Well-Balanced schemes for nonconservative and hyperbolic systems with source terms.

B. Nkonga[1]

Laboratoire J.A.D. & **INRIA/CASTOR**
Univ. de Nice Sophia-Antipolis
Nice Cote d'Azur, France.

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Shalow-water model with bottom topography: 1D

$$\left\{\begin{array}{llll} \partial_t h & +\partial_x (h\mathbf{u}) & & = 0 \\ \partial_t (h\mathbf{u}) & +\partial_x \left(h\mathbf{u}^2 + p(h)\right) & +gh\partial_x b & = 0 \end{array}\right., \quad p(h) = g\frac{h^2}{2}$$

Entropy Inequality[2]:

$$\partial_t (he) + \partial_x (\mathbf{u}(he + p(h))) \leq 0$$

with $e = \frac{\mathbf{u}^2}{2} + \frac{gh}{2} + gb$.

Reformulated system :

$$\left\{\begin{array}{llll} \partial_t h & +\partial_x (h\mathbf{u}) & & = 0 \\ \partial_t (h\mathbf{u}) & +\partial_x \left(h\mathbf{u}^2 + p(h)\right) & +gh\partial_x b & = 0 \\ \partial_t b & & & = 0 \end{array}\right.$$

Steady States

$\partial_x (h\mathbf{u}) = 0$ and $\partial_x \left(h\mathbf{u}^2 + p(h)\right) + gh\partial_x b = 0$

Lake at Rest : $\mathbf{u} = 0$ and $h + b = Cte$

[2] U. S. Fjordholm, S. Mishra, E. Tadmor, *Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography* , **Journal of Computational Physics 230(14), 5587-5609, 2011**

Shalow-water model with bottom topography and friction

$$\begin{cases} \partial_t \mathrm{h} & +\partial_x (\mathrm{h}\mathbf{u}) & & = & 0 \\ \partial_t (\mathrm{h}\mathbf{u}) & +\partial_x \left(\mathrm{h}\mathbf{u}^2 + p(\mathrm{h})\right) & +g\mathrm{h}\partial_x \mathrm{b} & = & \mathrm{c}_f \mathbf{u} |\mathbf{u}| \end{cases}$$

Augmented system[3]:

$$\begin{cases} \partial_t \mathrm{h} & +\partial_x (\mathrm{h}\mathbf{u}) & & = & 0 \\ \partial_t (\mathrm{h}\mathbf{u}) & +\partial_x \left(\mathrm{h}\mathbf{u}^2 + p(\mathrm{h})\right) & +g\mathrm{h}\partial_x \mathrm{b} - \mathrm{c}_f \mathbf{u} |\mathbf{u}| \partial_x x & = & 0 \\ \partial_t \mathrm{b} & & & = & 0 \\ \partial_t x & & & = & 0 \end{cases}$$

[3] Laurent Gosse, *A well-balanced scheme using non-conservative products designed for hyperbolic systems of conservation laws with source terms*, **M3AS, Vol. 11 (02), 339-365, 2001**

General setting

$$\partial_t \boldsymbol{\omega} + \partial_x \left(\mathbf{f}\left(\boldsymbol{\omega}\right) \right) + \underline{\mathbf{B}}\left(\boldsymbol{\omega}\right) \partial_x \boldsymbol{\omega} = 0$$

Model with bottom topography

$$\boldsymbol{\omega} = \begin{pmatrix} \mathrm{h} \\ \mathrm{h}\mathbf{u} \\ \mathrm{b} \end{pmatrix}, \quad \mathbf{f}\left(\boldsymbol{\omega}\right) = \begin{pmatrix} \mathrm{h}\mathbf{u} \\ \mathrm{h}\mathbf{u}^2 + p \\ 0 \end{pmatrix}, \quad \underline{\mathbf{B}}\left(\boldsymbol{\omega}\right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & g\mathrm{h} \\ 0 & 0 & 0 \end{pmatrix}$$

Model with bottom topography and friction

$$\boldsymbol{\omega} = \begin{pmatrix} \mathrm{h} \\ \mathrm{h}\mathbf{u} \\ \mathrm{b} \\ x \end{pmatrix}, \mathbf{f}\left(\boldsymbol{\omega}\right) = \begin{pmatrix} \mathrm{h}\mathbf{u} \\ \mathrm{h}\mathbf{u}^2 + p \\ 0 \\ 0 \end{pmatrix}, \underline{\mathbf{B}}\left(\boldsymbol{\omega}\right) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & g\mathrm{h} & -\mathrm{C}_f \mathbf{u} \left|\mathbf{u}\right| \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

General setting : $\partial_t \boldsymbol{\omega} + \partial_x (\mathbf{f}(\boldsymbol{\omega})) + \underline{\mathbf{B}}(\boldsymbol{\omega}) \partial_x \boldsymbol{\omega} = 0$

Finite volume approximation

$$\delta x \left(\boldsymbol{\omega}_i^{n+1} - \boldsymbol{\omega}_i^n\right) + \delta t \left(\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}\right) + \delta t \left(\mathbf{D}^i_{i-\frac{1}{2}} + \mathbf{D}_i^{i+\frac{1}{2}}\right) = 0$$

where

$$\begin{aligned}
\boldsymbol{\phi}_{i\pm\frac{1}{2}} &= \frac{1}{\delta t} \int_0^{\delta t} \mathbf{f}_{i\pm\frac{1}{2}} \\
\mathbf{D}^i_{i-\frac{1}{2}} &= \frac{1}{\delta t} \int_0^{\delta t} \int_{x_{i-\frac{1}{2}}}^{x_i} \underline{\mathbf{B}}(\boldsymbol{\omega}) \partial_x \boldsymbol{\omega} \\
\mathbf{D}_i^{i+\frac{1}{2}} &= \frac{1}{\delta t} \int_0^{\delta t} \int_{x_i}^{x_{i+\frac{1}{2}}} \underline{\mathbf{B}}(\boldsymbol{\omega}) \partial_x \boldsymbol{\omega}
\end{aligned}$$

Approximation of $\mathbf{D}_l^r = \frac{1}{\delta t}\int_0^{\delta t}\int_{x_l}^{x_r} \underline{\mathbf{B}}(\omega)\,\partial_x\omega\,dt\,dx$

Let us introduce a linear path in the phase space [4]:

$$\omega \simeq \mathcal{P}(\omega_l,\omega_r,s) = \omega_l + s\,(\omega_r - \omega_l) \quad \text{with} \quad s = \frac{x - x_l}{x_r - x_l}$$

Etimation associated to this path gives :

$$\mathbf{D}_l^r \simeq \frac{1}{\delta t}\int_0^{\delta t}\left[\int_0^1 \underline{\mathbf{B}}\left(\mathcal{P}(\omega_l,\omega_r,s)\right) ds\right](\omega_r - \omega_l)\,dt$$

Therefore $\boxed{\mathbf{D}_l^r \simeq \underline{\mathbf{B}}(\omega_l,\omega_r)(\omega_r - \omega_l)}$ where

$$\underline{\mathbf{B}}(\omega_l,\omega_r) \equiv \underline{\mathbf{B}}_{\mathcal{P}}(\omega_l,\omega_r) = \int_0^1 \underline{\mathbf{B}}\left(\mathcal{P}(\omega_l,\omega_r,s)\right) ds$$

[4] G. Dal Maso, P. LeFloch, F. Murat, *Definition and weak stability of a non-conservative product*, **J. Math. Pures et Appl., Vol. 74 No 6, 1995.**

HLL conservative fluxes $\boldsymbol{\phi}_{\frac{l+r}{2}} = \boldsymbol{\phi}(\boldsymbol{\omega}_l, \boldsymbol{\omega}_r) \equiv \boldsymbol{\phi}_{hll}(\sigma = 0)$

Riemann Problem (RP)

$$\begin{cases} \partial_t \boldsymbol{\omega} + \partial_x (\mathbf{f}(\boldsymbol{\omega})) + \underline{\mathbf{B}}(\boldsymbol{\omega}) \partial_x \boldsymbol{\omega} = 0 \\ \boldsymbol{\omega}(t = 0, x) = \begin{cases} \boldsymbol{\omega}_l & \text{if } x < 0 \\ \boldsymbol{\omega}_r & \text{if } x > 0 \end{cases} \end{cases}$$

RP for similarity variable $\sigma = \frac{x}{t}$

$$\begin{cases} \partial_\sigma (\mathbf{f} - \sigma \boldsymbol{\omega}) + \underline{\mathbf{B}}(\boldsymbol{\omega}) \partial_\sigma \boldsymbol{\omega} + \boldsymbol{\omega} = 0 \\ \boldsymbol{\omega}(\sigma = -\infty) = \boldsymbol{\omega}_l \\ \boldsymbol{\omega}(\sigma = +\infty) = \boldsymbol{\omega}_r \end{cases}$$

$$\boldsymbol{\omega}_{hll}(\sigma) = \begin{cases} \boldsymbol{\omega}_l & \text{if } \sigma < \mathrm{s}_l \leq 0 \\ \boldsymbol{\omega}_\star & \text{if } \mathrm{s}_l < \sigma < \mathrm{s}_r \\ \boldsymbol{\omega}_r & \text{if } \sigma > \mathrm{s}_r \geq 0 \end{cases}$$

$$\boldsymbol{\phi}_{hll}(\sigma) = \begin{cases} \mathbf{f}(\boldsymbol{\omega}_l) & \text{if } \sigma < \mathrm{s}_l \leq 0 \\ \boldsymbol{\phi}_\star & \text{if } \mathrm{s}_l < \sigma < \mathrm{s}_r \\ \mathbf{f}(\boldsymbol{\omega}_r) & \text{if } \sigma > \mathrm{s}_r \geq 0 \end{cases}$$

HLL: Consistency with the integral formulation

$$\partial_\sigma \left(\mathbf{f} - \sigma \boldsymbol{\omega}\right) + \underline{\mathbf{B}}\left(\boldsymbol{\omega}\right) \partial_\sigma \boldsymbol{\omega} + \boldsymbol{\omega} = 0$$

For any $\sigma_l < \sigma_r$

$$\boxed{\mathbf{f}\left(\sigma_r\right) - \sigma_r \boldsymbol{\omega}\left(\sigma_r\right) - \mathbf{f}\left(\sigma_l\right) + \sigma_l \boldsymbol{\omega}\left(\sigma_l\right) + \int_{\sigma_l}^{\sigma_r} \underline{\mathbf{B}}\left(\boldsymbol{\omega}\right) \partial_\sigma \boldsymbol{\omega} + \int_{\sigma_l}^{\sigma_r} \boldsymbol{\omega} = 0}$$

Straightforward application gives

$$\begin{cases} \boldsymbol{\phi}_\star - 0 - \mathbf{f}_l + \mathrm{s}_l \boldsymbol{\omega}_l + \mathbf{D}_l^\star - \mathrm{s}_l \boldsymbol{\omega}_\star & = & 0 \\ \mathbf{f}_r - \mathrm{s}_r \boldsymbol{\omega}_r - \boldsymbol{\phi}_\star + 0 + \mathbf{D}_\star^r + \mathrm{s}_r \boldsymbol{\omega}_\star & = & 0 \end{cases}$$

Therefore, $\boldsymbol{\omega}_\star$ is implicitly defined as

$$\boldsymbol{\omega}_\star = \boldsymbol{\omega}_\star^{(0)} - \frac{\mathbf{D}_l^\star + \mathbf{D}_\star^r}{\mathrm{s}_r - \mathrm{s}_l} \quad \text{with} \quad \boldsymbol{\omega}_\star^{(0)} = \frac{\mathrm{s}_r \boldsymbol{\omega}_r - \mathrm{s}_l \boldsymbol{\omega}_l - \left(\mathbf{f}_r - \mathbf{f}_l\right)}{\mathrm{s}_r - \mathrm{s}_l}$$

Indeed, $\mathbf{D}_\star^r$ and $\mathbf{D}_l^\star$ are functions of $\boldsymbol{\omega}_\star$.

HLL Continue

The flux can be formulated using one of these relations :

- $\boldsymbol{\phi}_\star = \dfrac{\mathrm{S}_r \mathbf{f}_l - \mathrm{S}_l \mathbf{f}_r + \mathrm{S}_r \mathrm{S}_l (\boldsymbol{\omega}_r - \boldsymbol{\omega}_l)}{\mathrm{S}_r - \mathrm{S}_l} - \dfrac{\mathrm{S}_r \mathbf{D}_l^\star + \mathrm{S}_l \mathbf{D}_\star^r}{\mathrm{S}_r - \mathrm{S}_l}$
- $\boldsymbol{\phi}_\star = \mathbf{f}_l - \mathrm{S}_l \boldsymbol{\omega}_l + \mathrm{S}_l \boldsymbol{\omega}_\star - \mathbf{D}_l^\star$
- $\boldsymbol{\phi}_\star = \mathbf{f}_r - \mathrm{S}_r \boldsymbol{\omega}_r + \mathrm{S}_r \boldsymbol{\omega}_\star + \mathbf{D}_\star^r$

The first order scheme is

$$\delta x \left(\boldsymbol{\omega}_i^{n+1} - \boldsymbol{\omega}_i^n\right) + \delta t \left(\boldsymbol{\phi}_{i+\frac{1}{2}} - \boldsymbol{\phi}_{i-\frac{1}{2}}\right) + \delta t \left(\mathbf{D}_{i-\frac{1}{2}}^i + \mathbf{D}_i^{i+\frac{1}{2}}\right) = 0$$

Then we use

$$\begin{aligned} \boldsymbol{\phi}_{i-\frac{1}{2}} &= \mathbf{f}_i + \boldsymbol{\psi}_{i-\frac{1}{2}}^i + \mathbf{D}_{i-\frac{1}{2}}^i \\ \boldsymbol{\phi}_{i+\frac{1}{2}} &= \mathbf{f}_i + \boldsymbol{\psi}_i^{i+\frac{1}{2}} - \mathbf{D}_i^{i+\frac{1}{2}} \end{aligned} \qquad \begin{aligned} \boldsymbol{\psi}_{i-\frac{1}{2}}^i &= (\mathrm{S}_r)_{i-\frac{1}{2}} \left(\boldsymbol{\omega}_{i-\frac{1}{2}} - \boldsymbol{\omega}_i\right) \\ \boldsymbol{\psi}_i^{i+\frac{1}{2}} &= (\mathrm{S}_l)_{i+\frac{1}{2}} \left(\boldsymbol{\omega}_{i+\frac{1}{2}} - \boldsymbol{\omega}_i\right) \end{aligned}$$

and obtain

$$\frac{\boldsymbol{\omega}_i^{n+1} - \boldsymbol{\omega}_i^n}{\delta t} + \frac{\boldsymbol{\psi}_i^{i+\frac{1}{2}} - \boldsymbol{\psi}_{i-\frac{1}{2}}^i}{\delta x} = 0$$

HLL Resumed : $\frac{\omega_i^{n+1} - \omega_i^n}{\delta t} + \frac{\psi_i^{i+\frac{1}{2}} - \psi_{i-\frac{1}{2}}^i}{\delta x} = 0$

with

$\psi_{i-\frac{1}{2}}^i = (\mathrm{S}_r)_{i-\frac{1}{2}} \left(\omega_{i-\frac{1}{2}} - \omega_i \right)$ and $\psi_i^{i+\frac{1}{2}} = (\mathrm{S}_l)_{i+\frac{1}{2}} \left(\omega_{i+\frac{1}{2}} - \omega_i \right)$.

For any i, $\omega_{i-\frac{1}{2}}$ is the solution of

$$\omega_\star = \omega_{i-\frac{1}{2}}^{(0)} - \frac{\underline{\mathbf{B}}(\omega_{i-1}, \omega_\star)(\omega_\star - \omega_{i-1}) + \underline{\mathbf{B}}(\omega_\star, \omega_i)(\omega_i - \omega_\star)}{(\mathrm{S}_r)_{i-\frac{1}{2}} - (\mathrm{S}_l)_{i-\frac{1}{2}}}$$

where

$$\omega_{i-\frac{1}{2}}^{(0)} = \frac{(\mathrm{S}_r)_{i-\frac{1}{2}} \omega_i - (\mathrm{S}_l)_{i-\frac{1}{2}} \omega_{i-1} - (\mathbf{f}_i - \mathbf{f}_{i-1})}{(\mathrm{S}_r)_{i-\frac{1}{2}} - (\mathrm{S}_l)_{i-\frac{1}{2}}}$$

Application to SW

$$\underline{\mathbf{B}}(\omega_l,\omega_r)(\omega_r-\omega_l) = \begin{pmatrix} 0 \\ g\dfrac{\mathrm{h}_l+\mathrm{h}_r}{2}(\mathrm{b}_r-\mathrm{b}_l) \\ 0 \end{pmatrix}$$

Therefore, $\boxed{\omega_{i-\frac{1}{2}} = \omega^{(0)}_{i-\frac{1}{2}} - \mathbf{G}_{i-\frac{1}{2}}}$ with

$$\mathbf{G}_{i-\frac{1}{2}} = g\begin{pmatrix} 0 \\ \dfrac{\left(\mathrm{h}_{i-1}+\mathrm{h}_{i-\frac{1}{2}}\right)\left(\mathrm{b}_{i-\frac{1}{2}}-\mathrm{b}_{i-1}\right)+\left(\mathrm{h}_{i-\frac{1}{2}}+\mathrm{h}_i\right)\left(\mathrm{b}_i-\mathrm{b}_{i-\frac{1}{2}}\right)}{2\left((\mathrm{S}_r)_{i-\frac{1}{2}}-(\mathrm{S}_l)_{i-\frac{1}{2}}\right)} \\ 0 \end{pmatrix}$$

Finally the numerical scheme is reformulated as :

$$\boxed{\frac{\omega_i^{n+1}-\omega_i^n}{\delta t} + \frac{\widetilde{\psi}_i^{i+\frac{1}{2}} - \widetilde{\psi}^i_{i-\frac{1}{2}}}{\delta x} + \frac{(\mathrm{S}_r)_{i-\frac{1}{2}}\mathbf{G}_{i-\frac{1}{2}} - (\mathrm{S}_l)_{i+\frac{1}{2}}\mathbf{G}_{i+\frac{1}{2}}}{\delta x} = 0}$$

where $\widetilde{\psi}$ is the classical conservative fluctuation.

Approximation of $\underline{\mathbf{B}}(\omega)\,\partial_x\omega$ is drived by the Riemann solver.

HLL Lake at rest : $\mathbf{u}_i = 0$ and $h_i + b_i = z_0$

In this context we have

$$\boldsymbol{\omega}^{(0)}_{i-\frac{1}{2}} = \frac{(S_r)_{i-\frac{1}{2}}\boldsymbol{\omega}_i - (S_l)_{i-\frac{1}{2}}\boldsymbol{\omega}_{i-1} - (\mathbf{f}_i - \mathbf{f}_{i-1})}{(S_r)_{i-\frac{1}{2}} - (S_l)_{i-\frac{1}{2}}} = \begin{pmatrix} h_{i-\frac{1}{2}} \\ -\frac{\frac{g}{2}\left(h_i^2 - h_{i-1}^2\right)}{(S_r)_{i-\frac{1}{2}} - (S_l)_{i-\frac{1}{2}}} \\ b_{i-\frac{1}{2}} \end{pmatrix}$$

Note that, in general, $\mathbf{u}^{(0)}_{i-\frac{1}{2}} \neq 0$ and will induce artificial motion.

However, as

$$\mathbf{G}_{i-\frac{1}{2}} = g\begin{pmatrix} 0 \\ \frac{(h_i + h_{i-1})(b_i - b_{i-1})}{2\left((S_r)_{i-\frac{1}{2}} - (S_l)_{i-\frac{1}{2}}\right)} \\ 0 \end{pmatrix}$$

we obtain

$$\boldsymbol{\omega}_{i-\frac{1}{2}} = \boldsymbol{\omega}^{(0)}_{i-\frac{1}{2}} - \mathbf{G}_{i-\frac{1}{2}} = \begin{pmatrix} h_{i-\frac{1}{2}} \\ 0 \\ b_{i-\frac{1}{2}} \end{pmatrix} \quad \text{and} \quad h_{i-\frac{1}{2}} + b_{i-\frac{1}{2}} = z_0$$

HLL Lake at rest : $\mathbf{u}_i = 0$ and $\mathrm{h}_i + \mathrm{b}_i = \mathrm{z}_0$

$$\omega_i^{n+1} = \omega_i^n - \frac{\delta t\, (\mathrm{s}_l)_{i+\frac{1}{2}}}{\delta x} \begin{pmatrix} \mathrm{h}_{i+\frac{1}{2}} - \mathrm{h}_i \\ 0 \\ \mathrm{b}_{i+\frac{1}{2}} - \mathrm{b}_i \end{pmatrix} + \frac{\delta t\, (\mathrm{s}_r)_{i-\frac{1}{2}}}{\delta x} \begin{pmatrix} \mathrm{h}_{i-\frac{1}{2}} - \mathrm{h}_i \\ 0 \\ \mathrm{b}_{i-\frac{1}{2}} - \mathrm{b}_i \end{pmatrix}$$

The desciption of the stared region by a constant state is not satisfactory. Indeed, $\omega_i^{n+1} \neq \omega_i^n$ in general We have

$$(\mathrm{s}_l)_{i+\frac{1}{2}} \left(\mathrm{h}_{i+\frac{1}{2}} - \mathrm{h}_i\right) = \frac{(\mathrm{s}_r)_{i+\frac{1}{2}} (\mathrm{s}_l)_{i+\frac{1}{2}}}{(\mathrm{s}_r)_{i+\frac{1}{2}} - (\mathrm{s}_l)_{i+\frac{1}{2}}} (\mathrm{h}_{i+1} - \mathrm{h}_i)$$

and

$$-(\mathrm{s}_r)_{i-\frac{1}{2}} \left(\mathrm{h}_{i-\frac{1}{2}} - \mathrm{h}_i\right) = -\frac{(\mathrm{s}_r)_{i-\frac{1}{2}} (\mathrm{s}_l)_{i-\frac{1}{2}}}{(\mathrm{s}_r)_{i-\frac{1}{2}} - (\mathrm{s}_l)_{i-\frac{1}{2}}} (\mathrm{h}_i - \mathrm{h}_{i-1})$$

HLL Lake at rest : $\mathbf{u}_i = 0$ and $h_i + b_i = z_0$

$$
\begin{aligned}
\omega_i^{n+1} &= \omega_i^n - \frac{\delta t}{\delta x} \frac{(\mathrm{S}_r)_{i+\frac{1}{2}} (\mathrm{S}_l)_{i+\frac{1}{2}}}{(\mathrm{S}_r)_{i+\frac{1}{2}} - (\mathrm{S}_l)_{i+\frac{1}{2}}} (\omega_{i+1} - \omega_i) \\
&\quad + \frac{\delta t}{\delta x} \frac{(\mathrm{S}_r)_{i-\frac{1}{2}} (\mathrm{S}_l)_{i-\frac{1}{2}}}{(\mathrm{S}_r)_{i-\frac{1}{2}} - (\mathrm{S}_l)_{i-\frac{1}{2}}} (\omega_i - \omega_{i-1})
\end{aligned}
$$

The missing contributions for well balanced are proportional to a local gradients $\frac{\omega_r - \omega_l}{\mathrm{S}_r - \mathrm{S}_l}$.

Well-balanced scheme by hydrostatic reconstruction[5]

$$\frac{\boldsymbol{\omega}_i^{n+1} - \boldsymbol{\omega}_i^n}{\delta t} + \frac{\boldsymbol{\phi}^{(0)}_{i+\frac{1}{2}} - \boldsymbol{\phi}^{(0)}_{i-\frac{1}{2}}}{\delta x} + \frac{\mathbf{S}^{i}_{i-} + \mathbf{S}^{i+}_{i}}{\delta x} = 0$$

where the state $i-$ and $i+$ are obtained by hydrostatic reconstruction:

$$\boldsymbol{\omega}_{i\pm} = \begin{pmatrix} \mathrm{h}_{i\pm} \\ \mathrm{h}_{i\pm}\mathbf{u}_i \\ \mathrm{b}_{i\pm\frac{1}{2}} \end{pmatrix}, \quad \boldsymbol{\phi}^{(0)}_{i+\frac{1}{2}} = \boldsymbol{\phi}^{(0)}\left(\boldsymbol{\omega}_{i+}, \boldsymbol{\omega}_{(i+1)-}\right), \quad \mathbf{S}_l^r = \begin{pmatrix} 0 \\ g\frac{\mathrm{h}_r + \mathrm{h}_l}{2}(\mathrm{h}_r - \mathrm{h}_r) \\ 0 \end{pmatrix}$$

where

$$\begin{aligned} \mathrm{h}_{i+} = \mathrm{h}_i + \tfrac{\delta x}{2}(\partial_x \mathrm{h}) \quad &\simeq \quad \mathrm{h}_i - \left(\mathrm{b}_{i+\frac{1}{2}} - \mathrm{b}_i\right) \\ \mathrm{h}_{i-} = \mathrm{h}_i - \tfrac{\delta x}{2}(\partial_x \mathrm{h}) \quad &\simeq \quad \mathrm{h}_i + \left(\mathrm{b}_i - \mathrm{b}_{i-\frac{1}{2}}\right) \end{aligned}$$

[5] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, B. Perthame *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows* **SIAM Journal on Scientific Computing 25 (6), 2050-2065, 2004**

HLLE by B. Einfeldt[6]

Introduce a linear variation in the interaction zone :

$$\omega_{hlle}(\sigma) = \begin{cases} \omega_l & \text{if } \sigma < s_l \leq 0 \\ \omega_\star + \dfrac{2\sigma - (s_r + s_l)}{s_r - s_l}\underline{\delta}(\omega_r - \omega_l) & \text{if } s_l < \sigma < s_r \\ \omega_r & \text{if } \sigma > s_r \geq 0 \end{cases}$$

where $\underline{\delta}$ is a projection matrix.

Note that

$$\int_{s_l}^{0} \frac{2\sigma - (s_r + s_l)}{s_r - s_l} d\sigma = \frac{s_r s_l}{s_r - s_l}$$

$$\int_{0}^{s_r} \frac{2\sigma - (s_r + s_l)}{s_r - s_l} d\sigma = -\frac{s_r s_l}{s_r - s_l}$$

[6] B. Einfeldt. *On Godunov-type methods for gas dynamics.* SIAM J. Numer. Anal., 25:294-318, 1988.

HLLE: Consistency with the integral formulation

For any $\sigma_l < \sigma_r$

$$\mathbf{f}(\sigma_r) - \sigma_r\boldsymbol{\omega}(\sigma_r) - \mathbf{f}(\sigma_l) + \sigma_l\boldsymbol{\omega}(\sigma_l) + \int_{\sigma_l}^{\sigma_r} \underline{\mathbf{B}}(\boldsymbol{\omega})\,\partial_\sigma\boldsymbol{\omega} + \int_{\sigma_l}^{\sigma_r} \boldsymbol{\omega} = 0$$

Straightforward application gives

$$\begin{cases} \boldsymbol{\phi}_\star - 0 - \mathbf{f}_l + \mathrm{s}_l\boldsymbol{\omega}_l + \mathbf{D}_l^\star - \mathrm{s}_l\boldsymbol{\omega}_\star + \dfrac{\mathrm{s}_r\mathrm{s}_l}{\mathrm{s}_r - \mathrm{s}_l}\underline{\boldsymbol{\delta}}(\boldsymbol{\omega}_r - \boldsymbol{\omega}_l) & = & 0 \\ \mathbf{f}_r - \mathrm{s}_r\boldsymbol{\omega}_r - \boldsymbol{\phi}_\star + 0 + \mathbf{D}_\star^r + \mathrm{s}_r\boldsymbol{\omega}_\star - \dfrac{\mathrm{s}_r\mathrm{s}_l}{\mathrm{s}_r - \mathrm{s}_l}\underline{\boldsymbol{\delta}}(\boldsymbol{\omega}_r - \boldsymbol{\omega}_l) & = & 0 \end{cases}$$

We then observe that the E-modification do not change the estimation of $\boldsymbol{\omega}_\star$. The associated fluxes are

$$\begin{aligned} \boldsymbol{\phi}_{i-\frac{1}{2}} &= \mathbf{f}_i + \boldsymbol{\psi}^i_{i-\frac{1}{2}} + \mathbf{D}^i_{i-\frac{1}{2}} - \frac{(\mathrm{s}_r)_{i-\frac{1}{2}}(\mathrm{s}_l)_{i-\frac{1}{2}}}{(\mathrm{s}_r)_{i-\frac{1}{2}} - (\mathrm{s}_l)_{i-\frac{1}{2}}}\underline{\boldsymbol{\delta}}(\boldsymbol{\omega}_i - \boldsymbol{\omega}_{i-1}) \\ \boldsymbol{\phi}_{i+\frac{1}{2}} &= \mathbf{f}_i + \boldsymbol{\psi}_i^{i+\frac{1}{2}} - \mathbf{D}_i^{i+\frac{1}{2}} - \frac{(\mathrm{s}_r)_{i+\frac{1}{2}}(\mathrm{s}_l)_{i+\frac{1}{2}}}{(\mathrm{s}_r)_{i+\frac{1}{2}} - (\mathrm{s}_l)_{i+\frac{1}{2}}}\underline{\boldsymbol{\delta}}(\boldsymbol{\omega}_{i+1} - \boldsymbol{\omega}_i) \end{aligned}$$

HLLE Scheme

$$\frac{\omega_i^{n+1}-\omega_i^n}{\delta t}+\frac{\widetilde{\psi_i^{i+\frac{1}{2}}}-\widetilde{\psi_{i-\frac{1}{2}}^{i}}}{\delta x}-\frac{(\mathrm{S}_l)_{i+\frac{1}{2}}}{\delta x}\left(\mathbf{G}_{i+\frac{1}{2}}+\frac{(\mathrm{S}_r)_{i+\frac{1}{2}}\,\underline{\boldsymbol{\delta}}\,(\omega_{i+1}-\omega_i)}{(\mathrm{S}_r)_{i+\frac{1}{2}}-(\mathrm{S}_l)_{i+\frac{1}{2}}}\right)$$
$$+\frac{(\mathrm{S}_r)_{i-\frac{1}{2}}}{\delta x}\left(\mathbf{G}_{i-\frac{1}{2}}+\frac{(\mathrm{S}_l)_{i-\frac{1}{2}}\,\underline{\boldsymbol{\delta}}\,(\omega_i-\omega_{i-1})}{(\mathrm{S}_r)_{i-\frac{1}{2}}-(\mathrm{S}_l)_{i-\frac{1}{2}}}\right)=0$$

HLLE Lake at rest : $\mathbf{u}_i = 0$ and $h_i + b_i = z_0$

$$\begin{aligned}\omega_i^{n+1} &= \omega_i^n - \frac{\delta t}{\delta x}\frac{(s_r)_{i+\frac{1}{2}}(s_l)_{i+\frac{1}{2}}}{(s_r)_{i+\frac{1}{2}} - (s_l)_{i+\frac{1}{2}}}\left(\underline{\mathrm{Id}} - \underline{\boldsymbol{\delta}}\right)(\omega_{i+1} - \omega_i) \\ &\quad + \frac{\delta t}{\delta x}\frac{(s_r)_{i-\frac{1}{2}}(s_l)_{i-\frac{1}{2}}}{(s_r)_{i-\frac{1}{2}} - (s_l)_{i-\frac{1}{2}}}\left(\underline{\mathrm{Id}} - \underline{\boldsymbol{\delta}}\right)(\omega_i - \omega_{i-1})\end{aligned}$$

Theorem

If $\underline{\boldsymbol{\delta}} = \underline{\mathrm{Id}}$ the associated HLLE scheme (fisrt order accurate) preserves the steady state of lake at rest.

well-balanced scheme by hydrostatic reconstruction[7]

$$\frac{\omega_i^{n+1}-\omega_i^n}{\delta t}+\frac{\phi^{(0)}_{i+\frac{1}{2}}-\phi^{(0)}_{i-\frac{1}{2}}}{\delta x}+\frac{\mathbf{S}^i_{i-}+\mathbf{S}^{i+}_i}{\delta x}=0$$

where the state $i-$ and $i+$ are obtained by hydrostatic reconstruction:

$$\omega_{i\pm}=\begin{pmatrix} \mathrm{h}_{i\pm} \\ \mathrm{h}_{i\pm}\mathbf{u}_i \\ \mathrm{b}_{i\pm\frac{1}{2}} \end{pmatrix}, \quad \mathbf{S}^r_l=\begin{pmatrix} 0 \\ g\frac{\mathrm{h}_r+\mathrm{h}_l}{2}(\mathrm{h}_r-\mathrm{h}_r) \\ 0 \end{pmatrix}$$

with $\mathrm{h}_{i+}=\mathrm{h}_i-\left(\mathrm{b}_{i+\frac{1}{2}}-\mathrm{b}_i\right)$ and $\mathrm{h}_{i-}=\mathrm{h}_i+\left(\mathrm{b}_i-\mathrm{b}_{i-\frac{1}{2}}\right)$.

$$\phi^{(0)}_{i+\frac{1}{2}}=\phi^{(0)}\left(\omega_{i+},\omega_{(i+1)-}\right)$$

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[7] E. Audusse, F. Bouchut, M.-O. Bristeau, R. Klein, B. Perthame *A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows* **SIAM Journal on Scientific Computing 25 (6), 2050-2065, 2004**