty method to simulate wave/structure interactions (*2017*).

# The governing equations

## Fluid dynamics

The water waves problem : Based on the Laplace equation

$$\begin{cases} \Delta\Phi = 0 \quad \text{in } \Omega_t \\ \Phi|_{z=\zeta} = \psi, \quad \sqrt{1+|\partial_x b|^2}\partial_{\mathbf{n}}\Phi_{\mathrm{bott}} = \partial_t b. \end{cases}$$

An evolution equation for $\zeta$, the surface elevation.
An evolution equation for $\psi$, the velocity potential on the free surface.

# The governing equations

## Fluid dynamics

$$
\begin{cases}
\partial_t \zeta + \partial_x (h\overline{V}) = \partial_t b, \\
\partial_t \psi + g\zeta + \dfrac{1}{2}|\partial_x \psi|^2 - \dfrac{(-\partial_x(h\overline{V}) + \partial_t b + \partial_x \zeta \cdot \partial_x \psi)^2}{2(1+|\partial_x \zeta|^2)} = 0,
\end{cases}
$$

where

$$
\overline{V} = \frac{1}{h} \int_{-H_0+b}^{\zeta} \partial_x \Phi(\cdot, z)\, dz.
$$

# The governing equations

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## Solid mechanics

By Newton's second law :

$$\mathbf{F}_{\text{total}} = \mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{solid}-\text{bottom interaction}} + \mathbf{F}_{\text{solid}-\text{fluid interaction}}.$$

# The governing equations

## Fluid dynamics

$$\begin{cases} \partial_t \zeta + \partial_x(h\overline{V}) = \partial_t b, \\ \partial_t \psi + g\zeta + \dfrac{1}{2}|\partial_x \psi|^2 - \dfrac{(-\partial_x(h\overline{V}) + \partial_t b + \partial_x \zeta \cdot \partial_x \psi)^2}{2(1+|\partial_x \zeta|^2)} = 0, \end{cases}$$

where

$$\overline{V} = \frac{1}{h}\int_{-H_0+b}^{\zeta} \partial_x \Phi(\cdot, z)\, dz.$$

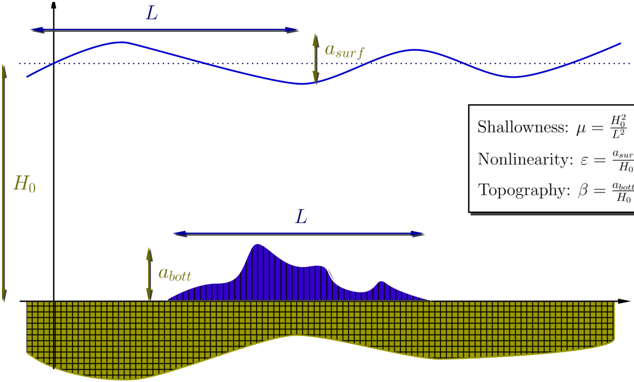
## Solid mechanics

The equation of motion for the solid

$$M\ddot{X}_S(t) = -c_{fric}\left(Mg + \int_{I(t)} P_{\mathrm{bott}}\, dx\right)\mathbf{e}_{\tan} + \int_{I(t)} P_{\mathrm{bott}}\partial_x b\, dx.$$

# Characteristic scales of the problem

- $L$, the characteristic horizontal scale of the wave motion,
- $H_0$, the base water depth,
- $a_{\mathrm{surf}}$, the order of the free surface amplitude,
- $a_{\mathrm{bott}}$, the characteristic height of the solid.




Shallowness: $\mu = \frac{H_0^2}{L^2}$

Nonlinearity: $\varepsilon = \frac{a_{surf}}{H_0}$

Topography: $\beta = \frac{a_{bott}}{H_0}$

# The coupled Boussinesq system

With an order $\mathcal{O}(\mu^2)$ approximation, we are going to work in the so called weakly nonlinear Boussinesq regime

$$0 \leqslant \mu \leqslant \mu_{max} \ll 1, \quad \varepsilon = \mathcal{O}(\mu), \beta = \mathcal{O}(\mu). \tag{BOUS}$$

The coupled Boussinesq system with an object moving at the bottom writes as

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{V}) = \frac{\beta}{\varepsilon}\partial_t b, \\ \partial_t \psi + \zeta + \dfrac{\varepsilon}{2}|\partial_x \psi|^2 - \varepsilon\mu \dfrac{(-\partial_x(h\overline{V}) + \frac{\beta}{\varepsilon}\partial_t b + \partial_x(\varepsilon\zeta)\cdot\partial_x\psi)^2}{2(1+\varepsilon^2\mu|\partial_x\zeta|^2)} = 0, \\ \ddot{X}_S(t) = -\dfrac{c_{fric}}{\sqrt{\mu}}\left(1 + \dfrac{1}{\beta\tilde{M}}\displaystyle\int_{I(t)} P_{\mathrm{bott}}\, dx\right)\mathbf{e}_{\mathrm{tan}} + \dfrac{1}{\tilde{M}}\displaystyle\int_{\mathbb{R}} P_{\mathrm{bott}}\partial_x b\, dx. \end{cases}$$

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$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{V}) = \frac{\beta}{\varepsilon} \partial_t b, \\ \left(1 - \dfrac{\mu}{3}\partial_{xx}\right) \partial_t \overline{V} + \partial_x \zeta + \varepsilon \overline{V} \cdot (\partial_x \overline{V}) = -\dfrac{\mu}{2} \partial_x \partial_{tt} b, \\ \ddot{X}_S(t) = -\dfrac{c_{fric}}{\sqrt{\mu}} \left( \dfrac{1}{\beta} c_{solid} + \dfrac{\varepsilon}{\beta \tilde{M}} \displaystyle\int_{I(t)} \zeta \, dx \right) \mathbf{e}_{\tan} + \dfrac{\varepsilon}{\tilde{M}} \displaystyle\int_{\mathbb{R}} \zeta(t,x) \partial_x b \, dx, \end{cases}$$

# $L^2$ estimates

$$E_B(t) = \frac{1}{2}\int_{\mathbb{R}} \zeta^2\,dx + \frac{1}{2}\int_{\mathbb{R}} h\overline{V}^2\,dx + \frac{1}{2}\int_{\mathbb{R}} \frac{\mu}{3} h(\partial_x \overline{V})^2\,dx + \frac{\tilde{M}}{2\varepsilon}\left|\dot{X}_S(t)\right|^2,$$

## Proposition

*Let $\mu \ll 1$ sufficiently small and let us take $s_0 > 1$. Any $\mathcal{U} \in \mathcal{C}^1([0,T]\times\mathbb{R}) \cap \mathcal{C}^1([0,T]; H^{s_0}(\mathbb{R}))$, $X_S \in \mathcal{C}^2([0,T])$ solutions to the coupled system, with initial data $\mathcal{U}(0,\cdot) = \mathcal{U}_{in} \in L^2(\mathbb{R})$ and $(X_S(0), \dot{X}_S(0)) = (0, v_{S_0}) \in \mathbb{R}\times\mathbb{R}$, verify*

$$\sup_{t\in[0,T]} \left\{ e^{-\sqrt{\varepsilon}c_0 t} E_B(t) \right\} \leqslant 2E_B(0) + \mu T c_0 \|\mathfrak{b}\|_{H^3},$$

*where*

$$c_0 = c(|||\mathcal{U}|||_{T,W^{1,\infty}}, |||\mathcal{U}|||_{T,H^{s_0}}, \|b\|_{W^{4,\infty}}).$$

# Long time existence for the Boussinesq system

## Theorem

*Let $\mu$ sufficiently small and $\varepsilon = \mathcal{O}(\mu)$. Let us suppose that the initial values $\zeta_{in}$ and $\mathfrak{b}$ satisfy the minimal water depth condition.*
*If $\zeta_{in}$ and $\overline{V}_{in}$ belong to $H^{s+1}(\mathbb{R})$ with $s \in \mathbb{R}$, $s > 3/2$, and that $X_{S_0}, v_{S_0} \in \mathbb{R}$, then there exists a maximal time $T > 0$ independent of $\varepsilon$ such that there exists a solution*

$$(\zeta, \overline{V}) \in C\left(\left[0, \frac{T}{\sqrt{\varepsilon}}\right]; H^{s+1}(\mathbb{R})\right) \cap C^1\left(\left[0, \frac{T}{\sqrt{\varepsilon}}\right]; H^{s}(\mathbb{R})\right),$$
$$X_S \in C^2\left(\left[0, \frac{T}{\sqrt{\varepsilon}}\right]\right)$$

*of the coupled system*

$$\left\{\begin{array}{l} D_\mu \partial_t \mathcal{U} + A(\mathcal{U}, X_S)\partial_x \mathcal{U} + B(\mathcal{U}, X_S) = 0, \\ \ddot{X}_S(t) = \mathcal{F}[\mathcal{U}]\left(t, X_S(t), \dot{X}_S(t)\right). \end{array}\right.$$

*with initial data $(\zeta_{in}, \overline{V}_{in})$ and $(X_{S_0}, v_{S_0})$.*

# The numerical scheme

The discretization in space : Adapting a staggered grid finite difference scheme, based on the work of P. Lin and Ch. Man (*Appl. Math. Mod. 2007*).

- finite difference scheme,
- surface elevation and bottom is defined on grid points, averaged velocity is defined on mid-points,
- order 4 central difference scheme,
- third order Simpson method for calculating the integrals.

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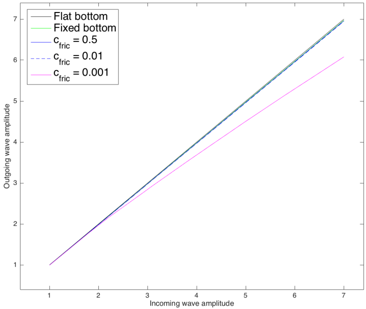
- order 4 central difference scheme,
- third order Simpson method for calculating the integrals.

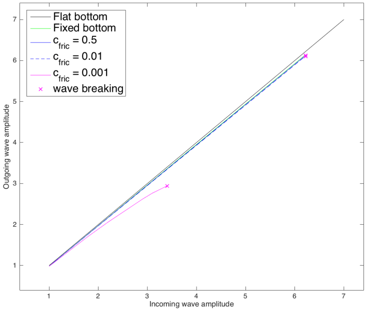
The discretization in time :

- Adams $4^{th}$ order predictor-corrector algorithm for the fluid dynamics
- An explicit scheme for the solid equation : an adapted second order central scheme
- preserves the dissipative property due to the friction,

# Amplitude variation for a passing wave

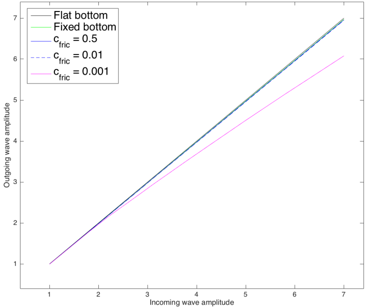



(a) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.3$




(b) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.5$

# Amplitude variation for a passing wave

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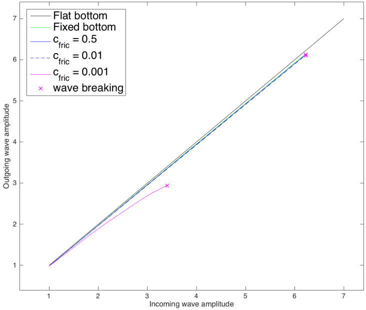



(b) Change in wave amplitude, $\mu = 0.25$, $\beta = 0.5$

Noticeable attenuation for the moving solid.
Observe the wave-breaking for the relatively large solid.

# The solid motion under the influence of the waves

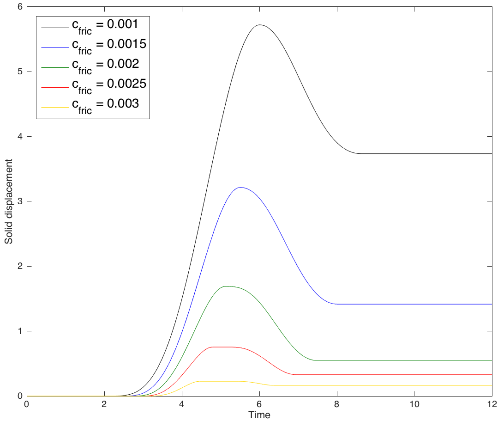
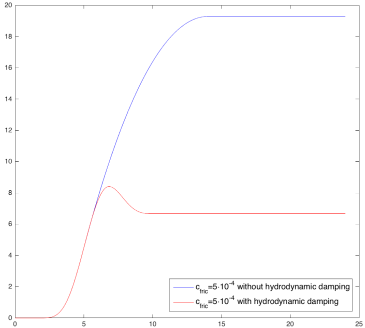


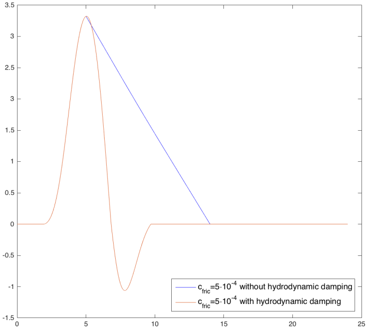

FIGURE – Solid position for varying coefficient of friction ($\mu = \varepsilon = 0.25$, $\beta = 0.3$)

Observable : *hydrodynamic* damping, *frictional* damping.

# The solid motion under the influence of the waves



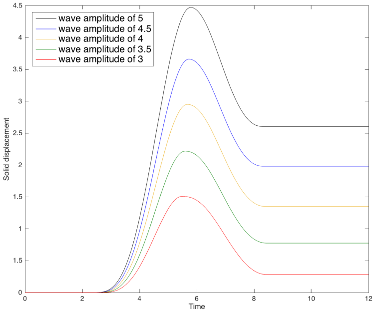

(a) Solid position, single wave, with and without hydrodynamic effects



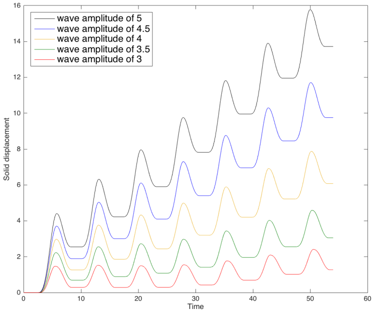

(b) Solid velocity, single wave, with and without hydrodynamic effects

Highlight : hydrodynamic damping effect

# The solid motion under the influence of the waves




(a) Solid position, single wave $\mu = 0.25$, $\beta = 0.3$, $c_{fric} = 0.001$




(b) Solid position, wavetrain $\mu = 0.25$, $\beta = 0.3$, $c_{fric} = 0.001$

# Influence over a long time scale

wave trains

# Conclusions

What we did :

- characterise mathematically the physical setting of an object on the bottom of an "oceanographic fluid domain",
- establish the coupled system,
- analyse the order 2 asymptotic system in $\mu$ (weakly nonlinear Boussinesq setting),
- create an accurate finite difference scheme for the coupled model,
- highlight the effects of a free solid motion on wave transformation as well as the effects of friction on the system.

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What we still have to do :

- treat the case of a non-horizontal bottom,
- generalize the notion of friction to a more realistic physical interpretation,
- ...

# Remerciements

Thank you for your attention !