Hybrid high-order methods for linear and nonlinear parabolic problems

Zhaonan Dong* Alexandre Ern†

HHO methods were introduced in [6] for locking-free linear elasticity and in [7] for linear diffusion. The main advantages of the HHO methods are (i) the support of polytopal meshes (the mesh cells can be polytopal as such or have a simple shape but contain hanging nodes); (ii) the static condensation to reduce the total number of global coupled basis functions. In the past decade, HHO methods have undergone vigorous development for a large class of linear and nonlinear problems. However, several outstanding topics remain, in particular for time-dependent problems and for the a posteriori error analysis. The goal of this project is to explore some of these topics.

1. **Inf-sup stability for the heat equation with HHO space semi-discretization**

   The inf-sup stability plays a key role in the numerical analysis of the linear parabolic PDEs. Inf-sup stability for the heat equation is well established at the continuous level and also with space semi-discretization using continuous finite elements (cFEM) . The goal of this first task is to extend these results to the HHO space semi-discretization, by considering first the time-continuous setting and then the fully discrete setting where time discretization hinges on some A-stable scheme, such as discontinuous Galerkin in time.

2. **A posteriori error analysis for linear parabolic problems**

   In [9], the author considered the a posteriori error analysis of linear parabolic problems, leveraging on the above inf-sup stability theory. The discretization therein hinges on conforming FEM in space and discontinuous Galerkin in time. The goal of this task is to extend the results from [9] to HHO space semi-discretization. To this purpose, the idea is to leverage on recent advances concerning the a posteriori error analysis of HHO methods in the steady case [2, 8].

3. **Extension to nonlinear parabolic problems: phase field models**

   The typical PDE used in the phase field modeling is the Allen–Cahn equation, which is a second-order, singularly perturbed, semilinear parabolic PDE. The singular pertubation parameter ε controls the length scale of the interface separating the two phases. HHO methods for this problem were considered in [4], but leading to an unfavorable dependency of the error bounds on the singular parameter. The goal of this task is to leverage on recent results from [1, 5] to explore their extension to the HHO setting.

4. **Data assimilation problem subject to the heat equation**

   In [3], the authors study the data assimilation problem subject to the heat equation. The main goal is to reconstruct the solution of the heat equation in a target space-time

*Inria, 2 rue Simone Iff, 75589 Paris, France, and CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée, France `zhaonan.dong@inria.fr`.

†CERMICS, Ecole des Ponts, 77455 Marne-la-Vallée, France, and Inria, 2 rue Simone Iff, 75589 Paris, France. `alexandre.ern@enpc.fr`.

subdomain given its (noised) value in a subset of the computational domain with unknown initial (and boundary) datum. This ill-posed problem can be discretized using a space-time discontinuous Galerkin method, including hybrid variables in space, and looking for a solution that minimizes a discrete Lagrangian. Owing to the ill-posed nature of the problem, it is not possible so far to fully exploit the hybrid variables to achieve fully optimal error estimates in terms of mesh size. The goal of this task is to explore whether optimal estimates can be reached if the conditional stability estimate provides Lipschitz continuity of the solution with respect to the data.

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