Projection-based Model Order Reduction techniques: error bounds and quantities of interest

Olivier Zahm*

Lecture notes M2:SIAM, 2019.

1 Projection-based model order reduction

Consider the computational chain

$$\underbrace{x \in \mathbb{R}^d}_{\text{parameter}} \xrightarrow{\text{linear system (1)}} \underbrace{u(x) \in \mathbb{R}^N}_{\text{model output}} \xrightarrow{\text{post-processing (2)}} \underbrace{Y(x) \in \mathbb{R}}_{\text{quantity of interest}}$$

where $u(x)$ is the solution to a parametrized algebraic equation

$$A(x)u(x) = b(x), \tag{1}$$

and where $Y(x) \in \mathbb{R}$ is a linear quantity of interest of $u(x)$ defined by

$$Y(x) = q(x)^T u(x). \tag{2}$$

The matrix $A(x) \in \mathbb{R}^{N\times N}$, the right-hand side $b(x) \in \mathbb{R}^N$ and the extractor $q(x) \in \mathbb{R}^N$ depend on a d-dimensional parameter $x = (x_1, \ldots, x_d)$ which belongs to some parameter set $\mathcal{P} \subset \mathbb{R}^d$. We consider a reduced space of the form of

$$V_r = \text{span}\{u(x_1), \ldots, u(x_r)\} \subset \mathbb{R}^N, \quad r \ll N,$$

where $x_1, \ldots, x_r$ are r points in $\mathcal{P}$ carefully chosen (for instance using the Reduced Basis method, see previous lecture note). We approximate $u(x)$ by an element $\widetilde{u}_r(x) \in V_r$ defined by Galerkin projection

$$v_r^T A(x)\widetilde{u}_r(x) = v_r^T b(x), \quad \forall v_r \in V_r. \tag{3}$$

This means, we construct $\widetilde{u}_r(x) \in V_r$ sucht that the residual $A(x)u(x) - b(x)$ is orthogonal to the reduced space V_r. As seen in the previous practical session, solving (7) is computationally much cheaper than solving the original equation (1). In this tutorial, we ask the question of the quality of this approximation (Section 2) and how to estimate the quantity of interest (Section **??**).

*olivier.zahm@inria.fr

2 Error bounds

We denote by $\|\cdot\| = \sqrt{(\cdot)^T(\cdot)}$ the canonical norm of $\mathbb{R}^N$. The following quantity, sometimes called the *inf-sup constant*, plays a central role in error analysis

$$\alpha(x) = \inf_{v\in\mathbb{R}^N} \frac{\|A(x)v\|}{\|v\|}. \tag{4}$$

The condition $\alpha(x) > 0$ for any $x \in \mathcal{P}$ is necessary[1] to ensure that $A(x)$ is invertible. Indeed, $\alpha(x) = 0$ means there exists a nonzero $v \in \mathbb{R}^N$ such that $\|A(x)v\| = 0$, meaning that the kernel of $A(x)$ is not reduced to 0.

> **Exercise 1.** Show that $\alpha(x) = \sigma_{\min}(A(x))$, the smallest singular value of $A(x)$.

2.1 *A posteriori* error bound

When approximating $u(x)$, a essential question is how good is the resulting approximation. In general, we would like to know rather $\|u(x) - \widetilde{u}_r(x)\|$ is of the order of 10^{-1} (crude approximation) or 10^{-8} (excellent approximation). However $\|u(x) - \widetilde{u}_r(x)\|$ is costly to compute because is requires to know the solution $u(x)$. We can write

$$\underbrace{\|u(x) - \widetilde{u}_r(x)\|}_{\text{not accessible}} \overset{(4)}{\leq} \frac{1}{\alpha(x)}\|A(x)(u(x) - \widetilde{u}_r(x))\| \overset{(1)}{=} \frac{1}{\alpha(x)} \underbrace{\|A(x)\widetilde{u}_r(x) - b(x)\|}_{\text{easily computable}}.$$

Then if we have access to $\alpha(x)$ (or to a lower bound of it) we can *certify* the error by computing the residual norm $\|A(x)\widetilde{u}_r(x) - b(x)\|$. This is an *a posteriori* error indicator, meaning an error indicator which can be computed *after* the approximation $\widetilde{u}_r(x)$ has been computed.

> **Exercise 2.** Assume the affine decomposition of $A(x)$ and $b(x)$:
>
> $$A(x) = \sum_{k=1}^{m_a} \Psi_k^A(x)\, A_k, \qquad b(x) = \sum_{k=1}^{m_b} \Psi_k^b(x)\, b_k,$$
>
> where $m_a, m_b \ll N$. Let $\mathbf{V}_r = [u_1, \dots, u_r]$ be the matrix containing the basis vectors of V_r so that $\widetilde{u}_r(x) = \mathbf{V}_r\lambda(x)$, where $\lambda(x) \in \mathbb{R}^r$ denotes the components of $\widetilde{u}_r(x)$ on the reduced basis. Show that
>
> $$\|A(x)\widetilde{u}_r(x) - b(x)\|^2 = \lambda(x)^T M(x)\lambda(x) - 2\lambda(x)^T F(x) + S(x),$$
>
> and that the matrix $M(x) \in \mathbb{R}^{r\times r}$, the vector $F(x) \in \mathbb{R}^r$ and the scalar $S(x) \in$

[1] it is even sufficient in finite dimension (via the rank-nullity theorem), but not in infinite dimension!

$\mathbb{R}$ admit an affine decomposition and give the corresponding number of terms m_M, m_F, m_S. Why is this kind of decomposition useful?

2.2 *A priori* error bound

The goal of *a priori* error analysis is to give the guaranty that, even before we compute it, the approximation $\widetilde{u}_r(x)$ *will be* a good approximation. In the present context, we want to guarantee that the Galerkin projection (7) is a good way to construct the approximation $\widetilde{u}_r(x)$ in the reduced space V_r. This is given by the Céa's Lemma.

Lemma 2.1 (Céa's Lemma)**.** Assume the matrix $A(x)$ is coercive, meaning

$$\alpha_c(x) = \inf_{v\in\mathbb{R}^N} \frac{v^T A(x) v}{\|v\|^2} > 0. \tag{5}$$

Then the Galerkin projection $\widetilde{u}_r(x)$ defined by (7) satisfies

$$\|u(x) - \widetilde{u}_r(x)\| \leq \frac{\beta(x)}{\alpha_c(x)} \min_{v_r\in V_r} \|u(x) - v_r\|, \tag{6}$$

where $\beta(x)$ is the continuity constant of $A(x)$ defined by $\beta(x) = \sup_{v\in\mathbb{R}^N} \|Av\|/\|v\|$.

The proof of Lemma 2.1 is given at the end of this section[2]. Inequality (6) shows that the error $\|u(x) - \widetilde{u}_r(x)\|$ is no greater than a constant times $\min_{v_r\in V_r} \|u(x) - v_r\|$, which is the smallest error we can achieve when approximating $u(x)$ by an element in V_r. This ensures the stability of the Galerkin projection.

Exercise 3. Recall that the reduced space is

$$V_r = \text{span}\{u(x_1), \dots, u(x_r)\},$$

where $x_1, \dots, x_r$ are some points in the parameter set $\mathcal{P}$. Using Céa's Lemma, show that $\widetilde{u}_r(x)$ interpolates $u(x)$ at the points $x_1, \dots, x_r$. What happens between the interpolation points if $\beta(x)/\alpha_c(x)$ is large, say of the order of 10^8 ?

Exercise 4. Show that $\beta(x) = \sigma_{\max}(A(x))$, the largest singular of $A(x)$.

Exercise 5. Assume the matrix $A(x)$ is symmetric, meaning $A(x)^T = A(x)$. Show that $\alpha_c(x) = \lambda_{\min}(A(x))$, the smallest eigenvalue of $A(x)$.

[2]it is not mandatory to know it for the exam...

Proof of Lemma 2.1. For simplicity we omit the dependence in x and we write A, b, u,... instead of $A(x)$, $b(x)$, $u(x)$,... By (5) we have that $\|v\|^2 \leq \alpha_c^{-1} v^T A v$ for any $v \in \mathbb{R}^N$. In particular for $v = u - \widetilde{u}_r$ we have

$$\|u - \widetilde{u}_r\|^2 \leq \frac{1}{\alpha_c}(u - \widetilde{u}_r)^T A(u - \widetilde{u}_r) = \frac{1}{\alpha_c}(u - \widetilde{u}_r)^T (b - A\widetilde{u}_r).$$

Next, for any $v_r \in V_r$ we can write

$$\begin{aligned}\|u - \widetilde{u}_r\|^2 &\leq \frac{1}{\alpha_c}\left((u - v_r)^T (b - A\widetilde{u}_r) + \underbrace{(\overbrace{v_r - \widetilde{u}_r}^{\in V_r})^T (b - A\widetilde{u}_r)}_{=0 \text{ by } (7)} \right) \\ &\overset{\text{C.S.}}{\leq} \frac{1}{\alpha_c}\|u - v_r\|\,\|b - A\widetilde{u}_r\| \\ &\leq \frac{\beta}{\alpha_c}\|u - v_r\|\,\|u - \widetilde{u}_r\|,\end{aligned}$$

where, for the last inequality, we used the definition of β. Finally, simplifying by $\|u - \widetilde{u}_r\|$ and taking the infimum over $v_r \in V_r$ gives the result. □

3 Regularity

Exercise 6. Assume $x \mapsto A(x)$ and $x \mapsto b(x)$ are differentiable and denote by

$$A'(x) = \begin{pmatrix} A'_{11}(x) & \dots & A'_{1N}(x) \\ \vdots & \ddots & \vdots \\ A'_{N1}(x) & \dots & A'_{NN}(x) \end{pmatrix} \quad \text{and} \quad b'(x) = \begin{pmatrix} b'_1(x) \\ \vdots \\ b'_N(x) \end{pmatrix}$$

their derivatives. Show that $x \mapsto u(x)$ is differentiable and that $u'(x) \in \mathbb{R}^N$ is the solution to

$$A(x)u'(x) = \ell(x),$$

for some right-hand side $\ell(x) \in \mathbb{R}^N$ which depends on $A'(x)$, $b'(x)$ and $u(x)$. (*hint: try to differentiate equation $A(x)u(x) = b(x)$*)

Exercise 7. Let $V_r \subset \mathbb{R}^N$ be a low-dimensional subspace. We denote by $\widetilde{u}_r(x) \in V_r$ the Galerkin projection of $u(x)$ onto V_r defined by

$$v_r^T A(x)\widetilde{u}_r(x) = v_r^T b(x), \tag{7}$$

for any $v_r \in V_r$. Show that the derivative $\widetilde{u}'_r(x) \in V_r$ is such that

$$v_r^T A(x)\widetilde{u}'_r(x) = v_r^T \widetilde{\ell}(x),$$

for any $v_r \in V_r$, where $\widetilde{\ell}(x) \in \mathbb{R}^N$ depends on $A'(x)$, $b'(x)$ and $\widetilde{u}(x)$.

Exercise 8. Assume $A(x)$ is coercive. We recall that Céa's Lemma ensures that $\widetilde{u}_r(x)$ defined by (7) satisfies

$$\|u(x) - \widetilde{u}_r(x)\| \leq C \inf_{v_r \in V_r} \|u(x) - v_r\|,$$

for some constant $C < \infty$. Show that if $u(x_0) \in V_r$ for some $x_0 \in \mathbb{R}$ then $\widetilde{\ell}(x_0) = \ell(x_0)$.

Exercise 9. Assume that $u(x_0) \in V_r$ and $u'(x_0) \in V_r$. Show that $\widetilde{u}_r(x)$ interpolates the derivative of $u(x)$ at x_0. How to define V_r such that $\widetilde{u}_r(x)$ interpolates $u(x)$ and its derivatives at arbitrary points $x_1, \ldots, x_r$?