Master of Science in Industrial and Applied Mathematics - Grenoble
9^{th} **semester**

Lecture on *Model exploration for approximation of complex, high-dimensional problems* taught by Clémentine Prieur and Olivier Zahm, 2018-2019
Final exam, 2019 February the 5^{th}

Duration: 2h.
Authorized documents: lecture notes.

Exercice 1 (5 pts) Let X_1, X_2 and X_3 be independent random variables distributed as $\mathcal{N}(0,1)$. We define the random process $\{Y(t,s)\,,\ (t,s) \in [0,1]^2\}$ as $Y(t,s) = tX_1 + sX_2 + (t-s)X_3$.

1. Justify carefully that Y is a gaussian process on $[0,1]^2$.
2. What is the mean function of Y?
3. Prove that the covariance function of Y, denoted by $K : [0,1]^2 \to \mathbb{R}$, is defined as

$$K\left((t_1,s_1),(t_2,s_2)\right) = t_1 t_2 + s_1 s_2 + (t_1 - s_1)(t_2 - s_2)\ .$$

4. What is the probability distribution of $Y(1,1)$?
5. What is the conditional probability distribution of $Y(1,1)$ on $Y(0,1) = 1$?

Exercice 2 (7 pts) Let us define the sequences of polynomials (U_n) and (P_n) as follows:

$$U_0 = 1 \qquad \forall\, n \geq 1\,,\ U_n(X) = \frac{X^n (X-1)^n}{n\,!} \qquad \forall\, n \geq 0\,,\ P_n = U_n^{(n)}$$

1. Give an explicit formula for polynomials P_0 and P_1.
2. Prove the two following relations:

$$\begin{aligned} U'_{n+1} &= (2X-1)U_n\,, \\ U''_{n+1} &= 2(2n+1)U_n + U_{n-1}\,. \end{aligned}$$

3. By differentiating n times the first relation and $n-1$ times the second one, prove that, for any positive integer n,

$$(n+1)P_{n+1} - (2n+1)(2X-1)P_n + nP_{n-1} = 0\,.$$

Clue: we recall the Leibniz formula

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}\,.$$

4. Deduce from the previous question an explicit formula for P_2.

5. Let $P(\mathbf{x}) = P(x_1, x_2, x_3) = 12x_1x_2^2 - 6x_2^2 - 12x_1x_2 + 14x_2 + 2x_1 + 2x_3 - 3$. Prove that:

$$P(\mathbf{x}) = P_1(x_1)P_2(x_2) + 4P_1(x_2) + P_1(x_3) + 3P_0\,.$$

6. The sequence (P_n) is known as Legendre polynomials on $[0,1]$. Check that

$$\int_0^1 P_j^2(x) = \frac{1}{2j+1} \quad j = 0,1,2$$

$$\int_0^1 P_j(x)P_k(x) = 0 \quad j,k = 0,1,2\,.$$

7. Compute all the Sobol' indices for the polynomials P if the uncertain input vector is modeled by the uniform probability distribution on $[0,1]^3$.

Exercice 3 (8 pts) Let $u(x) \in \mathbb{R}^N$ be the solution to

$$A(x)u(x) = b(x),$$

where $A(x) \in \mathbb{R}^{N\times N}$ is an invertible matrix and where $b(x) \in \mathbb{R}^N$. Here, the parameter $x \in \mathbb{R}$ is a scalar.

1. Assume that $x \mapsto A(x)$ and $x \mapsto b(x)$ are differentiable and denote by $A'(x) \in \mathbb{R}^{N\times N}$ and $b'(x) \in \mathbb{R}^N$ their derivatives. Show that $x \mapsto u(x)$ is differentiable and that $u'(x) \in \mathbb{R}^N$ is the solution to

$$A(x)u'(x) = \ell(x),$$

for some right-hand side $\ell(x) \in \mathbb{R}^N$ which depends on $A'(x)$, $b'(x)$ and $u(x)$. (*hint: try to differentiate equation* $A(x)u(x) = b(x)$)

2. Let $V_r \subset \mathbb{R}^N$ be a low-dimensional subspace. We denote by $\widetilde{u}_r(x) \in V_r$ the Galerkin projection of $u(x)$ onto V_r defined by

$$v_r^T A(x)\widetilde{u}_r(x) = v_r^T b(x), \tag{1}$$

for any $v_r \in V_r$. Show that the derivative $\widetilde{u}_r'(x) \in V_r$ is such that

$$v_r^T A(x)\widetilde{u}_r'(x) = v_r^T \widetilde{\ell}(x),$$

for any $v_r \in V_r$, where $\widetilde{\ell}(x) \in \mathbb{R}^N$ depends on $A'(x)$, $b'(x)$ and $\widetilde{u}(x)$.

3. Assume $A(x)$ is coercive. We recall that Céa's Lemma ensures that $\widetilde{u}_r(x)$ defined by (1) satisfies

$$\|u(x) - \widetilde{u}_r(x)\| \leq C \inf_{v_r \in V_r} \|u(x) - v_r\|,$$

for some constant $C < \infty$. Show that if $u(x_0) \in V_r$ for some $x_0 \in \mathbb{R}$ then $\widetilde{\ell}(x_0) = \ell(x_0)$.

4. Assume that $u(x_0) \in V_r$ and $u'(x_0) \in V_r$. Show that $\widetilde{u}_r(x)$ interpolates the derivative of $u(x)$ at x_0 (*hint: replace* $\widetilde{u}_r(x)$ *by* $\widetilde{u}_r'(x)$ *in the Céa's Lemma*). How to define V_r such that $\widetilde{u}_r(x)$ interpolates $u(x)$ and its derivatives at arbitrary points $x_1, \ldots, x_r$?