ON SOLITARY WAVES FOR BOUSSINESQ'S TYPE MODELS

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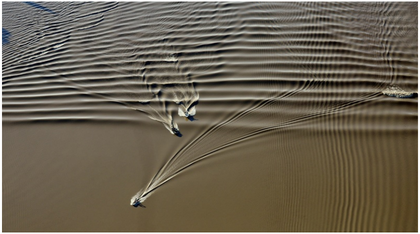
Context

Aim: Modeling wave transformation in near-shore zones.

- dispersive effects;
- nonlinear effects;

⇝ enhanced Boussinesq-type models, Green-Nadghi, ...

Sumatra 2004 tsunami reaching the coast of Thailand (from Madsen et al.2008)



Undular tidal bore — Garonne 2010 (from Bonneton et al.2011)

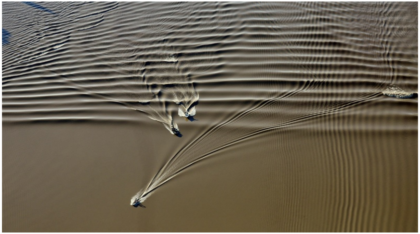
Example : Wave breaking

Question: recover energy dissipation effects ? :

- by locally reverting to NLSW equations;
- by locally adding eddy viscosity terms.

Challenge : the capture of these fronts needs to understand the wave shoaling.

Sumatra 2004 tsunami reaching the coast of Thailand (from Madsen et al.2008)



Undular tidal bore — Garonne 2010 (from Bonneton et al.2011)

Modeling

Several BT models : approximation of Euler equations

Design properties of these models :

- linear dispersion relation
- shoalling coefficients

Two kinds of BT models :

- amplitude-velocity system
- amplitude-volume flux system

$\rightsquigarrow$ degenerate to same linearized system but differ from high order derivatives terms !

cf : M.W. Dingemans

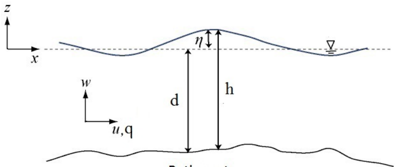
Euler system of equations

Euler system:

$$\begin{cases} \partial_t u + u\partial_x u + w\partial_z u + \dfrac{\partial_x p}{\rho} = 0 \\ \partial_t w + u\partial_x w + w\partial_z w + \dfrac{\partial_z p}{\rho} = 0 \\ \partial_x u + \partial_z w = 0 \\ \partial_z u - \partial_x w = 0 \end{cases} \tag{1}$$

B.C. :

- in $z = \eta$: $w = \partial_t \eta + u\partial_x \eta,\ P = 0$
- in $z = -d$: $w = -u\partial_x d$



Parameters

nonlinearity parameter $\varepsilon = a/d_0$

dispersion parameter $\sigma = d_0/L$

Weakly nonlinear models $\varepsilon = O(\sigma^2)$

Obtention of Nwogu equations :

$$\tilde{x} = \frac{x}{L},\ \tilde{z} = \frac{z}{d_0},\ \tilde{t} = \frac{\sqrt{gd_0}}{L}t,\ \tilde{\eta} = \frac{\eta}{a},\ \tilde{d} = \frac{d}{d_0},$$

$$\tilde{u} = \frac{d_0}{a\sqrt{gd_0}}u,\ \tilde{w} = \frac{L}{a}\frac{1}{\sqrt{gd_0}}v,\ \tilde{p} = \frac{p}{gd_0\rho},$$

Euler system of equations

Euler system:

$$\begin{cases} \varepsilon \tilde{u}_{\tilde{t}} + \varepsilon^2 \tilde{u}\tilde{u}_{\tilde{x}} + \varepsilon^2 \tilde{w}\tilde{u}_{\tilde{z}} + \tilde{p}_{\tilde{x}} = 0 \\ \varepsilon\sigma^2 \tilde{w}_{\tilde{t}} + \varepsilon^2\sigma^2 \tilde{u}\tilde{w}_{\tilde{x}} + \varepsilon^2\sigma^2 \tilde{w}\tilde{w}_{\tilde{z}} + \tilde{p}_{\tilde{z}} + 1 = 0 \\ \tilde{u}_{\tilde{x}} + \tilde{w}_{\tilde{z}} = 0 \\ \tilde{u}_{\tilde{z}} - \sigma^2 \tilde{w}_{\tilde{x}} = 0. \end{cases} \tag{2}$$

B.C. :

- in $\tilde{z} = \tilde{\eta}$: $\tilde{w} = \tilde{\eta}_{\tilde{t}} + \varepsilon \tilde{u}\tilde{\eta}_{\tilde{x}}$, $\tilde{P} = 0$
- in $\tilde{z} = -\tilde{d}$: $\tilde{w} = -\tilde{d}_{\tilde{x}}\tilde{u}$

Integrate incompressibility equation :

$$\tilde{w} = -\frac{\partial}{\partial \tilde{x}} \left(\int_{-\tilde{d}}^{\tilde{z}} \tilde{u} d\tilde{z} \right).$$

Nwogu

Plugging in the irrotational condition

$$\frac{\partial \tilde{u}}{\partial \tilde{z}} = -\sigma^2 \frac{\partial^2}{\partial \tilde{x}^2}\left(\int_{-\tilde{d}}^{\tilde{z}} \tilde{u} d\tilde{z}\right).$$

Taylor expansion of $\tilde{z} \mapsto \tilde{u}(\tilde{t},\tilde{x},\tilde{z})$ around $\tilde{z} = \tilde{z}_\alpha$ $(\tilde{U} = \tilde{u}(\tilde{t},\tilde{x},\tilde{z}_\alpha))$

$$\tilde{u} = \tilde{U} + (\tilde{z}-\tilde{z}_\alpha)\frac{\partial \tilde{u}}{\partial \tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \frac{(\tilde{z}-\tilde{z}_\alpha)^2}{2}\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots$$

Integrate

$$\int_{-\tilde{d}}^{\tilde{z}} \tilde{u} d\tilde{z} = (\tilde{z}+\tilde{d})\tilde{U} + \left(\frac{(\tilde{z}-\tilde{z}_\alpha)^2}{2} - \frac{(\tilde{d}+\tilde{z}_\alpha)^2}{2}\right)\frac{\partial \tilde{u}}{\partial \tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \left(\frac{(\tilde{z}-\tilde{z}_\alpha)^3}{6} + \frac{(\tilde{d}+\tilde{z}_\alpha)^3}{6}\right)\frac{\partial^2 \tilde{u}}{\partial \tilde{z}^2}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots$$

and plugg

$$\tilde{u}_{\tilde{z}} = -\sigma^2 \frac{\partial^2}{\partial \tilde{x}^2}\left((\tilde{z}+\tilde{d})\tilde{U} + \left[\frac{(\tilde{z}-\tilde{z}_\alpha)^2}{2} - \frac{(\tilde{d}+\tilde{z}_\alpha)^2}{2}\right]\frac{\partial \tilde{u}}{\partial \tilde{z}}|_{\tilde{z}=\tilde{z}_\alpha} + \cdots\right).$$

Nwogu

Apply $\partial_{\tilde z}$ and $\partial_{\tilde z}^2$:

$$\tilde u_{\tilde z\tilde z} = -\sigma^2 \frac{\partial^2 \tilde U}{\partial \tilde x^2} + \mathcal{O}(\sigma^4),\ \tilde u_{\tilde z\tilde z\tilde z} = \mathcal{O}(\sigma^4).$$

Coming back to Taylor expansion

$$\tilde u = \tilde U - \sigma^2 \left((\tilde z - \tilde z_\alpha) \frac{\partial^2}{\partial \tilde x^2} [(\tilde d + \tilde z_\alpha)\tilde U] + \frac{(\tilde z - \tilde z_\alpha)^2}{2} \frac{\partial^2 \tilde U}{\partial \tilde x^2} \right) + \mathcal{O}(\sigma^4).$$

and

$$\tilde w = -\frac{\partial}{\partial \tilde x} \left((\tilde d + \tilde z)\tilde U + \sigma^2 \Big(\frac{(\tilde d + \tilde z_\alpha)^2}{2} \frac{\partial^2}{\partial \tilde x^2} [(\tilde d + \tilde z_\alpha)\tilde U] - \frac{(\tilde d + \tilde z_\alpha)^3}{6} \frac{\partial^2 \tilde U}{\partial \tilde x^2} \Big) \right) + \mathcal{O}(\sigma^4).$$

Back to Euler :

$$-\varepsilon\sigma^2 \frac{\partial^2}{\partial \tilde t \partial \tilde x} \left((\tilde z + \tilde d)\tilde U \right) + \tilde p_{\tilde z} + 1 + \mathcal{O}(\varepsilon^2\sigma^2, \varepsilon\sigma^4) = 0. \tag{3}$$

Nwogu

and integrate between $\varepsilon\tilde{\eta}$ and $\tilde{z}$ + B.C.

$$\tilde{p} = \varepsilon\tilde{\eta} - \tilde{z} + \varepsilon\sigma^2 \frac{\partial^2}{\partial\tilde{t}\partial\tilde{x}}\left((\tilde{d}\tilde{z} + \frac{\tilde{z}^2}{2})\tilde{U}\right) + \mathcal{O}(\varepsilon\sigma^4, \varepsilon^2\sigma^2). \tag{4}$$

Plugg in the first equation of Euler

Nwogu 1:

$$\tilde{U}_{\tilde{t}} + \varepsilon\tilde{U}\tilde{U}_{\tilde{x}} + \tilde{\eta}_{\tilde{x}} + \sigma^2\left(\frac{\tilde{z}_\alpha^2}{2}\tilde{U}_{\tilde{t}\tilde{x}\tilde{x}} + \tilde{z}_\alpha[\tilde{d}\tilde{U}_{\tilde{t}}]_{\tilde{x}\tilde{x}}\right) = \mathcal{O}(\varepsilon\sigma^2, \sigma^4). \tag{5}$$

Nwogu

Integrate incompressibility equation :

$$\tilde{\eta}_{\tilde{t}} + \frac{\partial}{\partial \tilde{x}} \left(\int_{-\tilde{d}}^{\varepsilon \tilde{\eta}} \tilde{u} d\tilde{z} \right) = 0,$$

and using the expression of $\tilde{u}$

Nwogu 2:

$$\tilde{\eta}_{\tilde{t}} + \left[(\varepsilon \tilde{\eta} + \tilde{d}) \tilde{U} \right]_{\tilde{x}} + \sigma^2 \frac{\partial}{\partial \tilde{x}} \left(\left(\tilde{d} \tilde{z}_\alpha + \frac{(\tilde{d})^2}{2} \right) [\tilde{d} \tilde{U}]_{\tilde{x}\tilde{x}} + \left(\frac{\tilde{d} \tilde{z}_\alpha^2}{2} - \frac{(\tilde{d})^3}{6} \right) \tilde{U}_{\tilde{x}\tilde{x}} \right) = \mathcal{O}(\varepsilon \sigma^2, \sigma^4).$$

Nwogu-Abott

To go further :

$$\tilde{h} = \tilde{d} + \varepsilon\tilde{\eta}, \quad \tilde{Q} := \tilde{h}\tilde{U} = \tilde{d}\tilde{U} + \mathcal{O}(\varepsilon), \tag{6}$$

$$\tilde{U}_{\tilde{t}} = \left(\frac{\tilde{Q}}{\tilde{d} + \varepsilon\tilde{\eta}}\right)_{\tilde{t}} = \frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}} + \mathcal{O}(\varepsilon).$$

$$\tilde{U}_{\tilde{t}\tilde{x}\tilde{x}} = \left(\frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}}\right)_{\tilde{x}\tilde{x}} + \mathcal{O}(\varepsilon),$$

$$\tilde{U}_{\tilde{x}\tilde{x}\tilde{x}} = \left(\frac{\tilde{Q}}{\tilde{d}}\right)_{\tilde{x}\tilde{x}\tilde{x}} + \mathcal{O}(\varepsilon).$$

Nwogu-Abott

Multiply Nwogu 1 by $\tilde{h}$ and Nwogu 2 by $\varepsilon\tilde{u}$

$$\tilde{Q}_{\tilde{t}} + \varepsilon\left(\frac{\tilde{Q}^2}{\tilde{h}}\right)_{\tilde{x}} + \tilde{h}\tilde{\eta}_{\tilde{x}} + \sigma^2\tilde{d}\left(\tilde{z}_\alpha\tilde{Q}_{\tilde{t}\tilde{x}\tilde{x}} + \frac{\tilde{z}_\alpha^2}{2}\left(\frac{\tilde{Q}_{\tilde{t}}}{\tilde{d}}\right)_{\tilde{x}\tilde{x}}\right) = \mathcal{O}(\varepsilon\sigma^2,\sigma^4),$$

$$\tilde{\eta}_{\tilde{t}} + \tilde{Q}_{\tilde{x}} + \sigma^2\frac{\partial}{\partial\tilde{x}}\left(\left(\tilde{d}\tilde{z}_\alpha + \frac{(\tilde{d})^2}{2}\right)\tilde{Q}_{\tilde{x}\tilde{x}} + \left(\frac{\tilde{d}\tilde{z}_\alpha^2}{2} - \frac{(\tilde{d})^3}{6}\right)\left(\frac{\tilde{Q}}{\tilde{d}}\right)_{\tilde{x}\tilde{x}}\right) = \mathcal{O}(\varepsilon\sigma^2,\sigma^4).$$

Back to physical variables :

Nowgu-Abbott: (Bellec, Filippini, Ricchuito, C.)

$$\begin{cases} \eta_t + Q_x + \left[\beta_1 d^2 Q_{xx} + \beta_2 d^3\left(\frac{Q}{d}\right)_{xx}\right]_x = 0 \\ Q_t + \left(\frac{Q^2}{h}\right)_x + gh\eta_x + \alpha_1 d^2 Q_{txx} + \alpha_2 d^3\left(\frac{Q}{d}\right)_{txx} = 0, \end{cases}$$

$$\beta_1 = \theta + \frac{1}{2},\ \beta_2 = \frac{\theta^2}{2} - \frac{1}{6},\ \ \alpha_1 = \theta,\ \alpha_2 = \frac{\theta^2}{2}.$$

Remark : OK for Madsen-Sorensen, Beji-Nadaoka, ...

Nwogu-Abott

Dispersion Relation (constant bathymetry) :

$$\eta(t,x) = a\sin(kx-\omega t)$$
$$Q(t,x) = b\sin(kx-\omega t)$$

$$\frac{\omega^2}{k^2} = c_0^2\frac{1-\beta k^2 d^2}{1-\alpha k^2 d^2}$$
$$\alpha = \theta^2/2+\theta,\;\; \beta = \theta^2/2+\theta+\frac{1}{3}$$

Expansion :

$$c^2(k) = \frac{\omega^2}{k^2} = c_0^2(1+(\alpha-\beta)d^2k^2+\alpha(\alpha-\beta)d^4k^4)+O(k^6)$$

Optimized for :

$$\alpha = -\frac{2}{5},\;\; \beta = -\frac{1}{15}$$

Other asymptotic model

Peregrine :

$$\eta_t + [(\eta + d)\bar{u}]_x = 0. \tag{7}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x + \left(\frac{d^2}{6}\bar{u}_{txx} - \frac{d}{2}[d\bar{u}_t]_{xx}\right) = 0. \tag{8}$$

Abbott ((η, q) version of Peregrine) :

$$\eta_t + q_x = 0. \tag{9}$$

$$q_t + \left(\frac{q^2}{d+\eta}\right)_x + g(d+\eta)\eta_x + \left(\frac{d^3}{6}\left(\frac{q}{d}\right)_{txx} - \frac{d^2}{2}q_{txx}\right) = 0. \tag{10}$$

Beji-Nadaoka :

$$\eta_t + [(\eta + d)\bar{u}]_x = 0. \tag{11}$$

$$\bar{u}_t + \bar{u}\bar{u}_x + g\eta_x + (1+\alpha_B)\left(\frac{d^2}{6}\bar{u}_{txx} - \frac{d}{2}[d\bar{u}_t]_{xx}\right)$$
$$+ \alpha_B g\left(\frac{d^2}{6}\eta_{xxx} - \frac{d}{2}[d\eta_x]_{xx}\right) = 0. \tag{12}$$

Beji-Nadaoka-Abbott :

$$\eta_t + q_x = 0. \tag{13}$$

$$Q_t + \left(\frac{Q^2}{d+\eta}\right)_x + g(d+\eta)\eta_x + (1+\alpha_B)\left(\frac{d^3}{6}\left(\frac{Q}{d}\right)_{txx} - \frac{d^2}{2}Q_{txx}\right)$$
$$+ \alpha_B\left(\frac{d^3}{6}\eta_{xxx} - \frac{d}{2}[d\eta_x]_{xx}\right) = 0. \tag{14}$$

Cauchy Problem Nwogu-Abbott

$$\begin{cases} \partial_t \eta + Q_x + \beta\sigma^2 d^2 Q_{xxx} = 0, \\ Q_t + \varepsilon \left(\dfrac{Q^2}{d+\varepsilon\eta} \right)_x + g(d+\varepsilon\eta)\eta_x + \alpha\sigma^2 d^2 Q_{txx} = 0 \end{cases}$$

Nowgu-Abbott:

Theorem : Well posedness for $T = O(\frac{1}{\varepsilon})$.

$$\eta \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^2(\mathbb{R}))$$
$$Q \in C([0,T]; L^2(\mathbb{R})) \cap L^\infty(0,T; H^4(\mathbb{R}))$$

Conditions :

- constant bathymetry
- $d + \varepsilon\eta_0(x) > 0$ on $\mathbb{R}$
- $g(d+\varepsilon\eta_0) - \varepsilon \frac{Q_0^2}{(d+\varepsilon\eta_0)^2} > 0$ on $\mathbb{R}$

Two difficulties :

- loss of derivatives : Bona, Chen Saut [04]
- h do not vanish !

Motivations

Question : Why ?

- create a hierarchy of models
- verification of numerical schemes
- practical applications : shoaling, ...

$\rightsquigarrow$ provide existence and uniqueness result + free software to compute!

Main Contribution : exhibit the relation between c and A !

Condition : constant bathymetry

Beji-Nadaoka

Equations:

$$\begin{cases} \eta_t + [h\bar{u}]_x = 0 \\ \bar{u}_t + g\eta_x + \bar{u}\bar{u}_x - \gamma d^2 \bar{u}_{txx} - \alpha_B g d^2 \eta_{xxx} = 0 \end{cases}$$

$$\gamma = \alpha_B + \frac{1}{3}$$

Solitary wave

$$\eta(t,x) = \eta_c(x - ct)$$
$$\bar{u}(t,x) = \bar{u}_c(x - ct),$$

$$\begin{cases} \eta_c = \frac{d\bar{u}_c}{c - \bar{u}_c}, \\ -c\bar{u}_c + g\eta + \frac{\bar{u}_c^2}{2} + c\gamma d^2 \bar{u}_c'' - \alpha_B g d^2 \eta_c'' = 0 \end{cases}$$

Remark : $\bar{u}_c < c$!

Beji-Nadaoka

Case 1 : $\gamma = 0$.

$$-\eta'' = \frac{3}{gd^2}\Big(\frac{c^2d^2}{2(d+\eta)^2} - \frac{c^2}{2} + c_0^2\frac{\eta}{d}\Big).$$

$$g(s) = \frac{3}{gd^2}\left(\frac{c^2d^2}{2(d+s)^2} - \frac{c^2}{2} + c_0^2\frac{s}{d}\right), \quad G(s) := \int_0^s g(t)\, dt$$

Th : Beretycki-Lions.
Let f be a locally Lipschitz continuous real function with $f(0) = 0$ and $F(z) = \int_0^z f(s)ds$.

$$-u'' = f(u),\ u \in \mathcal{C}^2(\mathbb{R}),\ \lim_{x\to\pm\infty} u(x) = 0,\ u(x_0) > 0 \text{ for some } x_0 \in \mathbb{R}.$$

has a unique solution $u \in H^1(\mathbb{R}) \cap C^2(\mathbb{R})$ if and only if

$$\xi_0 = \inf\{\xi > 0,\ F(\xi) = 0\} \text{ exists, and satisfies } \xi_0 > 0,\ f(\xi_0) > 0.$$

Remark : $\xi_0 = \max_{\mathbb{R}} u$.

Beji-Nadaoka

$$g(s) = 0 \Longleftrightarrow \Big(s = 0 \text{ or } s = s_1 \text{ or } s = s_2 \Big),$$

where

$$s_1 := d \frac{(\frac{c^2}{c_0^2} - 4) - \sqrt{8\frac{c^2}{c_0^2} + \frac{c^4}{c_0^4}}}{4}, \; s_2 := d \frac{(\frac{c^2}{c_0^2} - 4) + \sqrt{8\frac{c^2}{c_0^2} + \frac{c^4}{c_0^4}}}{4}.$$

- $c \leq c_0$, we have $s_2 \leq 0$: NO SOLUTION !
- $c > c_0$: UNIQUE SOLUTION!

$$G(A) = 0 \Leftrightarrow c^2 = c_0^2 \frac{d + A}{d}$$

Beji-Nadaoka

Case 2 : $\gamma \neq 0$.

$$(c\gamma d^2(c-\bar{u}_c)^2 - \alpha_B d^2 c c_0^2)\bar{u}_c'' - \frac{2cc_0^2\alpha_B d^2}{(c-\bar{u}_c)}\bar{u}_c'^2 - c\bar{u}_c(c-\bar{u}_c)^2$$
$$+c_0^2\bar{u}_c(c-\bar{u}_c) + \frac{\bar{u}_c^2(c-\bar{u}_c)^2}{2} = 0.$$

$\rightsquigarrow$ QUASILINEAR !

Idea :

$$\bar{u}_c = f(v_c)$$

$$s = f(s) - \frac{c_0^2\alpha_B}{\gamma(c-f(s))} + \frac{c_0^2\alpha_B}{c\gamma}.$$

$$c^2 > c_0^2\frac{\alpha_B}{\gamma} \Rightarrow \text{ f is a diffeomorphism}$$

Beji-Nadaoka

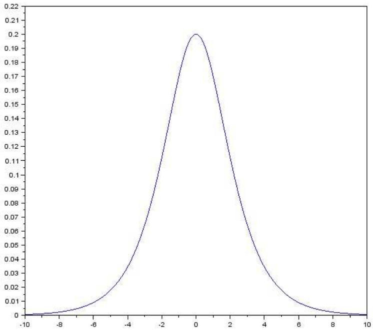
$$-v_c'' = \frac{1}{c\gamma d^2}\Big(- cf(v_c) + \frac{cc_0^2}{c - f(v_c)} - c_0^2 + \frac{f(v_c)^2}{2}\Big).$$

Conclusion :
Assume that one of the following alternative is satisfied :
i) $\gamma = 0,\ c > c_0,$
ii) $\alpha_B < 0,\ \gamma > 0,\ c > c_0,$
iii) $\alpha_B > 0,\ \gamma > 0,\ c \in (c_0, c_0 \frac{2 - \frac{\alpha_B}{\gamma}}{\sqrt{\frac{\alpha_B}{\gamma}}}).$
Then BN admits a unique solitary wave of the form $(\eta_c(x - ct), \bar{u}_c(x - ct))$.

$$c^4\gamma\left[\frac{A^3}{6(d+A)^3} - \frac{A^2}{2(d+A)^2}\right] + c^2c_0^2\left[\gamma\log(\frac{A+d}{d}) - \gamma\frac{A}{d+A} + \frac{\alpha_B}{2}\frac{A^2}{d(d+A)}\right] - c_0^4\frac{\alpha_B A^2}{2d^2} = 0.$$

Conversely, if $\gamma < 0$, $\alpha_B < 0$ and $c \geq 0$, then BN has no positive solutions of the previous form.

Beji-Nadaoka



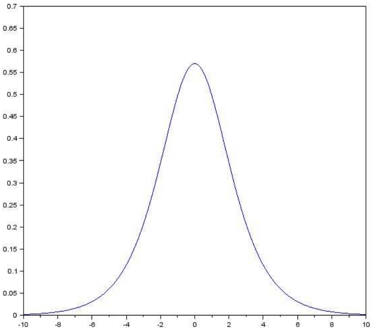


Figura: Solitary waves for Beji-Nadaoka equations (η_c is on the left and $\bar{u}_c$ on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\alpha_B = 1/15$.

Madsen-Sorensen

Equations:

$$\left\{ \begin{array}{l} \eta_t + q_x = 0, \\ \bar{q}_t + gh\eta_x + \left(\frac{\bar{q}^2}{h}\right)_x - Bd^2\bar{q}_{txx} - \beta g d^3 \eta_{xxx} = 0 \end{array} \right.$$

$$\begin{array}{c} \eta(t,x) = \eta_c(x-ct) \\ \bar{u}(t,x) = \bar{u}_c(x-ct), \end{array}$$

$$\left\{ \begin{array}{l} \bar{q}_c = c\eta_c, \\ -c\bar{q}_c + c_0^2\eta_c + \frac{g}{2}\eta_c^2 + \left(\frac{\bar{q}_c^2}{h}\right) + cBd^2\bar{q}_c'' - \beta g d^3 \eta_c'' = 0 \end{array} \right.$$

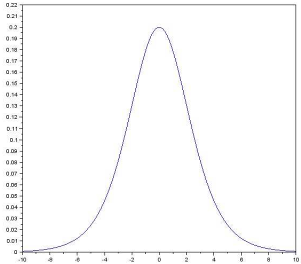
$$-\eta'' = \frac{1}{K}\left(-c^2\eta + c_0^2\eta + \frac{g}{2}\eta^2 + c^2\frac{\eta^2}{d+\eta}\right),$$

$$K = d^2(c^2B - \beta c_0^2)$$

Madsen-Sorensen

Conclusion : For all $c > c_0$, MS admit a unique solitary wave of the form $(\eta_c(x-ct), \bar{q}_c(x-ct))$. In addition, the relation between parameter c and the amplitude A of η_c is given by

$$c^2 = c_0^2 \frac{\frac{A^2}{2} + \frac{A^3}{6d}}{dA - d^2 \log(\frac{d+A}{d})}.$$



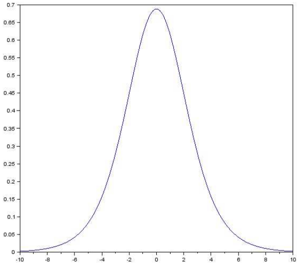


Figura: Solitary waves for Madsen and Sorensen equations (η_c is on the left and $\bar{q}_c$ on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\beta = 1/15$.

Nwogu

Equations:

$$\begin{cases} \eta_t + [(\eta + d)U]_x + \beta d^3 U_{xxx} = 0, \\ U_t + UU_x + g\eta_x + \alpha d^2 U_{txx} = 0. \end{cases}$$

remark : $\beta = 0$ or $\alpha_B = 0$ in BN $\rightsquigarrow$ Peregrine.

$$\begin{cases} -c\eta_c + [(\eta_c + d)U_c] + \beta d^3 U_c'' = 0, \\ -cU_c + \frac{U_c^2}{2} + g\eta_c - c\alpha d^2 U_c'' = 0 \end{cases}$$

implies

$$\eta_c = d\,\frac{cU_c/3 - \beta U_c^2/2}{c_0^2\beta - c^2\alpha + c\alpha U_c}.$$

Positive solutions $\Rightarrow 0 \le U_c < c - \frac{\beta}{\alpha}\frac{c_0^2}{c},\ \ 0 \le U_c < \frac{2c}{3\beta}$ and

$$c^2 > \frac{\beta}{\alpha}c_0^2.$$

Nwogu

$$-\frac{d^2}{3}U_c'' = \frac{(c_0^2-c^2)\frac{U_c}{3}+\frac{cU_c^2}{2}-\frac{U_c^3}{6}}{\alpha(c_0^2-c^2)+\frac{c_0^2}{3}+\alpha c U_c}.$$

Conclusion :
Assume that one of the following alternative is satisfied :
i) $\alpha \in (-\frac{1}{2}, -\frac{1}{9})$ and $c > c_0$,
ii) $\alpha \in (-\frac{1}{9}, 0)$ and $c > c_0\sqrt{\frac{\frac{\beta^2}{\alpha^2}}{2-\frac{\beta}{\alpha}}}$.
Then Nwogu admit a unique solitary wave of the form $(\eta_c(x-ct), U_c(x-ct))$. The amplitude A of U_c and parameter c satisfy

$$-\frac{1}{18\alpha}A^3+\left(\frac{c}{4\alpha}+\frac{\beta c_0^2-\alpha c^2}{12c\alpha}\right)A^2+\left(\frac{c_0^2-c^2}{3\alpha}-\frac{\beta c_0^2-\alpha c^2}{2\alpha^2}-\frac{(\beta c_0^2-\alpha c^2)^2}{6\alpha^2c^2}\right)A$$
$$+\left(\frac{(\beta c_0^2-\alpha c^2)^4}{6c^3\alpha^3}+\frac{(\beta c_0^2-\alpha c^2)^3}{2c\alpha^3}-\frac{(c_0^2-c^2)(\beta c_0^2-\alpha c^2)^2}{3c\alpha^2}\right)\log\left(1+\frac{c\alpha}{\beta c_0^2-\alpha c^2}A\right)=0.$$

Conversely, if $c \le c_0$, then Nwogu have no positive solutions of the previous form.

Nwogu

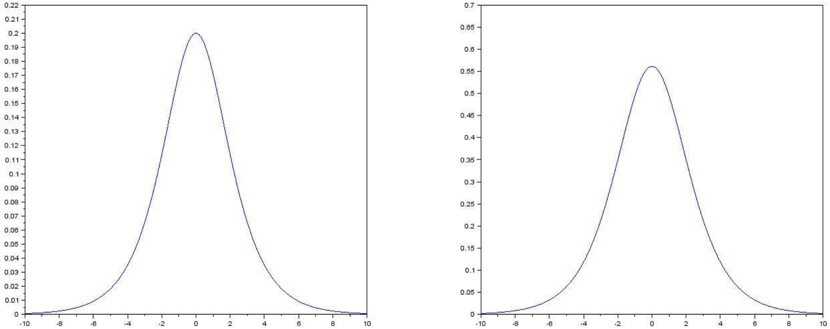


Figura: Solitary wave for Nwogu equations (η_c is on the left and U_c on the right), with $d = 1$, $\eta_c^0 = 0.2$ and $\beta = -1/15$.

remark : no result for Nwogu-Abott.....

Shoaling

A. Filippini, S. Bellec, M .Ricchuito, MC.

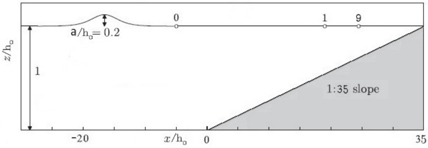


Figura: Shoaling of a solitary wave; computational configuration and gauges position (Grilli and *al*).

$$d_0 = 0.44m, \quad \frac{A}{d_0} = 0.2m \quad \text{slope} = 1:35, \quad \varepsilon = \frac{a_0}{h} \in [0.2; 2.2]$$

Shoaling

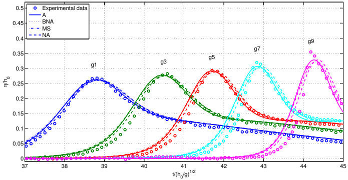


Figura: Nonlinear shoaling. Comparison between computed wave heights at gauges 1, 3, 5, 7 and 9 and data; models in amplitude-flux form.

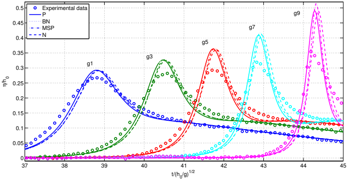


Figura: Nonlinear shoaling. Comparison between computed wave heights at gauges 1, 3, 5, 7 and 9 and data; models in amplitude-velocity form.

Shoaling

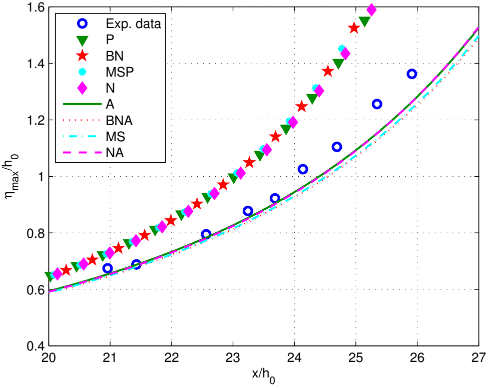


Figura: Nonlinear shoaling. Comparison between computed wave peak evolution and data.