Hyperbolic and dispersive models of shear shallow water

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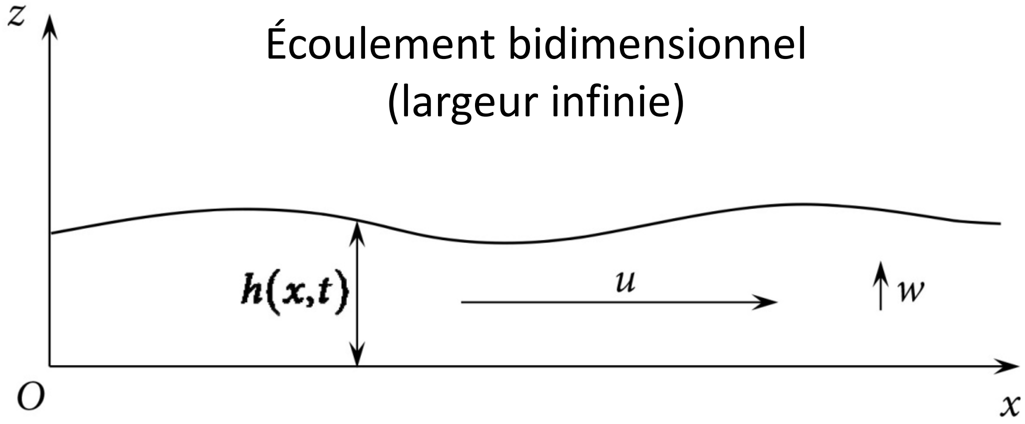
Modélisation

$\mathrm{Re} > 10^4$

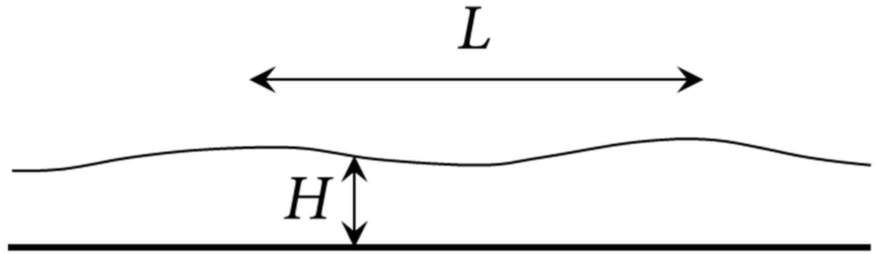
Écoulement bidimensionnel
(largeur infinie)

Fluide parfait incompressible



Faible profondeur

$$\varepsilon = \frac{H}{L} << 1.$$



Deux Modèles classiques

Saint-Venant (1871) : hyperbolique

(partie conservative)

$$\frac{\partial h}{\partial t}+\frac{\partial hU}{\partial x}=0,$$

Pression hydrostatique

$$\frac{\partial hU}{\partial t}+\frac{\partial\left(hU^{2}+gh^{2}/2\right)}{\partial x}=0.$$

Serre (1953), Su & Gardner (1969), Green & Naghdi (1976) : dispersif

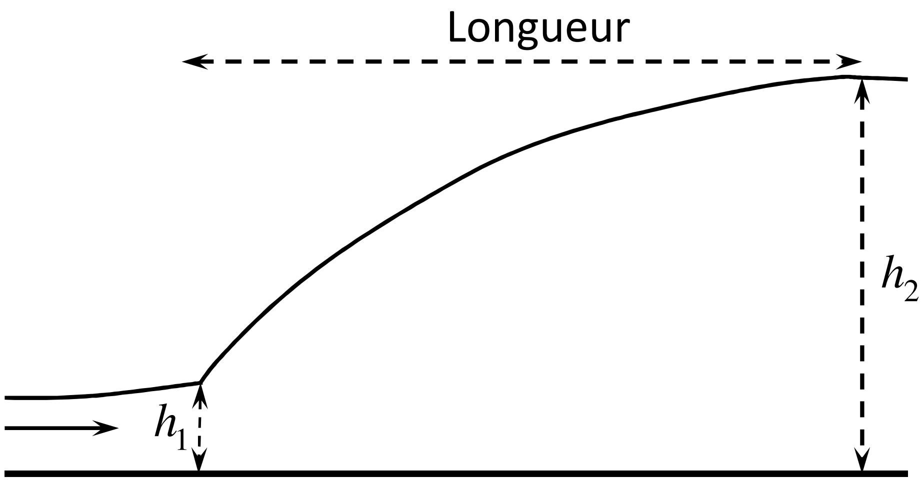
(partie conservative)

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(hU\right)=0,$$

Pression non hydrostatique

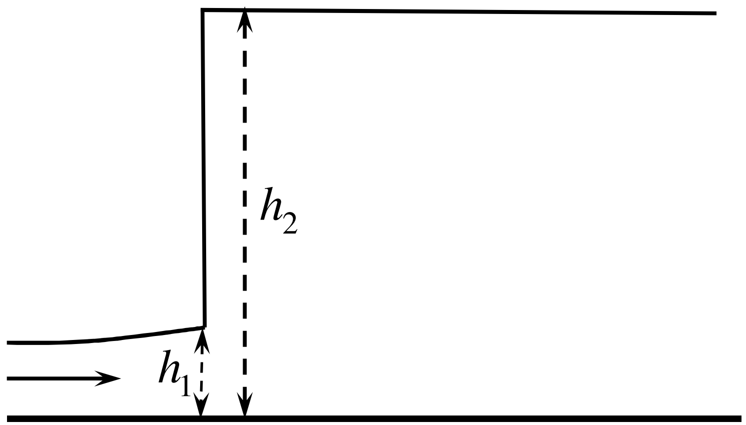
$$\frac{\partial hU}{\partial t}+\frac{\partial}{\partial x}\left(hU^{2}+\frac{gh^{2}}{2}+\frac{h^{2}}{3}\frac{D^{2}h}{Dt^{2}}\right)=0.$$

Ressaut réel



- Longueur non nulle
- Profil de profondeur
- Non stationnaire (oscillant)
- Faible écart à la relation de Bélanger (Hager *et al*. 1989)

Ressaut selon Saint-Venant



- Longueur nulle (choc)
- Pas de profil
- Stationnaire
- Relation de Bélanger

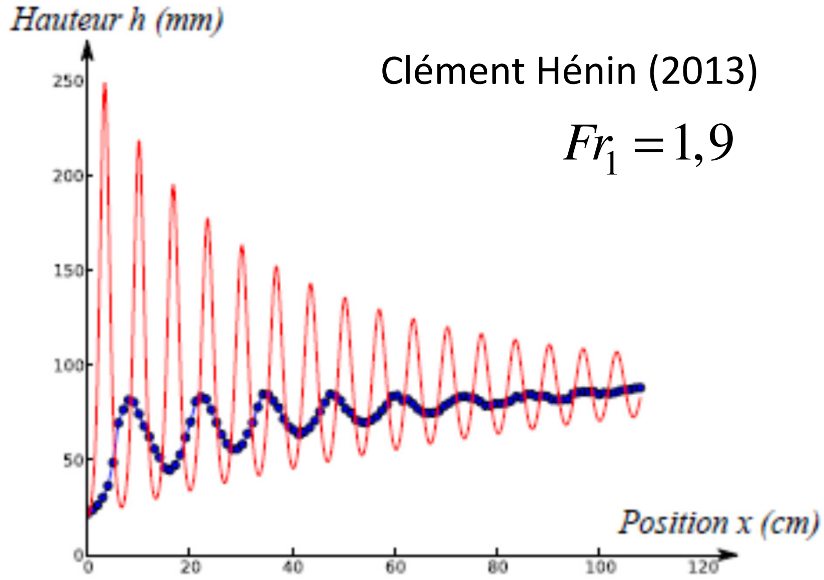
Modèle de Green-Naghdi et ressaut ondulaire

Clément Hénin (2013)

$Fr_1 = 1{,}9$

Amplitude : trop grande

Distance entre crêtes : trop petite



Modèle de Green-Naghdi et onde solitaire

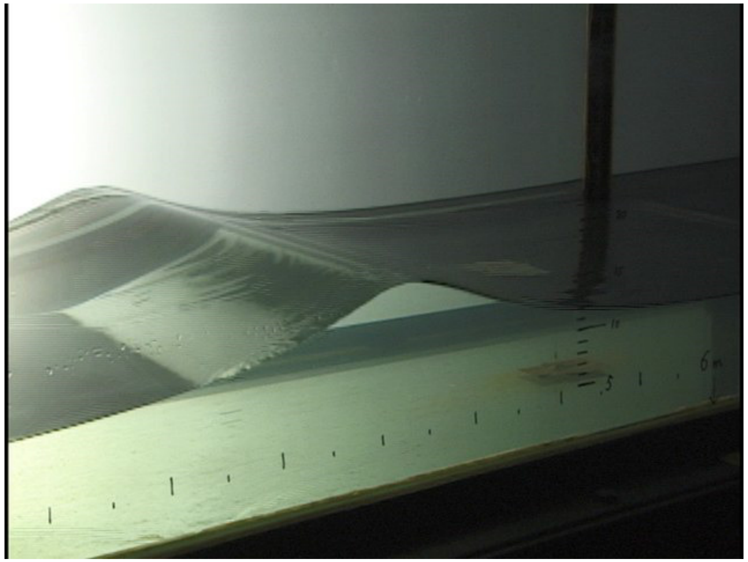
$F_\infty < 1{,}29$ Onde solitaire classique

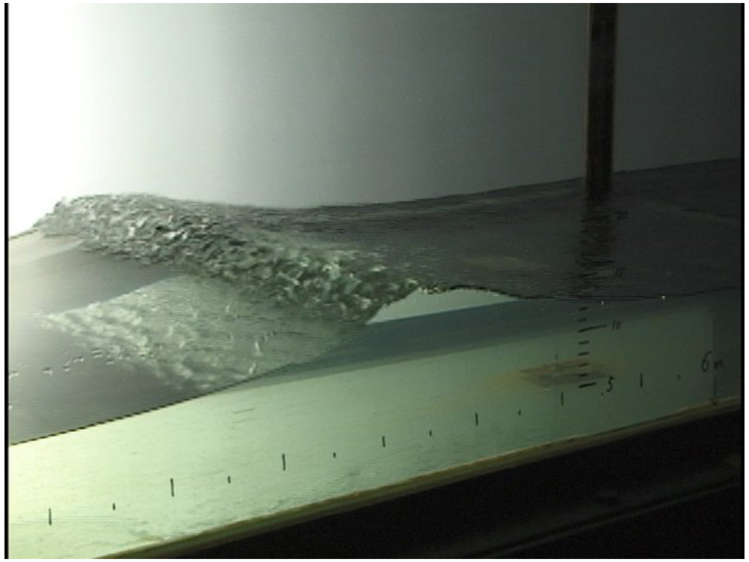
Profil : petit écart avec l'expérience

(Daily & Stephan 1952)

$F > 1{,}29$ Structure turbulente (micro-instabilité)

Watanabe *et al*. (2007)





Saint-Venant, Green-Naghdi

La vitesse horizontale est supposée **constante** dans la profondeur.

Pas de **cisaillement**.

Les modèles sont en désaccord avec l'expérience dès qu'apparaît une **structure turbulente**.

Pas **d'énergie turbulente**.

Équations d'Euler adimensionnées

Adimensionnement $\tilde{x}=\frac{x}{L}$; $\tilde{z}=\frac{z}{H}$; $\tilde{u}=\frac{u}{V}$; $\tilde{w}=\frac{w}{\varepsilon V}$; $\tilde{t}=\frac{Vt}{L}$; $\tilde{p}=\frac{p}{\rho g H}$; $V=\sqrt{gH}$.

$$\begin{cases} \dfrac{\partial \tilde{u}}{\partial \tilde{x}}+\dfrac{\partial \tilde{w}}{\partial \tilde{z}}=0 \\ \dfrac{\partial \tilde{u}}{\partial \tilde{t}}+\dfrac{\partial \tilde{u}^2}{\partial \tilde{x}}+\dfrac{\partial \tilde{u}\tilde{w}}{\partial \tilde{z}}=-\dfrac{\partial \tilde{p}}{\partial \tilde{x}} \\ \varepsilon^2\left(\dfrac{\partial \tilde{w}}{\partial \tilde{t}}+\dfrac{\partial \tilde{u}\tilde{w}}{\partial \tilde{x}}+\dfrac{\partial \tilde{w}^2}{\partial \tilde{z}}\right)=-1-\dfrac{\partial \tilde{p}}{\partial \tilde{z}} \\ \dfrac{\partial \tilde{E}}{\partial \tilde{t}}+\dfrac{\partial}{\partial \tilde{x}}\left(\tilde{u}\tilde{E}+\tilde{p}\tilde{u}\right)+\dfrac{\partial}{\partial \tilde{z}}\left(\tilde{w}\tilde{E}+\tilde{p}\tilde{w}\right)=0 \end{cases}$$

Masse

Quantité de mouvement

Énergie

où $\tilde{E}=\frac{\tilde{u}^2}{2}+\varepsilon^2\frac{\tilde{w}^2}{2}+\tilde{z}$

Conditions aux limites

$$\tilde{p}\left(\tilde{h}\right)=0\,;\quad \tilde{w}(0)=0$$

$$\tilde{w}\left(\tilde{h}\right)=\frac{\partial \tilde{h}}{\partial \tilde{t}}+\tilde{u}\left(\tilde{h}\right)\frac{\partial \tilde{h}}{\partial \tilde{x}}$$

Moyennisation sur la profondeur

$$\langle X \rangle = \frac{1}{h}\int_0^h X \,\mathrm{d}z \qquad \langle u \rangle = U$$

Masse
$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0$$
(exact)

Quantité de mouvement
$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left(\tilde{h}\langle \tilde{u}^2 \rangle + \int_0^{\tilde{h}} \tilde{p}\,\mathrm{d}\tilde{z} \right) = 0$$
(exact)

Énergie
$$\frac{\partial}{\partial \tilde{t}}\left[\tilde{h}\left(\frac{\langle \tilde{u}^2 \rangle}{2} + \frac{\tilde{h}}{2} + \varepsilon^2 \frac{\langle \tilde{w}^2 \rangle}{2} \right) \right] + \frac{\partial}{\partial \tilde{x}}\left[\frac{\tilde{h}\langle \tilde{u}^3 \rangle}{2} + \int_0^{\tilde{h}} \tilde{z}\tilde{u}\,\mathrm{d}\tilde{z} + \int_0^{\tilde{h}} \tilde{p}\tilde{u}\,\mathrm{d}\tilde{z} + \varepsilon^2 \frac{\tilde{h}\langle \tilde{u}\tilde{w}^2 \rangle}{2} \right] = 0$$
(exact)

Décomposition de la vitesse : $u(x,z,t) = U(x,t) + u_1(x,z,t)$

où $\langle u \rangle = U \qquad \langle u_1 \rangle = 0 \qquad u_1 = O(\varepsilon^\beta) \qquad \tilde{u} = \tilde{U} + \varepsilon^\beta \tilde{u}_1$

$$\langle \tilde{u}^2 \rangle = \tilde{U}^2 + \varepsilon^{2\beta}\langle \tilde{u}_1^2 \rangle \qquad \langle \tilde{u}^3 \rangle = \tilde{U}^3 + 3\varepsilon^{2\beta}\tilde{U}\langle \tilde{u}_1^2 \rangle + \varepsilon^{3\beta}\langle \tilde{u}_1^3 \rangle$$

Masse $$\frac{\partial \tilde{h}}{\partial \tilde{t}}+\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}}=0$$ (exact)

Quantité de mouvement $$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}}+\frac{\partial}{\partial \tilde{x}}\left(\tilde{h}\tilde{U}^2+\varepsilon^{2\beta}\tilde{h}^3\left\langle \tilde{\Psi}_1^2\right\rangle+\int_0^{\tilde{h}}\tilde{p}\,\mathrm{d}\tilde{z}\right)=0$$ (exact)

Énergie $$\frac{\partial}{\partial \tilde{t}}\left[\tilde{h}\left(\frac{\tilde{U}^2}{2}+\varepsilon^{2\beta}\frac{\tilde{h}^2\left\langle \tilde{\Psi}_1^2\right\rangle}{2}+\frac{\tilde{h}}{2}+\varepsilon^2\frac{\left\langle \tilde{w}^2\right\rangle}{2}\right)\right]+$$ (exact)

$$+\frac{\partial}{\partial \tilde{x}}\left[\frac{\tilde{h}\tilde{U}^3}{2}+\frac{3}{2}\varepsilon^{2\beta}\tilde{h}^3\tilde{U}\left\langle \tilde{\Psi}_1^2\right\rangle+\varepsilon^{3\beta}\frac{\tilde{h}\left\langle \tilde{u}_1^3\right\rangle}{2}+\int_0^{\tilde{h}}\tilde{z}\tilde{u}\,\mathrm{d}\tilde{z}+\int_0^{\tilde{h}}\tilde{p}\tilde{u}\,\mathrm{d}\tilde{z}+\varepsilon^2\frac{\tilde{h}\left\langle \tilde{u}\tilde{w}^2\right\rangle}{2}\right]=0$$

Pression

Enstrophie

$$\tilde{p}(\tilde{z})=\underbrace{\tilde{h}-\tilde{z}}_{\text{hydrostatique}}+\underbrace{\varepsilon^2\tilde{p}_{NH}}_{\text{non-hydrostatique}}$$

$$\Psi=\frac{\left\langle u_1^2\right\rangle}{h^2}\qquad \tilde{\Psi}=\frac{\Psi H}{g}$$

$$\int_0^{\tilde{h}}\left(\tilde{h}-\tilde{z}\right)\mathrm{d}\tilde{z}=\frac{\tilde{h}^2}{2}$$

Masse $$\frac{\partial \tilde{h}}{\partial \tilde{t}}+\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}}=0$$ (exact)

Quantité de mouvement $$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}}+\frac{\partial}{\partial \tilde{x}}\left(\tilde{h}\tilde{U}^2+\varepsilon^{2\beta}\tilde{h}^3\tilde{\Psi}+\frac{\tilde{h}^2}{2}+\varepsilon^2\int_0^{\tilde{h}}\tilde{p}_{NH}\ \mathrm{d}\tilde{z}\right)=0$$ (exact)

Énergie $$\frac{\partial}{\partial \tilde{t}}\left[\tilde{h}\left(\frac{\tilde{U}^2}{2}+\varepsilon^{2\beta}\frac{\tilde{h}^2\tilde{\Psi}}{2}+\frac{\tilde{h}}{2}+\varepsilon^2\frac{\left\langle\tilde{w}^2\right\rangle}{2}\right)\right]+$$ (exact)

$$+\frac{\partial}{\partial \tilde{x}}\left[\frac{\tilde{h}\tilde{U}^3}{2}+\frac{3}{2}\varepsilon^{2\beta}\tilde{h}^3\tilde{U}\,\tilde{\Psi}+\varepsilon^{3\beta}\frac{\tilde{h}\left\langle\tilde{u}_1^3\right\rangle}{2}+\right.$$

$$\left.+\tilde{h}^2\tilde{U}+\varepsilon^2\tilde{U}\int_0^{\tilde{h}}\tilde{p}_{NH}\ \mathrm{d}\tilde{z}+\varepsilon^{2+\beta}\int_0^{\tilde{h}}\tilde{p}_{NH}\tilde{u}_1\ \mathrm{d}\tilde{z}+\varepsilon^2\frac{\tilde{h}\left\langle\tilde{u}\tilde{w}^2\right\rangle}{2}\right]=0$$

$$\tilde{p}_{NH}=-\int_{\tilde{h}}^{\tilde{z}}\left(\frac{\partial \tilde{w}}{\partial \tilde{t}}+\frac{\partial \tilde{u}\tilde{w}}{\partial \tilde{x}}+\frac{\partial \tilde{w}^2}{\partial \tilde{z}}\right)\mathrm{d}\tilde{z}$$

$\tilde{w}(\tilde{z})$?

$$\frac{\partial \tilde{u}}{\partial \tilde{x}} + \frac{\partial \tilde{w}}{\partial \tilde{z}} = 0$$

$$\tilde{w}(\tilde{z}) = -\int_0^{\tilde{z}} \frac{\partial \tilde{u}}{\partial \tilde{x}}\, \mathrm{d}\tilde{z}$$

$$\tilde{u} = \tilde{U} + \varepsilon^{\beta} \tilde{u}_1$$

$$\tilde{w}(\tilde{z}) = -\tilde{z}\frac{\partial \tilde{U}}{\partial \tilde{x}} - \varepsilon^{\beta} \int_0^{\tilde{z}} \frac{\partial \tilde{u}_1}{\partial \tilde{x}}\, \mathrm{d}\tilde{z}$$

$$\dot{h} = \frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} = -h\frac{\partial U}{\partial x}$$

$$\tilde{w}(\tilde{z}) = \tilde{z}\frac{\dot{\tilde{h}}}{\tilde{h}} - \varepsilon^{\beta} \int_0^{\tilde{z}} \frac{\partial \tilde{u}_1}{\partial \tilde{x}}\, \mathrm{d}\tilde{z}$$

$$\tilde{p}_{NH} = \underbrace{\frac{\tilde{h}^2 - \tilde{z}^2}{2}\frac{\ddot{\tilde{h}}}{\tilde{h}}}_{\text{Green-Naghdi}} + \varepsilon^{\beta} \tilde{p}_{NGN} + O\left(\varepsilon^{2\beta}\right)$$

$$\varepsilon^2 \int_0^{\tilde{h}} \tilde{p}_{NH}\, \mathrm{d}\tilde{z} = \underbrace{\varepsilon^2 \frac{\tilde{h}^2 \ddot{\tilde{h}}}{3}}_{\text{Green-Naghdi}} + \varepsilon^{2+\beta} \int_0^{\tilde{h}} \tilde{p}_{NGN}\, \mathrm{d}\tilde{z} + O\left(\varepsilon^{2+2\beta}\right)$$

Termes $O\left(\varepsilon^{2+\beta}\right)$

$$\int_0^{\tilde{h}} \tilde{p}_{NGN}\ \mathrm{d}\tilde{z} = \frac{\partial}{\partial \tilde{x}}\frac{\mathrm{D}\tilde{A}}{\mathrm{D}\tilde{t}} + 2\tilde{A}\frac{\partial}{\partial \tilde{x}}\left(\frac{\dot{\tilde{h}}}{\tilde{h}}\right)$$

$$\text{où}\quad \tilde{A} = \int_0^{\tilde{h}} \mathrm{d}\tilde{z}\int_{\tilde{h}}^{\tilde{z}} \mathrm{d}\tilde{z}\int_0^{\tilde{z}} \tilde{u}_1\ \mathrm{d}\tilde{z} = \int_0^{\tilde{h}} \frac{\tilde{z}^2}{2}\tilde{u}_1\ \mathrm{d}\tilde{z}$$

$$\varepsilon^2\left\langle \tilde{w}^2\right\rangle = \varepsilon^2\frac{\dot{\tilde{h}}^2}{3} + 2\varepsilon^{2+\beta}\frac{\dot{\tilde{h}}}{\tilde{h}^2}\frac{\partial \tilde{A}}{\partial \tilde{x}} + O\left(\varepsilon^{2+2\beta}\right)$$

$$\varepsilon^{2+\beta}\int_0^{\tilde{h}} \tilde{p}_{NH}\tilde{u}_1\ \mathrm{d}\tilde{z} = -\varepsilon^{2+\beta}\tilde{A}\frac{\ddot{\tilde{h}}}{\tilde{h}} + O\left(\varepsilon^{2+2\beta}\right)$$

$$\varepsilon^2\tilde{h}\left\langle \tilde{u}\tilde{w}^2\right\rangle = \varepsilon^2\frac{\tilde{h}\tilde{U}\dot{\tilde{h}}^2}{3} + 2\varepsilon^{2+\beta}\tilde{A}\left(\frac{\dot{\tilde{h}}}{\tilde{h}}\right)^2 + 2\varepsilon^{2+\beta}\frac{\dot{\tilde{h}}}{\tilde{h}}\tilde{U}\frac{\partial \tilde{A}}{\partial \tilde{x}} + O\left(\varepsilon^{2+2\beta}\right)$$

Finalement :

$$\frac{\partial \tilde{h}}{\partial \tilde{t}}+\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}}=0$$

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}}+\frac{\partial}{\partial \tilde{x}}\left[\tilde{h}\tilde{U}^2+\tilde{P}\right]=O\left(\varepsilon^{2+2\beta}\right)$$

$$\frac{\partial \tilde{h}\tilde{e}}{\partial \tilde{t}}+\frac{\partial}{\partial \tilde{x}}\left[\tilde{h}\tilde{U}\left(\tilde{e}+\frac{\tilde{P}}{\tilde{h}}\right)+\varepsilon^{2+\beta}\tilde{A}\left(\frac{\dot{\tilde{h}}^2}{\tilde{h}^2}-\frac{\ddot{\tilde{h}}}{\tilde{h}}\right)+\varepsilon^{3\beta}\frac{\tilde{h}\left\langle \tilde{u}_1^3\right\rangle}{2}\right]=O\left(\varepsilon^{2+2\beta}\right)$$

où

$$\tilde{e}=\frac{\tilde{U}^2}{2}+\varepsilon^{2\beta}\frac{\tilde{h}^2\tilde{\Psi}}{2}+\frac{\tilde{h}}{2}+\varepsilon^2\frac{\dot{\tilde{h}}^2}{6}+\varepsilon^{2+\beta}\frac{\dot{\tilde{h}}}{\tilde{h}^2}\frac{\partial \tilde{A}}{\partial \tilde{x}}$$

$$\tilde{P}=\frac{\tilde{h}^2}{2}+\varepsilon^{2\beta}\tilde{h}^3\tilde{\Psi}+\varepsilon^2\frac{\tilde{h}^2\ddot{\tilde{h}}}{3}+\varepsilon^{2+\beta}\left[\frac{\partial}{\partial \tilde{x}}\frac{\mathrm{D}\tilde{A}}{\mathrm{D}\tilde{t}}+2\tilde{A}\frac{\partial}{\partial \tilde{x}}\left(\frac{\dot{\tilde{h}}}{\tilde{h}}\right)\right]$$

Valeur de β

Vorticité : $\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$

Équation d'évolution de la vorticité : $\frac{\mathrm{D}\omega}{\mathrm{D}t} = 0$

Si, à $t = 0$, $\omega = O\left(\varepsilon^{\alpha}\right)$, alors pour $t > 0$, $\omega = O\left(\varepsilon^{\alpha}\right)$.

$$\tilde{\omega} = \frac{\omega}{\varepsilon^{\alpha}} \frac{H}{V}$$

$$\varepsilon^{\beta} \frac{\partial \tilde{u}_1}{\partial \tilde{z}} = \underbrace{\varepsilon^{\alpha} \tilde{\omega}}_{\substack{\text{terme} \\ \text{rotationnel}}} + \underbrace{\varepsilon^{2} \frac{\partial \tilde{w}}{\partial \tilde{x}}}_{\substack{\text{terme} \\ \text{irrotationnel}}}$$

$$\beta = \min(\alpha, 2)$$

Si $\alpha \geqslant 2$ $\quad \beta = 2$

Si $\alpha < 2$ $\quad \beta = \alpha$ $\quad \frac{\partial \tilde{u}_1}{\partial \tilde{z}} = \tilde{\omega} + O\left(\varepsilon^{2-\alpha}\right)$ Le cisaillement est essentiellement dû à la vorticité.

Si $\alpha = 0 \quad \beta = 0$ Le problème n'est pas fermé.

Si $\alpha > 0 \quad \beta > 0$ Alors : à l'ordre $O(1)$ ⟶ Saint-Venant (1871)

à un ordre supérieur :

Si $\alpha < 1$ à l'ordre $O\left(\varepsilon^{2\beta}\right)$ ⟶ Teshukov (2007)
Richard & Gavrilyuk (2013)

Si $\alpha > 1$ à l'ordre $O\left(\varepsilon^{2}\right)$ ⟶ Su-Gardner (1969),
Green-Naghdi (1976)

à l'ordre $O\left(\varepsilon^{2+\beta}\right)$ ⟶ Nouveau modèle

Si $\alpha = 1$ à l'ordre $O\left(\varepsilon^{2}\right)$ ⟶ Teshukov-Green-Naghdi

à l'ordre $O\left(\varepsilon^{3}\right)$ ⟶ Termes $O\left(\varepsilon^{2+\beta}\right)$

+ Termes $O\left(\varepsilon^{3\beta}\right)$

Cf. Castro & Lannes (2014)

$$\frac{\partial \tilde{h}}{\partial \tilde{t}} + \frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{x}} = 0$$

$$\frac{\partial \tilde{h}\tilde{U}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left[\tilde{h}\tilde{U}^2 + \tilde{P}\right] = O\left(\varepsilon^{2+2\beta}\right)$$

$$\frac{\partial \tilde{h}\tilde{e}}{\partial \tilde{t}} + \frac{\partial}{\partial \tilde{x}}\left[\tilde{h}\tilde{U}\left(\tilde{e} + \frac{\tilde{P}}{\tilde{h}}\right) + \varepsilon^{2+\beta}\tilde{A}\left(\frac{\dot{\tilde{h}}^2}{\tilde{h}^2} - \frac{\ddot{\tilde{h}}}{\tilde{h}}\right) + \varepsilon^{3\beta}\frac{\tilde{h}\left\langle \tilde{u}_1^3\right\rangle}{2}\right] = O\left(\varepsilon^{2+2\beta}\right)$$

$$\tilde{e} = \frac{\tilde{U}^2}{2} + \varepsilon^{2\beta}\frac{\tilde{h}^2\tilde{\Psi}}{2} + \frac{\tilde{h}}{2} + \varepsilon^2\frac{\dot{\tilde{h}}^2}{6} + \varepsilon^{2+\beta}\frac{\dot{\tilde{h}}}{\tilde{h}^2}\frac{\partial \tilde{A}}{\partial \tilde{x}}$$

$$\tilde{P} = \frac{\tilde{h}^2}{2} + \varepsilon^{2\beta}\tilde{h}^3\tilde{\Psi} + \varepsilon^2\frac{\tilde{h}^2\ddot{\tilde{h}}}{3} + \varepsilon^{2+\beta}\left[\frac{\partial}{\partial \tilde{x}}\frac{\mathrm{D}\tilde{A}}{\mathrm{D}\tilde{t}} + 2\tilde{A}\frac{\partial}{\partial \tilde{x}}\left(\frac{\dot{\tilde{h}}}{\tilde{h}}\right)\right]$$

Le Modèle hyperbolique

$0<\beta<1$ Teshukov (2007) Richard & Gavrilyuk (2013)

$$\frac{\partial h}{\partial t}+\frac{\partial hU}{\partial x}=0$$

$$\frac{\partial hU}{\partial t}+\frac{\partial}{\partial x}\left[hU^2+h^3(\varphi+\Phi)+\frac{gh^2}{2}\right]=0$$

$$\frac{\partial he}{\partial t}+\frac{\partial}{\partial x}\left[hU\left(e+\frac{P}{h}\right)\right]=-\frac{\Phi}{\varphi+\Phi}C_r U^2|U|$$

$$e\equiv\frac{1}{2}\left[U^2+gh+h^2(\varphi+\Phi)\right]$$

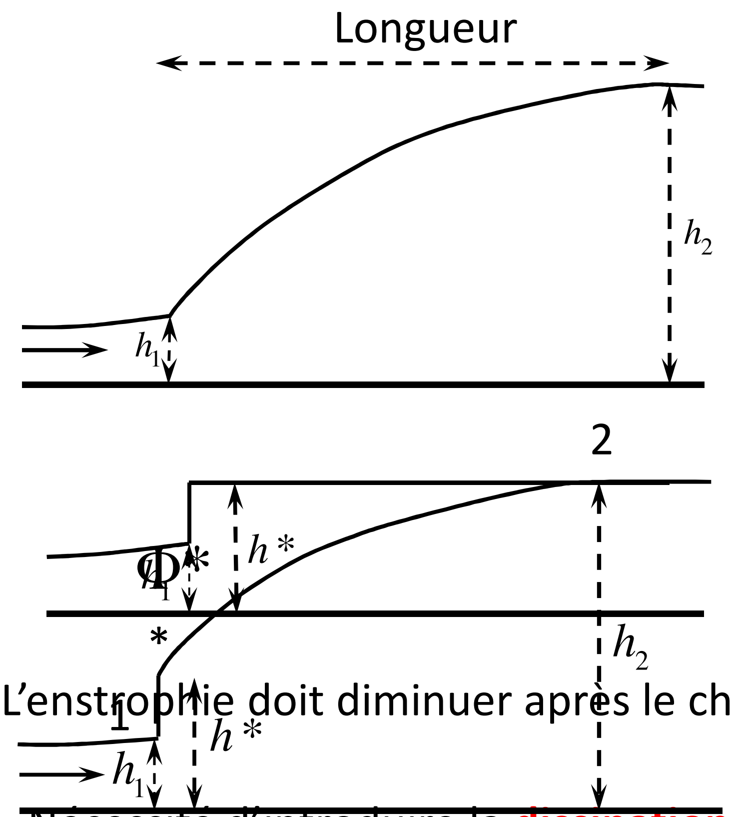
$$P\equiv\frac{gh^2}{2}+h^3(\varphi+\Phi)$$

Équation d'évolution de l'enstrophie

$$\frac{D\Phi}{Dt}=-2C_r\frac{\Phi}{\varphi+\Phi}\frac{U^2|U|}{h^3} \qquad \varphi=Cte$$

L'enstrophie est créée par les **chocs**.

Ressaut hydraulique



L'enstrophie doit diminuer après le choc

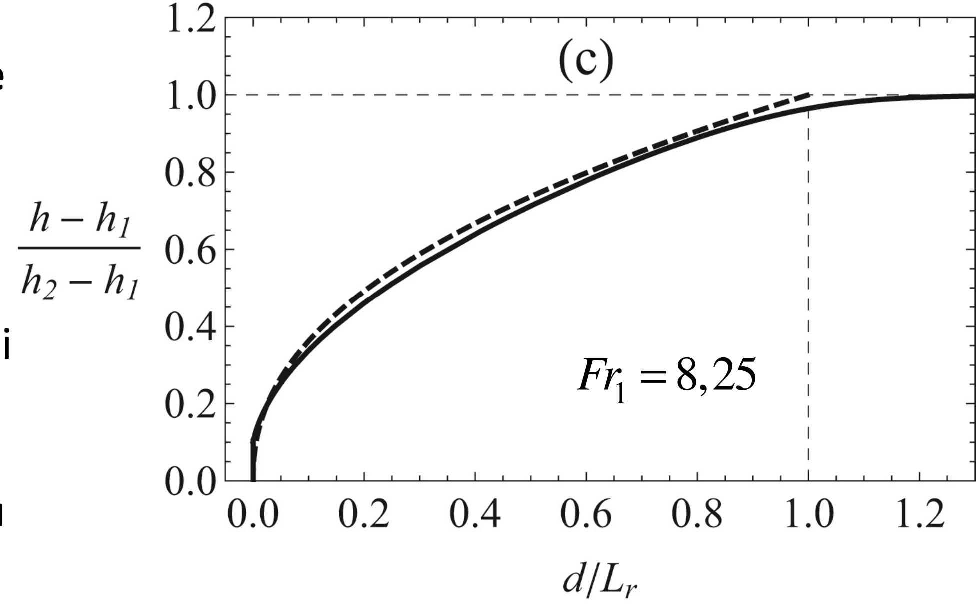
Nécessité d'introduire la **dissipation**

$\Phi_1=0$ $\Phi_2 \approx \varphi$

Résultats

Profil du ressaut hydraulique

Comparaison avec la loi expérimentale de Chanson (2011)



Disparition du rouleau turbulent si

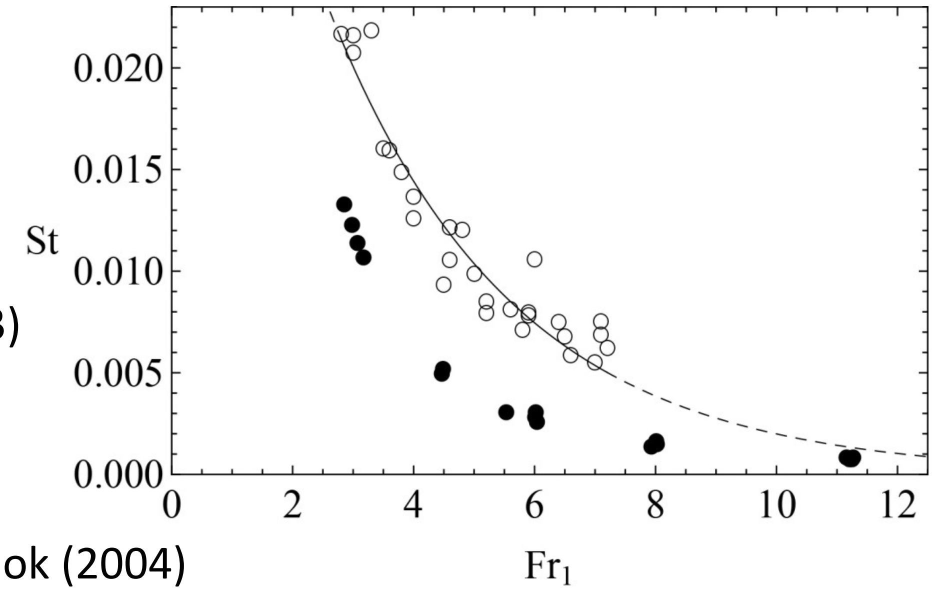
$$\tilde{\Phi} < \tilde{\Phi}_{cr} \approx 2.5 \cdot 10^{-3}$$

Oscillations de la position du ressaut hydraulique

Fréquence adimensionnée (nombre de Strouhal)

$$St = \frac{h_1^2 f}{q}$$

Comparaison avec les résultats expérimentaux de Zhang *et al*. (2013)



Pas d'oscillations pour $Fr_1 < 1,5$

Oscillations pour $Fr_1 > 1,5$

En accord avec les expériences de Mok (2004)

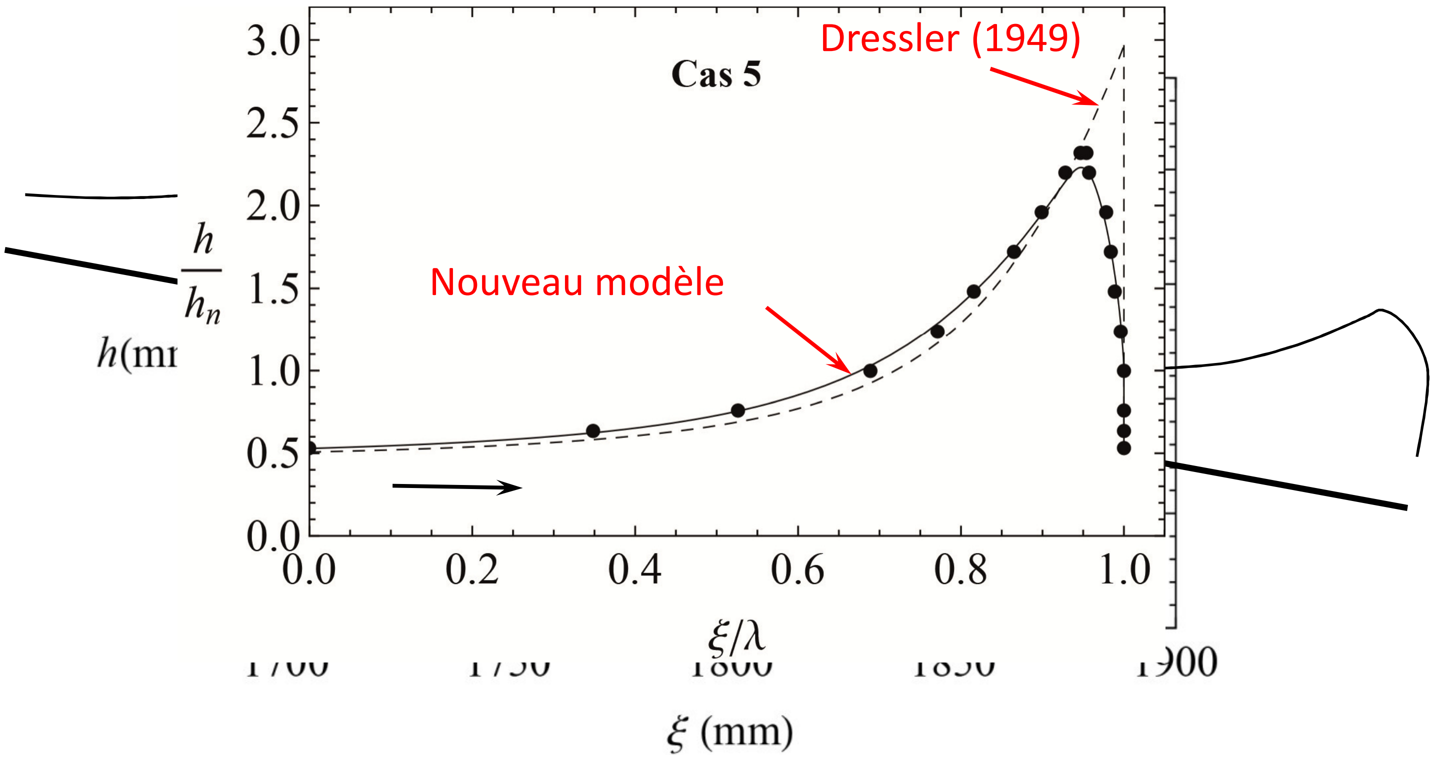
Application aux trains de rouleaux (roll-waves)

$g \to g\cos\theta$

Terme dissipatif supplémentaire : $gh\sin\theta - C_f U|U|$

$\left(gh\sin\theta - C_f U|U|\right)U$

pente $\mathrm{tg}\theta$



Le Modèle dispersif

$1 < \beta \leqslant 2$ On garde les termes $O\left(\varepsilon^{2+\beta}\right)$

$$3\beta > 2+\beta$$

$$2\beta \leqslant 2+\beta$$

(cas conservatif)

$$\frac{\partial h}{\partial t} + \frac{\partial hU}{\partial x} = 0$$

$$\frac{\partial hU}{\partial t} + \frac{\partial}{\partial x}\left[hU^2 + P\right] = 0$$

$$\frac{\partial he}{\partial t} + \frac{\partial}{\partial x}\left[hU\left(e + \frac{P}{h}\right) + A\left(\frac{\dot{h}^2}{h^2} - \frac{\ddot{h}}{h}\right)\right] = 0$$

$$e = \frac{U^2}{2} + \frac{h^2\Psi}{2} + \frac{gh}{2} + \frac{\dot{h}^2}{6} + \frac{\dot{h}}{h^2}\frac{\partial A}{\partial x}$$

$$P = \frac{gh^2}{2} + h^3\Psi + \frac{h^2\ddot{h}}{3} + \left[\frac{\partial}{\partial x}\frac{\mathrm{D}A}{\mathrm{D}t} + 2A\frac{\partial}{\partial x}\left(\frac{\dot{h}}{h}\right)\right]$$

Équation d'évolution de l'enstrophie

$$\frac{\mathrm{D}\Psi}{\mathrm{D}t} = \frac{2A}{h^3}\frac{\partial}{\partial x}\left(\frac{\ddot{h}}{h}\right)$$

Si on néglige les termes $O\left(\varepsilon^{2+\beta}\right)$, l'enstrophie se conserve.

Profil de l'onde solitaire

Su-Gardner

$$h = h_\infty + a\,\mathrm{sech}^2\left[\frac{x}{h_\infty}\sqrt{\frac{3a}{4(h_\infty + a)}}\right]$$

Boussinesq (1872)

$$h = h_\infty + a\,\mathrm{sech}^2\left[\frac{x}{h_\infty}\sqrt{\frac{3a}{4h_\infty}}\right]$$

$$\tilde{a} = \frac{a}{h_\infty} = F^2 - 1$$

Nouveau Modèle

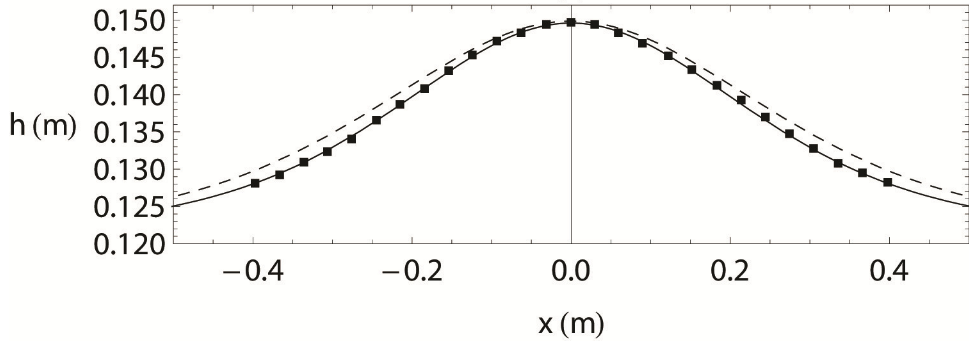
$$A = A_0 h^2 \qquad \varphi = \Phi_\infty \qquad \tilde{A}_0 = A_0 \Big/ \sqrt{g h_\infty^3}$$

$$h = h + \frac{2a\left(F^2 - 1 - 3\tilde{\varphi}\right)}{F^2 - 1 - \left(3 + \tilde{a}^2\right)\tilde{\varphi} + \left[F^2 - 1 - \left(3 - \tilde{a}^2\right)\tilde{\varphi}\right]\mathrm{ch}\left[\frac{x}{h_\infty}\sqrt{3\frac{F^2 - 1 - 3\tilde{\varphi}}{F^2 + 18\tilde{A}_0 F}}\right]}$$

Si $\varphi = 0$, $h = h_\infty + a\,\mathrm{sech}^2\left[\frac{x}{h_\infty}\sqrt{\frac{3}{4}\frac{a}{h_\infty + a + 18\tilde{A}_0\sqrt{h_\infty}\sqrt{a + h_\infty}}}\right]$

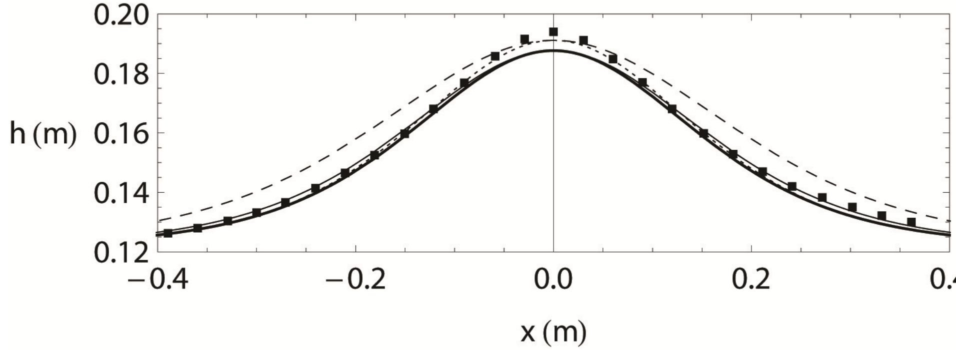
$F = 1{,}11\,;\quad \tilde{a} = 0{,}23\,;\quad \tilde{A}_0 = -0{,}0116\,;\quad \tilde{\varphi} = 5{,}33\cdot 10^{-4}.$

(a)



$F = 1{,}25\,;\quad \tilde{a} = 0{,}6\,;\quad \tilde{A}_0 = -0{,}0252(-0{,}020)\,;\quad \tilde{\varphi} = 5{,}19\cdot 10^{-3}.$

(b)

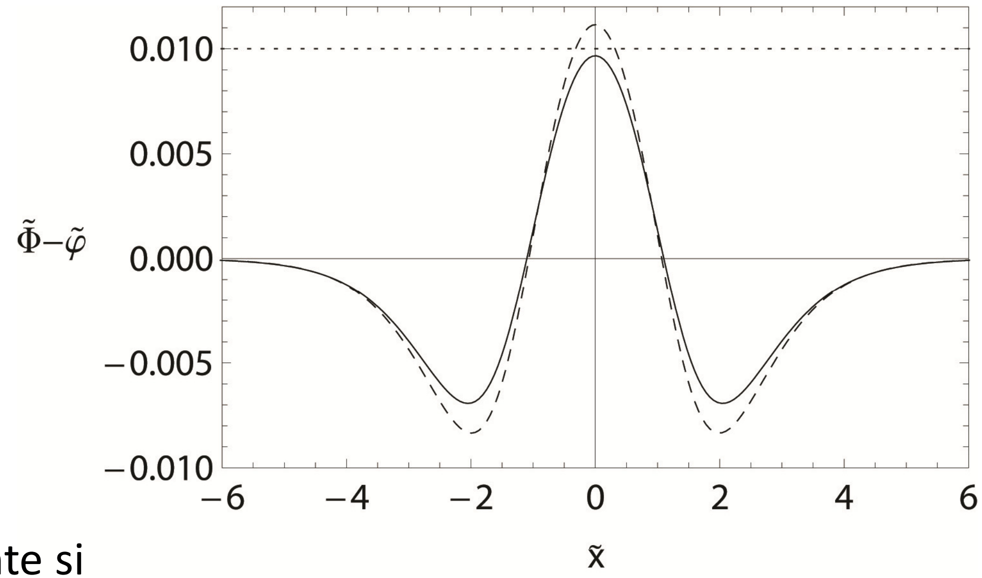


$F = 1,28$

Pas de structure turbulente
onde solitaire stable

$F = 1,30$

Structure turbulente
Onde solitaire instable



Apparition d'une structure turbulente si

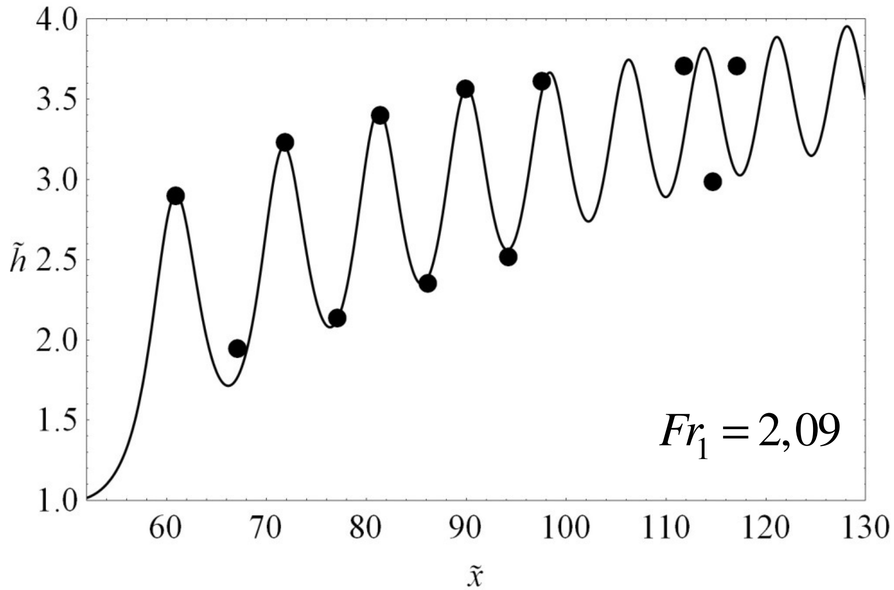
$$\tilde{\Phi} - \tilde{\varphi} > 10^{-2}$$

Profil du ressaut ondulaire

(avec dissipation)

Comparaison avec les résultats expérimentaux de Chanson (1993)

Le bon accord des distances crête à crête est dû **aux termes $O\left(\varepsilon^{2+\beta}\right)$ en *A***.



Conclusion et Perspectives

- Importance de la prise en compte du cisaillement ;

- Structures turbulentes (rouleaux, etc.) ;

- Perspective : modélisation du déferlement dans le cadre d’un modèle dispersif ;

- Modèle 2D à élaborer.