Exam MSIAM, Model Exploration

2019-02-05

Corrigé

Exo 2

1) $P_0 = U_0^{(0)} = 1$

$P_1 = U_1^{(1)} = (X(X-1))' = X-1+X = 2X-1$

2) $U_n(X) = \frac{X^n(X-1)^n}{n!}$

$$U_{n+1}'(X) = \left(\frac{X^{n+1}(X-1)^{n+1}}{(n+1)!}\right)' = \frac{(n+1)X^n(X-1)^n(X-1) + XX^n(n+1)(X-1)^n}{(n+1)!}$$

$$= \frac{X^n(X-1)^n}{n!}(2X-1) = (2X-1)U_n(X).$$

$$U_{n+1}''(X) = 2U_n(X) + (2X-1)U_n'(X)$$

$$= 2U_n(X) + (2X-1)(2X-1)U_{n-1}(X)$$

$$= 2U_n(X) + (4X^2-4X+1)\frac{X^{n-1}(X-1)^{n-1}}{(n-1)!}$$

$$= 2U_n(X) + \frac{4nX(X-1)}{n}\frac{X^{n-1}(X-1)^{n-1}}{(n-1)!} + U_{n-1}(X)$$

$$= 2(2n+1)U_n(X) + U_{n-1}(X).$$

$$U_{n+1}^{(n+1)} = \left((2X-1)U_n\right)^{(n)}$$

$$U_{n+1}^{(n+1)} = 2(2n+1)U_n^{(n-1)} + U_{n-1}^{(n-1)}$$

3) $P_{n+1} = \left[(2X-1)\,U_n\right]^{(n)}$

$P_{n+1} = 2(2n+1)\,U_n^{(n-1)} + U_{n-1}^{(n-1)}$

Formule de Leibniz

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}$$

$g = 2X-1 \qquad g' = 2 \quad g'' = 0$

$f = U_n \qquad f^{(n-1)} = U_n^{(n-1)}$

$P_{n+1} = \binom{n}{0} U_n^{(n)}(2X-1) + \binom{n}{1} U_n^{(n-1)}\, 2$

$= P_n(2X-1) + 2n\, U_n^{(n-1)}$

$= P_n(2X-1) + \cancel{2}n\left[\dfrac{P_{n+1}}{\cancel{2}(2n+1)} - \dfrac{P_{n-1}}{\cancel{2}(2n+1)}\right]$

$(2n+1)P_{n+1} - nP_{n+1} = (2n+1)P_n(2X-1) - P_{n-1}$

$$\boxed{(n+1)P_{n+1} - (2n+1)(2X-1)P_n + nP_{n-1} = 0}$$

4) $P_0 = 1,\ P_1 = 2X-1$

$2P_2 - 3(2X-1)^2 + 1 = 0 \iff P_2 = \frac{3}{2}(2X-1)^2 - \frac{1}{2}$

$= \frac{3}{2}(4X^2+1-4X) - \frac{1}{2}$

$= 6X^2 - 6X + 1$

5) $P_1(x_1)P_2(x_2) + 4P_1(x_2) + P_1(x_3) + 3P_0$

$= (2x_1-1)(6x_2^2 - 6x_2 + 1) + 4(2x_2-1) + (2x_3-1) + 3$

$= 12x_1x_2^2 - 6x_2^2 - 12x_1x_2 + 14x_2 + 2x_1 - 1 - 4 + 2x_3 - 1 + 3$

$= 12x_1x_2^2 - 6x_2^2 - 12x_1x_2 + 14x_2 + 2x_1 + 2x_3 - 3.$

$= 12x_1x_2^2 - 6x_2^2 - 12x_1x_2 + 14x_2 + 2x_1 + 2x_3 - 3$

6) $\int_0^1 P_0^2\,dx = 1,\quad \int_0^1 (2x-1)^2\,dx = \frac{4}{3} - \frac{4}{2} + 1 = \frac{1}{3},\quad \int_0^1 (6x^2-6x+1)^2\,dx$

$= \int_0^1 (36x^4 + 36x^2 + 1 - 72x^3 + 12x^2 - \cancel{12x})\,dx = \frac{36}{5} + \frac{36}{3} + 1 - \frac{72}{4} + \frac{12}{3} - \frac{12}{2}$

$= \frac{1}{5}$

$$\int_0^1 (2x-1)\,dx = \frac{2}{2} - 1 = 0 \qquad P_0 P_1$$

$$\int_0^1 (6x^2-6x+1)\,dx = \frac{6}{3} - \frac{6}{2} + 1 = 0 \qquad P_0 P_2$$

$P_1 P_2$

$$\int_0^1 (2x-1)(6x^2-6x+1)\,dx$$
$$= \int_0^1 (12x^3 - 6x^2 - 12x^2 + 6x + 2x - 1)\,dx$$
$$= \frac{12}{4} - \frac{6}{3} - \frac{12}{3} + \frac{6}{2} + \cancel{\frac{2}{2}} - \cancel{1}$$
$$= 3 - 2 - 4 + 3 = 0 .$$

On normalise $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2$ $\qquad \tilde{P}_0 = P_0,\ \tilde{P}_1 = \frac{P_1}{\sqrt{3}},\ \tilde{P}_2 = \frac{P_2}{\sqrt{5}}$

c) $P = \sqrt{15}\,\tilde{P}_1(x_1)\tilde{P}_2(x_2) + 4\sqrt{3}\,\tilde{P}_1(x_2) + \sqrt{3}\,\tilde{P}_1(x_3) + 3\tilde{P}_0$

$\operatorname{Var} P(X) = 15 + 16 \times 3 + 3 = 66$

$S_{12} = \frac{15}{66} = \frac{5}{22}$

$S_2 = \frac{16\times 3}{66} = \frac{16}{22} = \frac{8}{11}$

$S_3 = \frac{3}{66} = \frac{1}{22}$

$S_1 = 0$

$S_{13} = S_{23} = 0$

$S_{123} = 0$.

Exo 1

1) $\begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, I\right)$

$Y(t,s)$ CL $X_1, X_2, X_3 \Rightarrow \mathcal{N}$

$\Rightarrow Y$ processus $\mathcal{N}$

2) $E\,Y(t,s) = t\,EX_1 + s\,EX_2 + (t-s)\,EX_3 = m(t,s) = 0$.

3) $E\,Y(t_1,s_1)\,Y(t_2,s_2)$
$= E\left(t_1X_1 + s_1X_2 + (t_1-s_1)X_3\right)\left(t_2X_1 + s_2X_2 + (t_2-s_2)X_3\right)$
$= E\big\{t_1t_2X_1^2 + s_1t_2X_1X_2 + (t_1-s_1)X_3X_1 + t_1s_2X_1X_2 + s_1s_2X_2^2 + (t_1-s_1)s_2X_3X_2 + (t_1-s_1)(t_2-s_2)X_3^2\big)$
$+ s_1(t_2-s_2)X_3 + (t_1-s_1)t_2X_3X_1$
$= t_1t_2 + s_1s_2 + (t_1-s_1)(t_2-s_2) = 2(t_1t_2 + s_1s_2) - (s_1t_2 + t_1s_2)$
$= K((t_1,s_1),(t_2,s_2))$

4)

$$Y(1,1) = X_1 + X_2$$
$$\sim \mathcal{N}(0, 2).$$

<u>5)</u> $Y(1,1) \mid Y(0,1) = 1$
$\sim \mathcal{N}$

$$\begin{pmatrix} Y(0,1) \\ Y(1,1) \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \right)$$

$E\ Y(1,1) \mid Y(0,1) = 1$

$$= m(1,1) + K((1,1),(0,1)) \frac{1}{K((0,1),(0,1))} \left(Y(0,1) - 0 \right)$$

$$= \frac{1}{2}(1-0) = \frac{1}{2}$$

$$\mathrm{Var}\left(Y(1,1) \mid Y(0,1) = 1\right) = 2 - K((1,1),(0,1)) \frac{1}{K((0,1),(1,1))} K((1,1),(0,1))$$

$$= 2 - \frac{1}{2} = \frac{3}{2}$$

$$Y(1,1) \mid Y(0,1) = 1 \sim \mathcal{N}\left(\frac{1}{2}, \frac{3}{2}\right).$$

(formules krigeage)