Modified shallow water equations for significant bathymetry variations

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Motivation

Understanding waves and surface flows $\Rightarrow$ Simplified models.

Saint-Venant (SV) simple model is often good enough, but not always.

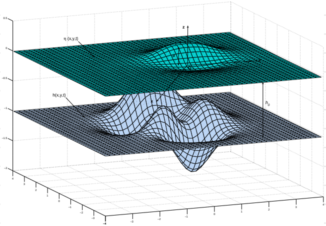
Can SV be improved at minimum cost (keeping the hyperbolicity)?

We propose a modified Saint-Venant (mSV) model when bathymetry gradient is significant.

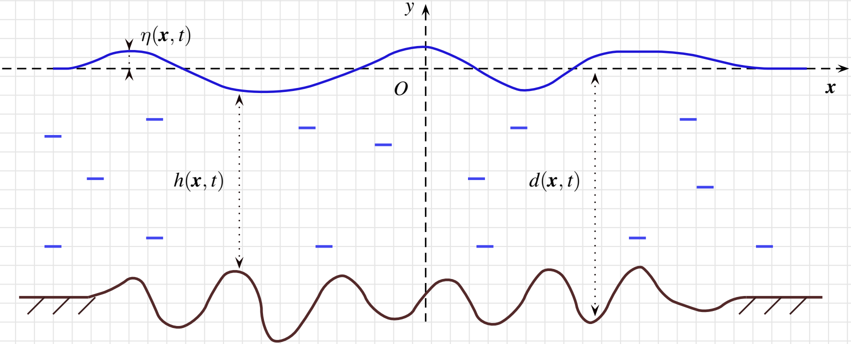
Hypothesis

Physical assumptions:

- Fluid is ideal, homogeneous & incompressible;
- Flow is irrotational, i.e., $\vec{V} = \operatorname{grad} \phi$;
- Free surface is a graph;
- Above free surface there is void;
- Atmospheric pressure is constant.
- Surface and bottom are both impermeable.
- Bottom can vary in space and time.



Definition Sketch



Notations

- $\boldsymbol{x} = (x_1, x_2)$: Horizontal Cartesian coordinates.
- y : Upward vertical coordinate.
- t : Time.
- $\boldsymbol{u} = (u_1, u_2)$: Horizontal velocity.
- v : Vertical velocity.
- ϕ : Velocity potential.
- $y = \eta(\boldsymbol{x}, t)$: Equation of the free surface.
- $y = -d(\boldsymbol{x}, t)$: Equation of the seabed.
- $\boldsymbol{\nabla}$: Horizontal gradient.
- Over tildes : Quantities at the surface, e.g., $\tilde{\boldsymbol{u}} = \boldsymbol{u}(y = \eta)$.
- Over check : Quantities at the surface, e.g., $\check{\boldsymbol{u}} = \boldsymbol{u}(y = -d)$.
- Over bar : Quantities averaged over the depth, e.g.,

$$\bar{\boldsymbol{u}} \;=\; \frac{1}{h}\int_{-d}^{\eta} \boldsymbol{u}\,\mathrm{d}y, \qquad h \;=\; \eta + d.$$

Basic shallow water models (2D + flat bottom)

Columnar flow: uniform horizontal velocity

$$u(x, y, t) \;\approx\; \bar{u}(x, t)$$

Vertical velocity (2 classical possibilities):

(1) Incompressibility: $u_x + v_y = 0 \quad \Rightarrow \quad v(x, y, t) \approx -(y + d)\,\bar{u}_x$

(2) Irrotationality: $v_x - u_y = 0 \quad \Rightarrow \quad v(x, y, t) \approx 0$

Kinetic and Potential energy densities:

$$\mathscr{K} = \int_{-d}^{\eta} \frac{u^2 + v^2}{2}\,\mathrm{d}y \approx \frac{h\bar{u}^2}{2} + \boldsymbol{\frac{h^3 \bar{u}_x^{\,2}}{6}}, \quad \mathscr{V} = \int_{-d}^{\eta} g(y + d)\,\mathrm{d}y = \frac{gh^2}{2}$$

Equations of motion

Lagrangian density:

$$\mathscr{L} \;=\; \mathscr{K} \;-\; \mathscr{V} \;+\; \{\, h_t \,+\, [\, h\,\bar{u}\,]_x \,\}\,\phi$$

ϕ: Lagrange multiplier.

Euler–Lagrange equations:

$$\begin{aligned} h_t \;+\; \partial_x[\, h\,\bar{u}\,] \;&=\; 0 \\ \partial_t[\, h\,\bar{u}\,] \;+\; \partial_x\!\left[\, h\,\bar{u}^2 \;+\; \tfrac{1}{2}\,g\,h^2 \;+\; \tfrac{1}{3}\,h^2\,\gamma \,\right] \;&=\; 0 \\ 2\,h\,\bar{u}_x^{\,2} \;-\; h\,\partial_x[\, \bar{u}_t \;+\; \bar{u}\,\bar{u}_x \,] \;&=\; \gamma \end{aligned}$$

With red terms: Serre–Green–Naghdi equations.
Without red terms: Saint-Venant equations.

Serre equations

Pros:

- Dispersive.
- Admit permanent solutions.
- Regular.

Cons:

- High-order derivatives.
- Hard to solve numerically.
- Not hyperbolic.

Saint-Venant equations

Pros:

- Hyperbolic.
- Characteristics.
- Fast numerical solvers.

Cons:

- Non-dispersive.
- No smooth permanent solutions.
- Limited to very slowly varying bottoms.

Modified Saint-Venant (mSV) equations

Choice of the ansatz (columnar flow):

$$\boldsymbol{u} \approx \bar{\boldsymbol{u}}(\boldsymbol{x}, t), \qquad v \approx \check{v}(\boldsymbol{x}, t) = -d_t - \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} d$$

Equations of motion:

$$0 = \partial_t h + \boldsymbol{\nabla} \cdot [\, h \bar{\boldsymbol{u}} \,]$$

$$\partial_t [\, h \bar{\boldsymbol{u}} \,] + \boldsymbol{\nabla} [\, h \bar{\boldsymbol{u}}^{\,2} + \tfrac{1}{2} g h^{2} \,] = (g + \gamma) h \boldsymbol{\nabla} d + h \bar{\boldsymbol{u}} \wedge (\boldsymbol{\nabla} \check{v} \wedge \boldsymbol{\nabla} d)$$

DUTYKH & CLAMOND 2011. *J. Phys. A: Math. & Theor.* 44, 332001.

Model properties

Hyperbolic equations.

Method of characteristics is usable.

Waves propagation speed in SV and mSV:

$$c_{\text{SV}} = \sqrt{gh} \qquad\qquad c_{\text{mSV}} = \frac{\sqrt{gh}}{\sqrt{1 + |\boldsymbol{\nabla} d|^2}}$$

Wave propagation over oscillating bottom

Initial surface:

$$\eta(x, t = 0) \; = \; b \; \text{sech}^2(\kappa x), \qquad u(x, t = 0) \; = \; 0.$$

Bottom profile:

$$d(x) \; = \; d_0 \; + \; a \sin(kx).$$

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

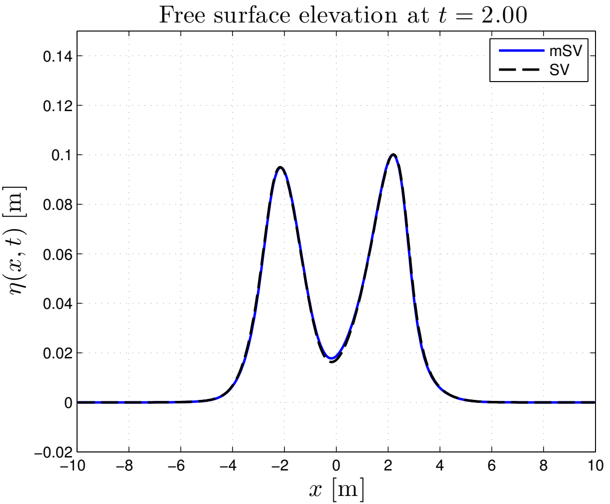


Figure: $t = 2$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

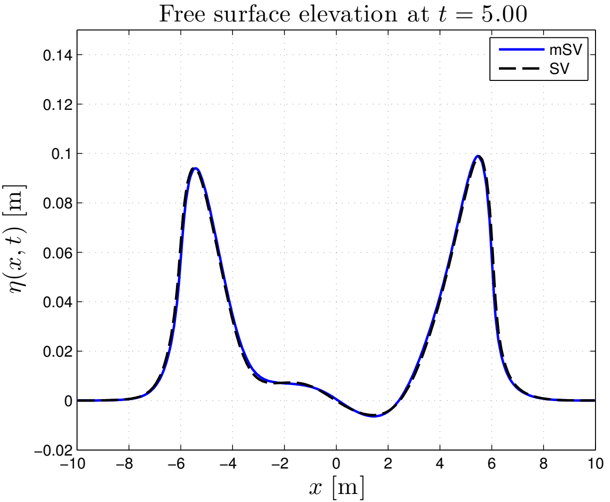


Figure: $t = 5$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

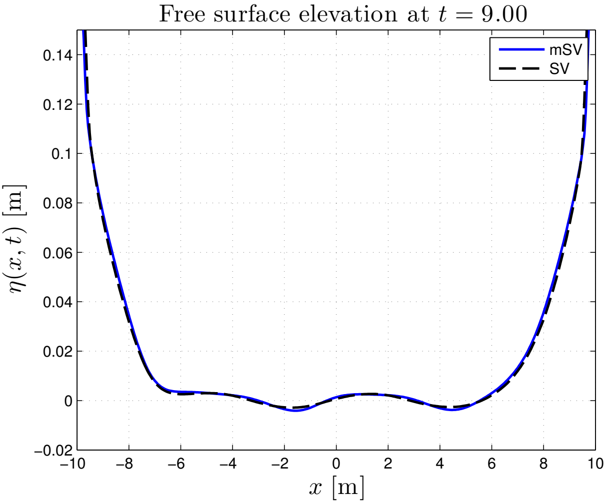


Figure: $t = 9$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

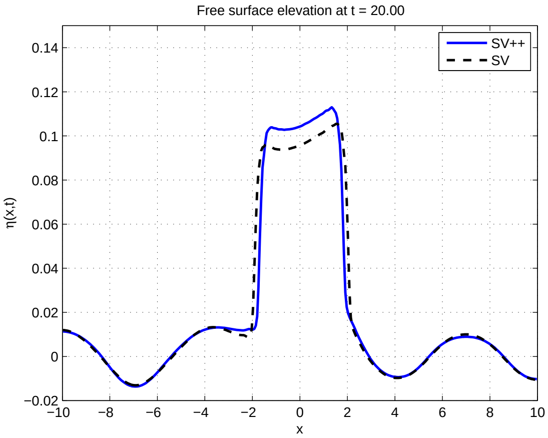


Figure: $t = 20$ s

Wave propagation over oscillating bottom: low freq.

Comparison with classical Saint-Venant equations

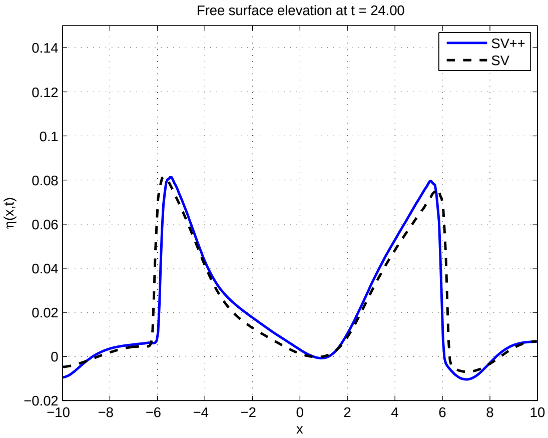


Figure: $t = 24$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

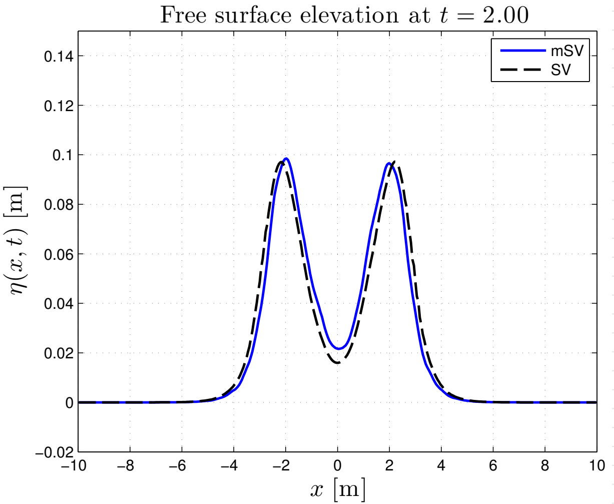


Figure: $t = 2$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

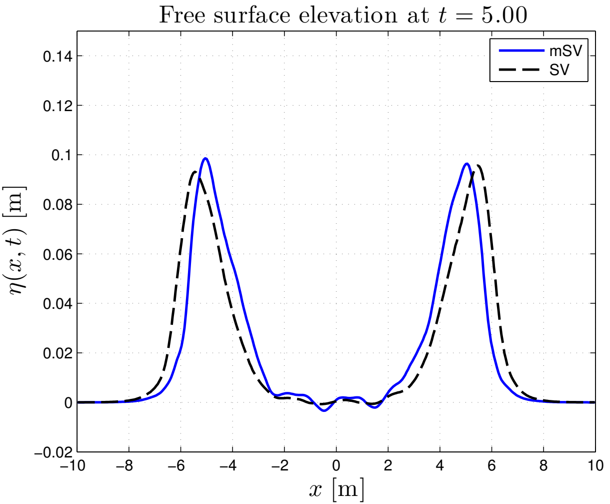


Figure: $t = 5$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

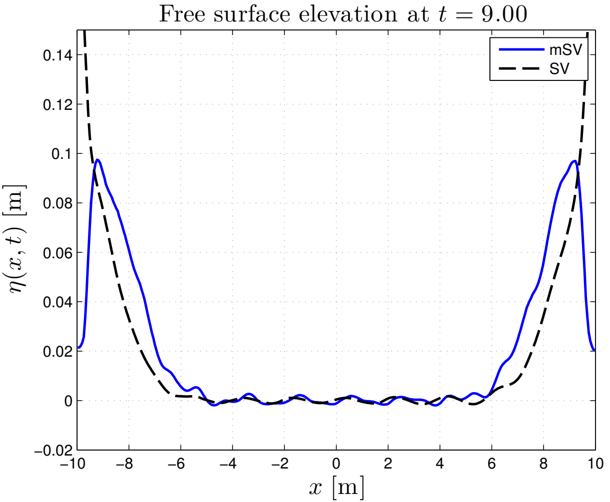


Figure: $t = 9$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

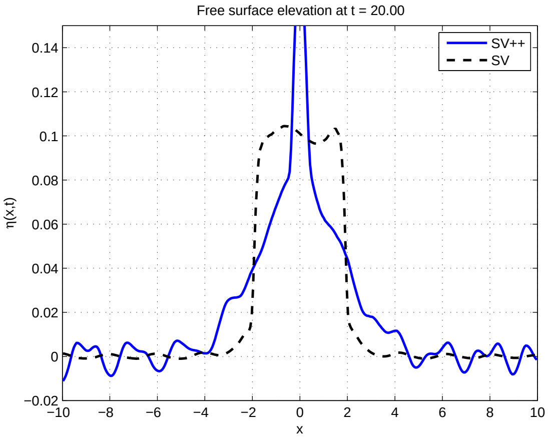


Figure: $t = 20$ s

Wave propagation over oscillating bottom: high freq.

Comparison with classical Saint-Venant equations

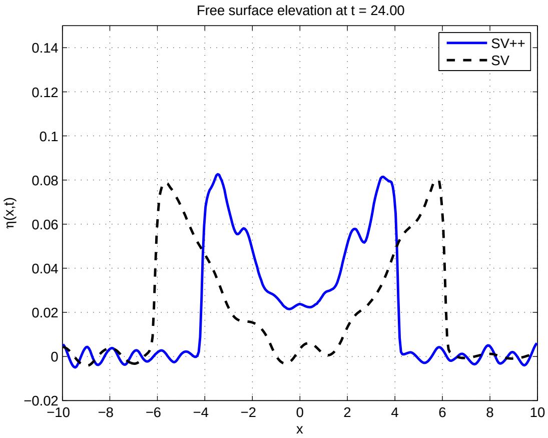


Figure: $t = 24$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

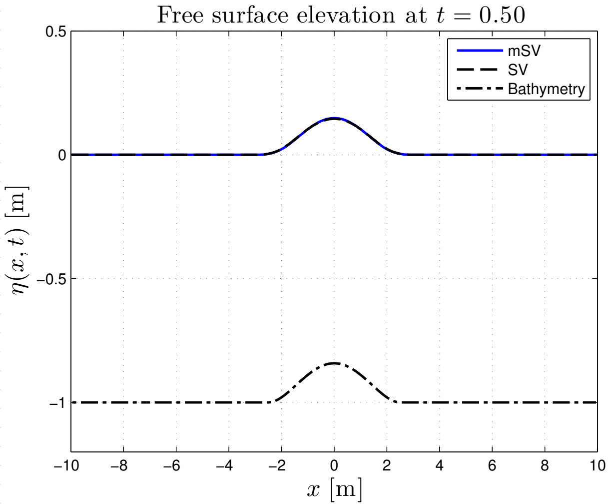


Figure: $t = 0.5$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

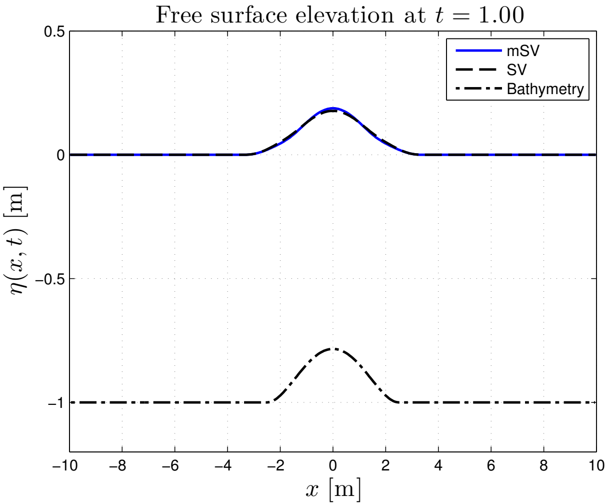


Figure: $t = 1.0$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

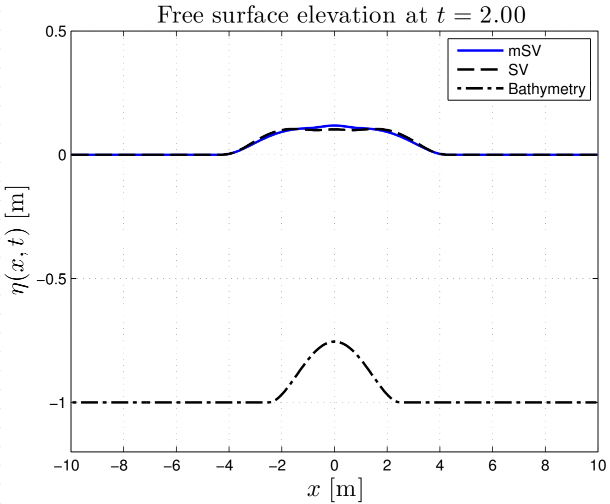


Figure: $t = 2.0$ s

Moving bottom test-case: slow uplift

Comparison with classical Saint-Venant equations

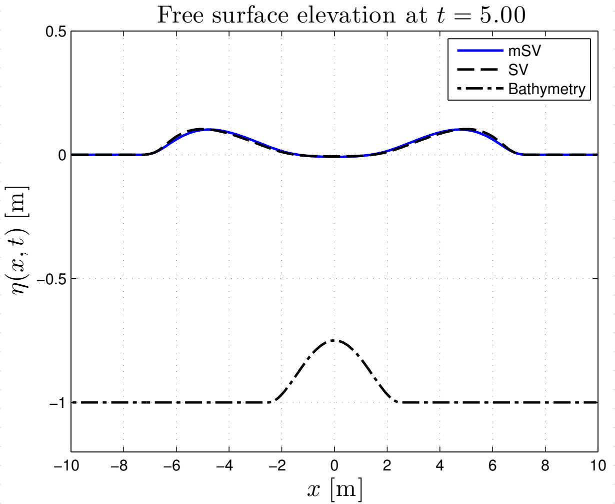


Figure: $t = 5.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

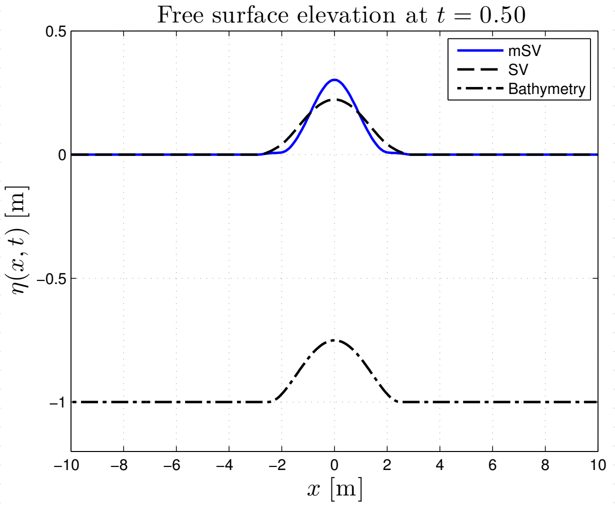


Figure: $t = 0.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

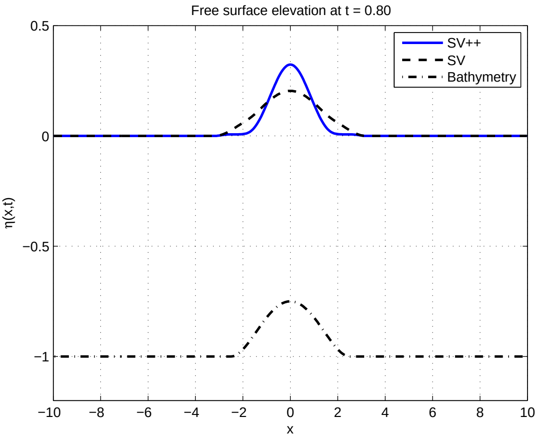


Figure: $t = 0.8$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

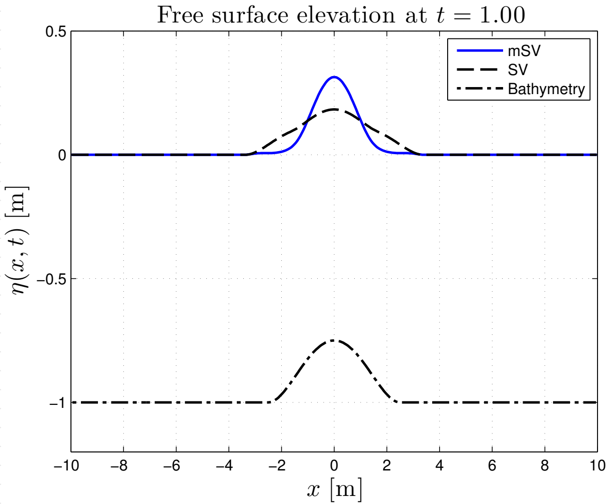


Figure: $t = 1.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

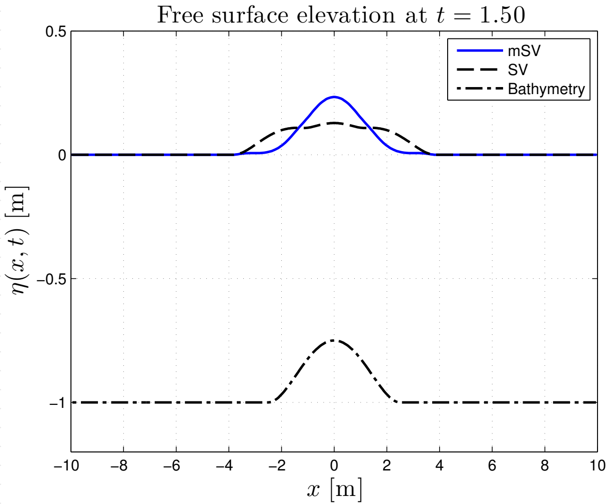


Figure: $t = 1.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

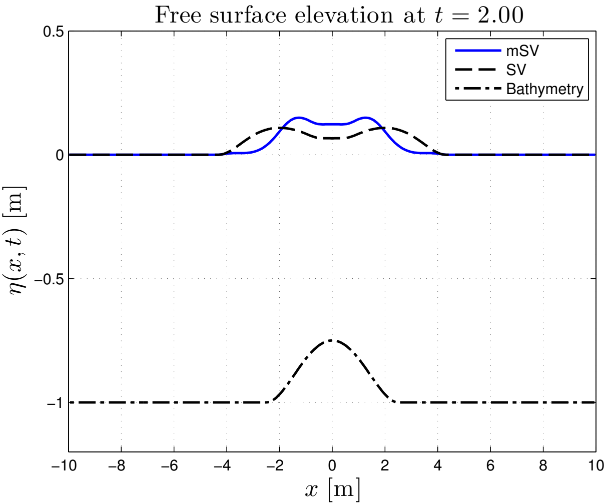


Figure: $t = 2.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

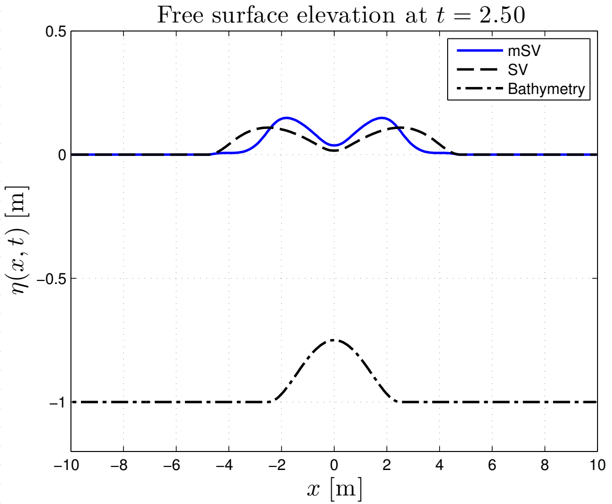


Figure: $t = 2.5$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

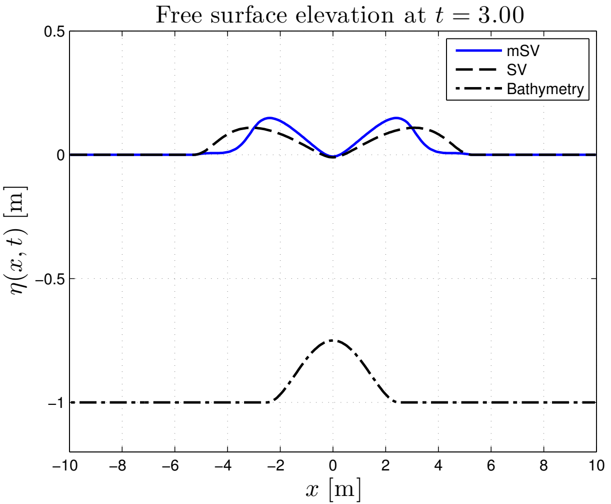


Figure: $t = 3.0$ s

Moving bottom test-case: fast uplift

Comparison with classical Saint-Venant equations

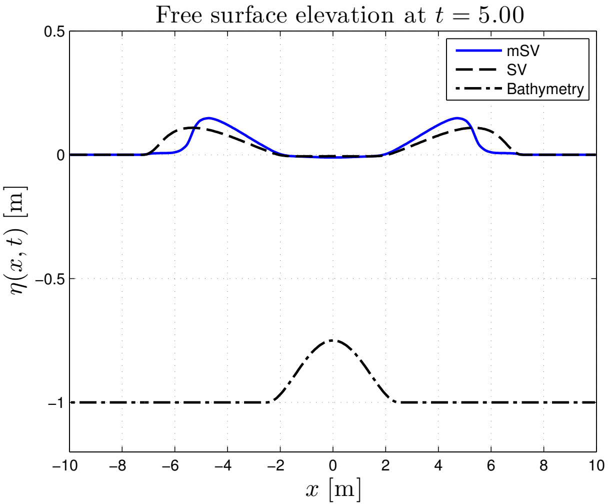


Figure: $t = 5.0$ s

Partial validations

More numerical examples:
http://arxiv.org/pdf/1202.6542.pdf

Observations: Reported tsunami travel time often exceeds slightly the values predicted by the classical shallow water theory (WESSEL 2009, *Pure Appl. Geophys.* 166).

Experimental validations: Require a very long flume.

Numerical validations: Currently under development.

Shortcomings

More validations needed from experiments and field data.

The model cannot be mathematically justified via classical perturbation techniques.

This variational approach is not common in water wave theories.

But it is well-known in quantum mechanics.

Variational principle: Practical use

Choose an ansatz for, e.g., the vertical variations of ϕ, $\boldsymbol{u}$ and v from the linear theory.

Add constraints if necessary (e.g., enforce impermeability, incompressibility).

Derive the Euler–Lagrange equations.

Advantage: A small parameter is not required.

CLAMOND & DUTYKH 2012. Practical use of variational principles for modeling water waves. *Physica D* 241, 1, 25-36.

Berkeley's course on Quantum Mechanics

The perturbation theory is useful when there is a small dimensionless parameter in the problem, and the system is exactly solvable when the small parameter is sent to zero.

... it is not required that the system has a small parameter, nor that the system is exactly solvable in a certain limit. Therefore it has been useful in studying strongly correlated systems, such as the fractional Quantum Hall effect.

... the success of the variational method depends on the initial "guess" ... and an excellent physical intuition is required for a successful application.

H. Murayama. 221A Lecture Notes: Variational Method.

Caution (extract from Murayama's lecture notes)

... there is no way to judge how close your result is to the true result. The only thing you can do is to try out many Ansätze and compare them.

For example, R. Laughlin proposed a trial wave function that beat other wave functions that had been proposed earlier, such as "Wigner crystal".

... Once your wave function gives a lower energy than your rival's, you won the race.

R. Laughlin *et al.* earned the 1998 Physics Nobel price for this work.

Conclusions & Perspectives

Conclusions:

- ‘Cheap’ improved SV-like model.
- Predicted new physical features.
- Physical observations of tsunamis seem to support it.
- Definitive confirmations are still lacking.

Perspectives:

- Quest for improved SV models in this framework: many variants can easily be derived.
- Easy adaptation of existing numerical schemes.
- Relevance for real-life applications.
- Variational methods apply to floating bodies as well.
- Can be modified to incorporate dissipation effects.

Mathematical formulation

- Continuity equation $\boldsymbol{x} \in \Omega$, $-d \leqslant y \leqslant \eta$ (Laplace)

$$\boldsymbol{\nabla}^2 \phi \;+\; \partial_y^2 \phi \;=\; 0$$

- Bottom's impermeability condition at $y = -d(\boldsymbol{x}, t)$

$$\partial_t d \;+\; (\boldsymbol{\nabla} \phi) \cdot (\boldsymbol{\nabla} d) \;=\; -\partial_y \phi$$

- Free surface's impermeability condition at $y = \eta(\boldsymbol{x}, t)$

$$\partial_t \eta + (\boldsymbol{\nabla} \phi) \cdot (\boldsymbol{\nabla} \eta) \;=\; \partial_y \phi$$

- Dynamic free surface condition at $y = \eta(\boldsymbol{x}, t)$ (Bernoulli)

$$\partial_t \phi \;+\; \tfrac{1}{2} |\boldsymbol{\nabla} \phi|^2 \;+\; \tfrac{1}{2} (\partial_y \phi)^2 \;+\; g\eta \;=\; 0$$

Variational principle: Luke's Lagrangian

Functional $\mathfrak{L} = \iiint \mathscr{L}\, \mathrm{d}^2\boldsymbol{x}\, \mathrm{d}t$ with

$$\mathscr{L} = -\int_{-d}^{\eta} \left[gy + \partial_t\phi + \frac{|\boldsymbol{\nabla}\phi|^2}{2} + \frac{(\partial_y\phi)^2}{2} \right] \mathrm{d}y$$

and such that $\delta\mathfrak{L} = 0$.

The Euler–Lagrange equations yield the equations of the previous slide.

Shortcoming: Only two dependent variables (ϕ and η).

Variational principle: Hamilton's principle

Functional $\mathfrak{L} = \iiint \mathscr{L}\, \mathrm{d}^2\boldsymbol{x}\, \mathrm{d}t$ with

$$\mathscr{L} = (\eta_t + \tilde{\boldsymbol{u}} \cdot \boldsymbol{\nabla}\eta - \tilde{v})\tilde{\phi} + (d_t + \check{\boldsymbol{u}} \cdot \boldsymbol{\nabla} d + \check{v})\check{\phi} + \int_{-d}^{\eta} \left[\frac{\boldsymbol{u}^2 + v^2}{2} + (\boldsymbol{\nabla} \cdot \boldsymbol{u} + v_y)\phi\right] \mathrm{d}y - \frac{g\,\eta^2}{2}$$

and such that $\delta\mathfrak{L} = 0$.

The Euler–Lagrange equations yield the exact equations.

Advantage: Four dependent variables (ϕ, η, $\boldsymbol{u}$ and v).

Straightforward generalisation with floating and sumerged bodies.