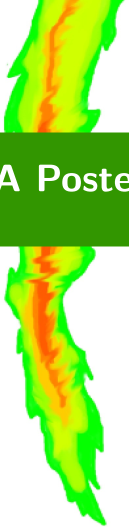
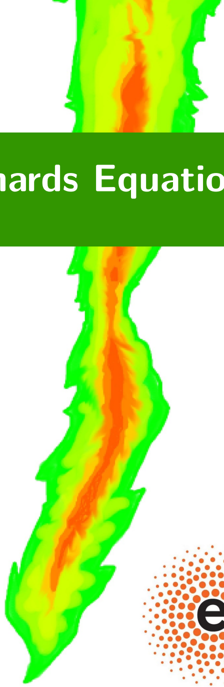
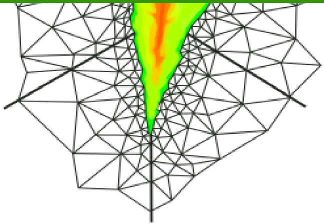




A Posteriori Estimates for the Richards Equation



Koondanibha (**Koondi**) Mitra

work with Prof. Martin Vohralík

INRIA Paris

Dec 2020

1 Outline

Why a posteriori estimates?

For $\partial_t u - \Delta u = f$ and its f.e. solution $u_{h\tau}$ a priori analysis gives:

$$\|(u - u_{h\tau})(T)\|^2 + \int_0^T \|\nabla(u - u_{h\tau})\|^2 \leq C(h, \tau, \|f\|, \|u(0) - u_{h\tau}(0)\|)$$

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A posteriori analysis gives fully computable $C_{\mathrm{M}} \geq C_{\mathrm{m}} > 0$ and $\eta(\cdot,\cdot)$ s.t.:

$$C_{\mathrm{m}}\, \eta(u_{h\tau}, \mathcal{T}) \leq \mathrm{dist}(u - u_{h\tau}) \leq C_{\mathrm{M}}\, \eta(u_{h\tau}, \mathcal{T})$$

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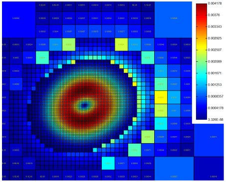
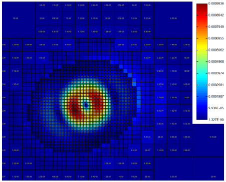
- Distinguish and estimating separately the different error components
- Adapting the numerical parameters based on the estimates



Estimated Error



Actual Error

1 Nonlinear advection-reaction-diffusion equation

Richards equation[1]: modelling flow of water through soil

$$\partial_t S(p) = \nabla \cdot [\bar{\mathbf{K}} \kappa(S(p))(\nabla p + \boldsymbol{g})] + f(S(p), \boldsymbol{x}, t)$$

p is pressure, $s := S(p)$ is saturation

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- the *Darcy Law*[1]

$$\boldsymbol{\sigma} = -\bar{\mathbf{K}} \kappa(s)(\nabla p + \boldsymbol{g}),$$

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- $S \in \mathrm{Lip}(\mathbb{R})$ is increasing in $(-\infty, p_{\mathrm{M}})$, $S(-\infty) = 0$ and

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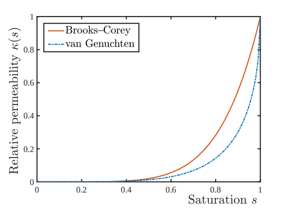

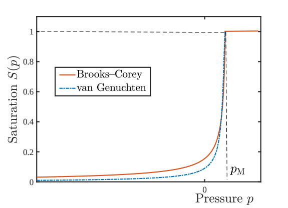

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2. Degeneracy 

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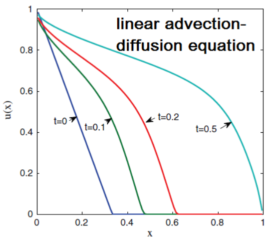

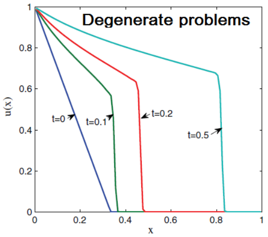

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Literature: [Dolejší *et al* (2013)][Bernardi *et al* (2014)][Cancès *et al* (2014)] [Verfürth (2004)]; [Di Pietro *et al* (2015)]; [Ohlberger (2001)]

1 Alternative formulations

Pressure formulation

$$\partial_t S(p) = \nabla \cdot [\bar{\mathbf{K}} \kappa(S(p))(\nabla p + \boldsymbol{g})] + f(S(p), \boldsymbol{x}, t)$$

The Kirchhoff transform and some definitions

$$\mathcal{K}(p) = \int_0^p \kappa(S(\varrho))\,\mathrm{d}\varrho, \quad \theta = S \circ \mathcal{K}^{-1}, \quad D = 1/\theta'$$

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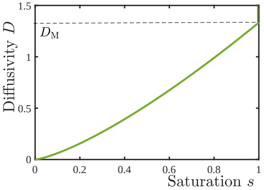

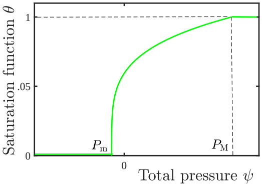

Observe that D is a graph

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Total pressure formulation

For $\Psi = \mathcal{K}(p)$,

$$\partial_t \theta(\Psi) = \nabla \cdot [\bar{\mathbf{K}}(\nabla \Psi + \kappa(\theta(\Psi))\boldsymbol{g})] + f(\theta(\Psi), \boldsymbol{x}, t)$$

1 Well-posedness

Weak total pressure formulation

For the initial condition s_0 bounded in $(0,1]$ a.e., find $\Psi \in L^2(0,T;H_0^1(\Omega))$, $s = \theta(\Psi) \in H^1(0,T;H^{-1}(\Omega))$, $s(0) = s_0$ satisying $\forall\, \varphi \in L^2(0,T;H_0^1(\Omega))$,

$$\int_0^T [\langle \partial_t s, \varphi \rangle + (\bar{\mathbf{K}}[\nabla\Psi + \kappa(s)\boldsymbol{g}], \nabla\varphi)] = \int_0^T (f(s,\boldsymbol{x},t),\varphi)$$

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Theorem [Alt & Luckhaus (1983)][Otto (1991)]

There exists a unique weak solution Ψ for the total pressure formulations.

1 Maximum principle

To avoid the parabolic-hyperbolic degeneracy we need $s \geq S_{\mathrm{m}} > 0$

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Proposition

If s_0 is bounded in $[\varepsilon, 1]$ for some $\varepsilon > 0$, then there exists *saturation lower-bound function* $S_{\mathrm{m}} : [0, T] \to (0, 1]$ such that for almost all $(\boldsymbol{x}, t) \in \Omega \times [0, T]$,

$$s(\boldsymbol{x}, t) = S(p(\boldsymbol{x}, t)) \geq S_{\mathrm{m}}(t) > 0.$$

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$$s(\boldsymbol{x}, t) = S(p(\boldsymbol{x}, t)) \geq S_{\mathrm{m}}(t) > 0.$$

Computing S_{m}

For example, under minor restrictions

$$S_{\mathrm{m}}(t) = \min_{\boldsymbol{x} \in \Omega}\{s_0(\boldsymbol{x})\} + \int_0^t \min_{\boldsymbol{x} \in \Omega, \varrho > 0}\{f(S_{\mathrm{m}}(\varrho), \boldsymbol{x}, \varrho)\} \, \mathrm{d}\varrho$$

is a saturation lower-bound function.

2 Outline

2 Residual and dual norms

Residual

For $\Psi_{h\tau} \in L^2(0,T;H_0^1(\Omega))$, $s_{h\tau} = \theta(\Psi_{h\tau}) \in H^1(0,T;H^{-1}(\Omega))$ the residual $\mathcal{R}(\Psi_{h\tau}) \in L^2(0,T;H^{-1}(\Omega))$ is

$$\int_0^T \langle \mathcal{R}(\Psi_{h\tau}), \varphi \rangle = \int_0^T [(f(s_{h\tau}, \boldsymbol{x}, t), \varphi) - \langle \partial_t s_{h\tau}, \varphi \rangle - (\bar{\mathbf{K}}[\nabla \Psi_{h\tau} + \kappa(s_{h\tau})\boldsymbol{g}], \nabla \varphi)]$$

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The $H_{\bar{\mathbf{K}}}^{-1}$ norm

For $\omega \subseteq \Omega$,

$$\|\varrho\|_{H_{\bar{\mathbf{K}}}^{-1}(\omega)} := \sup_{\varphi \in H_0^1(\omega)} \frac{\langle \varrho, \varphi\rangle_{H^{-1}(\omega), H_0^1(\omega),}}{\|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla\varphi\|_{L^2(\omega)}}.$$

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The $H^{-1}_{\bar{\mathbf{K}}}$ norm

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Observe that,

$$K_{\mathrm{m}} \|\varrho\|_{H^{-1}_{\bar{\mathbf{K}}}(\omega)} \leq \|\varrho\|_{H^{-1}(\omega)} \leq K_{\mathrm{M}} \|\varrho\|_{H^{-1}_{\bar{\mathbf{K}}}(\omega)}$$

2 Lower bound on error by residual

Theorem 1

For a time-interval $I \in [0, T]$, $\omega \subseteq \Omega$,

$$\|\mathcal{R}(\Psi_{h\tau})\|_{L^2(I;H^{-1}_{\bar{\mathbf{K}}}(\omega))} \leq \|\partial_t(s - s_{h\tau})\|_{L^2(I;H^{-1}_{\bar{\mathbf{K}}}(\omega))} + \|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi - \Psi_{h\tau})\|_{L^2(\omega\times I)}$$

$$+\|\bar{\mathbf{K}}^{\frac{1}{2}}\boldsymbol{g}(\kappa(s) - \kappa(s_{h\tau}))\|_{L^2(\omega\times I)} + \|f(s,\cdot,\cdot) - f(s_{h\tau},\cdot,\cdot)\|_{L^2(I;H^{-1}_{\bar{\mathbf{K}}}(\omega))}$$

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proof: Use triangle inequality for the norm $\|\cdot\|_{L^2(I;H^{-1}_{\bar{\mathbf{K}}}(\omega))}$

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Assumptions on numerical solution

- $s_{h\tau} \geq \epsilon > 0$ a.e.
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Other definitions

Degeneracy estimator for parabolic-elliptic degeneracy

$$\eta^{\mathrm{deg}}(t) := D(1)^{-\frac{1}{2}} \|[f(1, \boldsymbol{x}, t)]_+\|_{H^{-1}_{\bar{\mathbf{K}}}(\{s=1\})}.$$

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- The estimator is evaluated only in a sub-domain $\{s = 1\}$

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Degeneracy estimator for parabolic-elliptic degeneracy

$$\eta^{\mathrm{deg}}(t) := D(1)^{-\frac{1}{2}} \|[f(1, \boldsymbol{x}, t)]_+\|_{H^{-1}_{\bar{\mathbf{K}}}(\{s=1\})}.$$

- The estimator is evaluated only in a sub-domain $\{s = 1\}$
- The estimator vanishes if $f(1, \boldsymbol{x}, t) \leq 0$ or $D(1) \to \infty$

Theorem 2 (a)

Estimate in the $\mathbf{H}^1(0,\mathbf{T};\mathbf{H}^{-1}(\Omega))$ norm:

$$\|\partial_t(s - s_{h\tau})\|_{L^2(0,T;H^{-1}_{\bar{\mathbf{K}}}(\Omega))} \leq \|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi - \Psi_{h\tau})\|_{L^2(\Omega\times(0,T))}$$
$$+\mathfrak{C}_1(t)\|s - s_{h\tau}\|_{L^2(\Omega\times(0,T))} + \|\mathcal{R}(\Psi_{h\tau})\|_{L^2(0,T;H^{-1}_{\bar{\mathbf{K}}}(\Omega))}.$$

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Estimate in the $\mathbf{L}^2(\Omega\times[0,\mathbf{T}])$ and $\mathbf{L}^\infty(0,\mathbf{T};\mathbf{H}^{-1}(\Omega))$ norm:

$$\|(s-s_{h\tau})(T)\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}+$$
$$\int_0^T\left[\frac{\|(s-s_{h\tau})(t)\|^2}{\theta_{\partial,\mathrm{M}}(t)}+(1+\mathfrak{C}_2(t))\,e^{\int_t^T(1+\mathfrak{C}_2)}\int_0^t\frac{\|(s-s_{h\tau})\|^2}{\theta_{\partial,\mathrm{M}}}\right]\mathrm{d}t$$
$$\le\left(1+\int_0^T(1+\mathfrak{C}_2(t)\,e^{\int_t^T(1+\mathfrak{C}_2)}\right)\|s_0-s_{h\tau}(0)\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}$$
$$+\int_0^T\left[\|\mathcal{R}(\Psi_{h\tau}(t))\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}+(1+\mathfrak{C}_2(t))\,e^{\int_t^T(1+\mathfrak{C}_2)}\int_0^t\|\mathcal{R}(\Psi_{h\tau})\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}\right]\mathrm{d}t.$$

proof uses $G\in H^1_0(\Omega)$ as test function, where $(\bar{\mathbf{K}}\nabla G,\nabla\varphi)=((s-s_{h\tau}),\varphi)$

Theorem 2 (b)

Estimate in the $\mathbf{L^2(0,T;H_0^1(\Omega))}$ and $\mathbf{L^\infty(0,T;L^2(\Omega))}$ norm:

$$\|(s-s_{h\tau})(T)\|^2 + \int_0^T \left[\int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi-\Psi_{h\tau})(t)|^2}{2D(\theta(\Psi(t)))} + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t \int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi-\Psi_{h\tau})|^2}{2D(\theta(\Psi))}\right] \mathrm{d}t$$

$$\leq e^{\int_t^T \mathfrak{C}_3} \|s_0 - s_{h\tau}(0)\|^2 + \int_0^T \left[[\eta^{\deg}(t)]^2 + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t [\eta^{\deg}]^2\right] \mathrm{d}t$$

$$+\int_0^T \left[\frac{4\|\mathcal{R}(\Psi_{h\tau}(t))\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}}{D_{\mathrm{m}}(t)} + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t \frac{4\|\mathcal{R}(\Psi_{h\tau})\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}}{D_{\mathrm{m}}}\right] \mathrm{d}t.$$

proof uses $s - s_{h\tau} \in L^2(0,T;H_0^1(\Omega))$ as test function

Theorem 2 (b)

Estimate in the $\mathbf{L}^2(\mathbf{0},\mathbf{T};\mathbf{H}_0^1(\Omega))$ and $\mathbf{L}^\infty(\mathbf{0},\mathbf{T};\mathbf{L}^2(\Omega))$ norm:

$$\|(s-s_{h\tau})(T)\|^2 + \int_0^T \left[\int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi-\Psi_{h\tau})(t)|^2}{2D(\theta(\Psi(t)))} + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t \int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi-\Psi_{h\tau})|^2}{2D(\theta(\Psi))}\right] \mathrm{d}t$$

$$\leq e^{\int_t^T \mathfrak{C}_3} \|s_0 - s_{h\tau}(0)\|^2 + \int_0^T \left[[\eta^{\mathrm{deg}}(t)]^2 + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t [\eta^{\mathrm{deg}}]^2\right] \mathrm{d}t$$

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The constants $\mathfrak{C}_j$ are calculated from D, f, θ, $C^\infty_{h\tau}$,S_{m}, ϵ. If the problem is linear then $\mathfrak{C}_j = 0$ making the exponential terms vanish (the heat equation case).

proof uses $s - s_{h\tau} \in L^2(0, T; H_0^1(\Omega))$ as test function

3 Outline

3 Finite element solution

- Let $\{t_0 := 0, t_1, \ldots, t_N := T\}$ be the time-discretization, with $\tau_n := t_n - t_{n-1}$, and $I_n := (t_{n-1}, t_n]$

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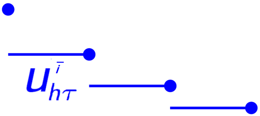
- Let $\Psi_{h\tau}^{\bar{i}}, s_{h\tau}^{\bar{i}} \in L^2(0, T; H_0^1(\Omega))$ be the time-discrete interpolations of $\Psi_{h\tau}^{\bar{i}}, s_{h\tau}^{\bar{i}}$:

$$\Psi_{h\tau}^{\bar{i}}|_{I_n} := \Psi_{n,h}^{i_n}, \quad s_{h\tau}^{\bar{i}}|_{I_n} := \theta(\Psi_{n,h}^{i_n})$$



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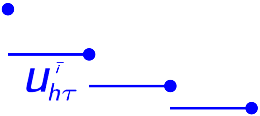
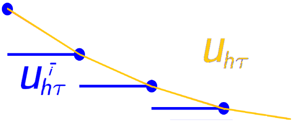
- Let $\Psi_{h\tau}, s_{h\tau} \in C(0,T;L^2(\Omega))$ be their time-continuous interpolations:





Error measure

- For $\omega \subseteq \Omega$, $n \in \{1, \ldots, N\}$,

$$\mathrm{dist}^{\mathcal{E}_N}_{\alpha,n,\omega}(\Psi^{\bar{i}}_{h\tau}, \Psi)^2 := \int_{I_n} \|\partial_t (s_{h\tau} - s)\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\omega)} + \int_{I_n} \|\bar{\mathbf{K}}^{\frac{1}{2}} \nabla (\Psi^{\bar{i}}_{h\tau} - \Psi)\|^2_{L^2(\omega)} + \int_{I_n} \alpha(t) \, \|s_{h\tau} - s\|_{L^2(\omega)}^2$$

3 Error measure and equilibrated flux

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- The global error $\mathrm{dist}^{\mathcal{E}_N}_{\alpha}(\Psi^{\bar{i}}_{h\tau}, \Psi)$ is similarly defined

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Equilibrated flux

The equilibrated flux $\boldsymbol{\sigma}_{n,h} \in \boldsymbol{H}(\mathrm{div}, \Omega)$ such that $\forall K \in \mathcal{T}_n$,

$$\int_K [\partial_t s_{h\tau} + \nabla \cdot \boldsymbol{\sigma}_{n,h}] = \int_K f(s^{\bar{i}}_{h\tau}, \boldsymbol{x}, t) + \text{ linearisation}$$

3 A posteriori estimator

- For the numerical flux $\boldsymbol{F}_{n,h}$, the flux non-conformity estimator is

$$\eta^{\mathrm{F}}_{n,h,K} := \|\bar{\mathbf{K}}^{\frac{1}{2}}(\bar{\mathbf{K}}^{-1}\boldsymbol{\sigma}_{n,h} + \boldsymbol{F}_{n,h})\|_K,$$

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- The time non-conformity estimators are

$$\eta_{n,h,K}^{\mathrm{J},H^1}(t) := \|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi_{h\tau} - \Psi_{h\tau}^{\bar{i}})\|_K, \quad \eta_{n,h,K}^{\mathrm{J},L^2}(t) := \|s_{h\tau} - s_{h\tau}^{\bar{i}}\|_K,$$

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$$\eta^{\mathrm{J},H^1}_{n,h,K}(t) := \|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi_{h\tau} - \Psi^{\bar{i}}_{h\tau})\|_K, \quad \eta^{\mathrm{J},L^2}_{n,h,K}(t) := \|s_{h\tau} - s^{\bar{i}}_{h\tau}\|_K,$$

- Let the data oscillation error be

$$\eta^{\mathrm{osc}}_{n,\omega}(t) := \|f(s^{\bar{i}}_{h\tau}, \boldsymbol{x}, t_n) - f(s^{\bar{i}}_{h\tau}, \boldsymbol{x}, t)\|_{H^{-1}_{\bar{\mathbf{K}}}(\omega)}.$$

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- Let $\eta^{\mathrm{qd},\mathcal{S}}_{n,h,K}$ and $\eta^{\mathrm{qd},\boldsymbol{F}}_{n,h,K}$ be quadrature errors for numerical source and flux terms

3 A posteriori estimator

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- Let the data oscillation error be

$$\eta^{\mathrm{osc}}_{n,\omega}(t) := \|f(s^{\bar{i}}_{h\tau}, \boldsymbol{x}, t_n) - f(s^{\bar{i}}_{h\tau}, \boldsymbol{x}, t)\|_{H^{-1}_{\bar{\mathbf{K}}}(\omega)}.$$

- Let $\eta^{\mathrm{qd},\mathcal{S}}_{n,h,K}$ and $\eta^{\mathrm{qd},\boldsymbol{F}}_{n,h,K}$ be quadrature errors for numerical source and flux terms
- Let $\eta^{\mathrm{lin},1}_{n,\omega}$ and $\eta^{\mathrm{lin},2}_{n,\omega}$ be linearisation errors for numerical source and flux terms

Lemma 1

For

$$\eta_{\mathcal{R}}(t) := C_{P,\Omega}\, h_\Omega\, \|\partial_s f\|_{L^\infty} \left(\sum_{K\in\mathcal{T}_n} [\eta^{\mathrm{J},L^2}_{n,h,K}(t)]^2 \right)^{\frac{1}{2}} + \left[\eta^{\mathrm{osc}}_{n,\Omega}(t) + \eta^{\mathrm{lin},1}_{n,\Omega} \right]$$

$$+ \left[\sum_{K\in\mathcal{T}_n} [\eta^{\mathrm{F}}_{n,h,K} + \eta^{\mathrm{J},H^1}_{n,h,K}(t) + K_{\mathrm{M}}^{\frac{1}{2}} |\boldsymbol{g}| \|\kappa'\|_{L^\infty}\, \eta^{\mathrm{J},L^2}_{n,h,K}(t) \right.$$

$$\left. + \eta^{\mathrm{qd},\mathcal{S}}_{n,h,K} + \eta^{\mathrm{qd},\boldsymbol{F}}_{n,h,K} + \eta^{\mathrm{lin},2}_{n,\Omega}]^2 \right]^{\frac{1}{2}}$$

3 Global reliability

Lemma 1

For

$$\eta_{\mathcal{R}}(t) := C_{P,\Omega}\, h_\Omega \, \|\partial_s f\|_{L^\infty} \left(\sum_{K\in\mathcal{T}_n} [\eta^{\mathrm{J},L^2}_{n,h,K}(t)]^2 \right)^{\frac{1}{2}} + \left[\eta^{\mathrm{osc}}_{n,\Omega}(t) + \eta^{\mathrm{lin},1}_{n,\Omega}\right]$$

$$+ \left[\sum_{K\in\mathcal{T}_n} [\eta^{\mathrm{F}}_{n,h,K} + \eta^{\mathrm{J},H^1}_{n,h,K}(t) + K_{\mathrm{M}}^{\frac{1}{2}} |\boldsymbol{g}| \|\kappa'\|_{L^\infty}\, \eta^{\mathrm{J},L^2}_{n,h,K}(t) \right.$$

$$\left. + \eta^{\mathrm{qd},\mathcal{S}}_{n,h,K} + \eta^{\mathrm{qd},\boldsymbol{F}}_{n,h,K} + \eta^{\mathrm{lin},2}_{n,\Omega}]^2 \right]^{\frac{1}{2}}$$

we have

$$\|\mathcal{R}(\Psi_{h\tau}(t))\|_{H^{-1}_{\bar{\kappa}}(\Omega)} \le \eta_{\mathcal{R}}(t)$$

Theorem 3 (a)

Estimate in the $\mathbf{L}^2(\boldsymbol{\Omega} \times [\mathbf{0}, \mathbf{T}])$ and $\mathbf{L}^\infty(\mathbf{0}, \mathbf{T}; \mathbf{H}^{-1}(\boldsymbol{\Omega}))$ norm:

$$\|(s - s_{h\tau})(T)\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)} +$$

$$\int_0^T \left[\frac{\|(s-s_{h\tau})(t)\|^2}{\theta_{\partial,\mathrm{M}}(t)} + (1 + \mathfrak{C}_2(t))\, e^{\int_t^T (1+\mathfrak{C}_2)} \int_0^t \frac{\|(s-s_{h\tau})\|^2}{\theta_{\partial,\mathrm{M}}} \right] \mathrm{d}t$$

$$\leq (1 + \int_0^T (1 + \mathfrak{C}_2(t)\, e^{\int_t^T (1+\mathfrak{C}_2)}) \|s_0 - s_{h\tau}(0)\|^2_{H^{-1}_{\bar{\mathbf{K}}}(\Omega)}$$

$$+ \int_0^T \left[[\eta_{\mathcal{R}}(t)]^2 + (1 + \mathfrak{C}_2(t))\, e^{\int_t^T (1+\mathfrak{C}_2)} \int_0^t [\eta_{\mathcal{R}}]^2 \right] \mathrm{d}t.$$

Theorem 3 (b)

Estimate in the $\mathbf{L}^2(0,T;\mathbf{H}_0^1(\Omega))$ and $\mathbf{L}^\infty(0,T;\mathbf{L}^2(\Omega))$ norm:

$$\|(s - s_{h\tau})(T)\|^2 + \int_0^T \left[\int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi - \Psi_{h\tau})(t)|^2}{2D(\theta(\Psi(t)))} + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t \int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi - \Psi_{h\tau})|^2}{2D(\theta(\Psi))}\right] \mathrm{d}t$$

$$\leq e^{\int_t^T \mathfrak{C}_3} \|s_0 - s_{h\tau}(0)\|^2 + \int_0^T \left[[\eta^{\mathrm{deg}}(t)]^2 + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t [\eta^{\mathrm{deg}}]^2\right] \mathrm{d}t$$

$$+ \int_0^T \left[\frac{4[\eta_{\mathcal{R}}(t)]^2}{D_{\mathrm{m}}(t)} + \mathfrak{C}_3(t)\, e^{\int_t^T \mathfrak{C}_3} \int_0^t \frac{4[\eta_{\mathcal{R}}]^2}{D_{\mathrm{m}}}\right] \mathrm{d}t.$$

3 Local space-time efficiency

Theorem 4

For $n \in \{1, \dots, N\}$, $K \in \mathcal{T}_n$ and a patch ω_K covering K,

$$[\eta^{\mathrm{F}}_{n,h,K}]^2 \lesssim [\eta^{\mathrm{qd},\mathcal{S}}_{n,h,\omega_K}]^2 + [\eta^{\mathrm{qd},\boldsymbol{F}}_{n,h,\omega_K}]^2 + \sum_{j=1,2} [\eta^{\mathrm{lin},j}_{n,\omega_K}]^2 + \frac{1}{\tau_n} \int_{I_n} [\eta^{\mathrm{osc}}_{n,\omega_K}(t)]^2$$

$$+ \mathrm{dist}^{\mathcal{E}_N}_{\alpha,n,\omega_K}(\Psi^{\bar{i}}_{h\tau}, \Psi).$$

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$$+ \mathrm{dist}^{\mathcal{E}_N}_{\alpha,n,\omega_K}(\Psi^{\bar{i}}_{h\tau}, \Psi).$$

Similar estimates hold for the global error $\mathrm{dist}^{\mathcal{E}_N}_{\alpha}(\Psi^{\bar{i}}_{h\tau}, \Psi)$.

4 Outline

4 Test 1: non-degenerate case

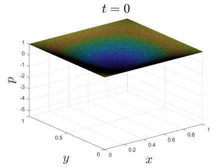
Solution

$p_{\text{exact}}(x, y, t) = 2 - e^{16(1+t^2)xy(1-x)(1-y)}$

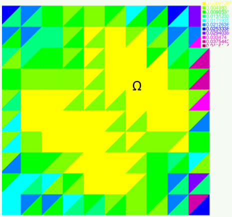
$k(s) = s^3,\ S(p) = \frac{1}{(2-p)^{\frac{1}{3}}}$ (Brooks-Corey type)

$\bar{\boldsymbol{K}} = \mathbb{I},\ \boldsymbol{g} = -\boldsymbol{e}_x$

$f(x, y, t)$ set accordingly



Domain and discretization



Domain: $\Omega = (0, 1)^2$
Discretization: $h = h_0/\ell,\ \tau = \tau_0/\ell$
$h_0 = 0.2,\ \tau_0 = 0.04,\ \ell \in \{1, 2, 4\}$,

4 Reliability of H^1-estimates

Error measures

- $\mathrm{Err}_{H^1}(\Psi, \Psi_{h\tau}^{\bar{i}})^2 :=$
$$e^{-\int_0^T \mathfrak{C}_3} \int_0^T \left[\int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}} \nabla(\Psi - \Psi_{h\tau}^{\bar{i}})(t)|^2}{2D(\theta(\Psi_{h\tau}^{\bar{i}}(t)))} + \mathfrak{C}_3(t) e^{\int_t^T \mathfrak{C}_3} \int_0^t \int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}} \nabla(\Psi - \Psi_{h\tau}^{\bar{i}})|^2}{2D(\theta(\Psi_{h\tau}^{\bar{i}}))} \right]$$

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- Effectivity Index= Estimator $/\mathrm{Err}_{H^1}(\Psi, \Psi_{h\tau}^{\bar{i}})$

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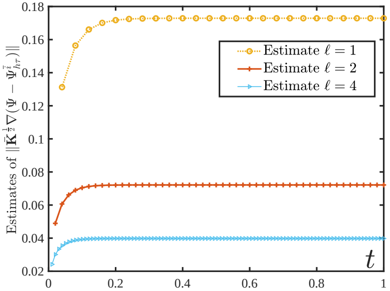
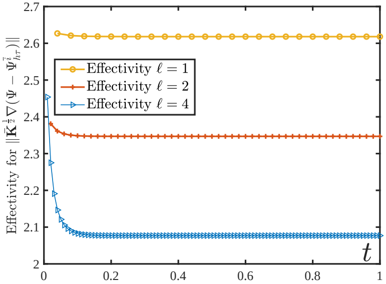
Error measures

- $\mathrm{Err}_{H^1}(\Psi, \Psi^{\bar{i}}_{h\tau})^2 :=$
$$e^{-\int_0^T \mathfrak{C}_3} \int_0^T \left[\int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}} \nabla (\Psi - \Psi^{\bar{i}}_{h\tau})(t)|^2}{2D(\theta(\Psi^{\bar{i}}_{h\tau}(t)))} + \mathfrak{C}_3(t) e^{\int_t^T \mathfrak{C}_3} \int_0^t \int_\Omega \frac{|\bar{\mathbf{K}}^{\frac{1}{2}} \nabla (\Psi - \Psi^{\bar{i}}_{h\tau})|^2}{2D(\theta(\Psi^{\bar{i}}_{h\tau}))} \right]$$
- Effectivity Index= Estimator $/\mathrm{Err}_{H^1}(\Psi, \Psi^{\bar{i}}_{h\tau})$





4 Reliability of L^2-estimates

Error measures

- $\mathrm{Err}_{L^2}(s, s_{h\tau}^{\bar{i}})$ is defined similar to Err_{H^1}
- Effectivity Index= Estimator $/\mathrm{Err}_{L^2}(s, s_{h\tau}^{\bar{i}})$

Error measures

- $\mathrm{Err}_{L^2}(s, s^{\bar{i}}_{h\tau})$ is defined similar to Err_{H^1}
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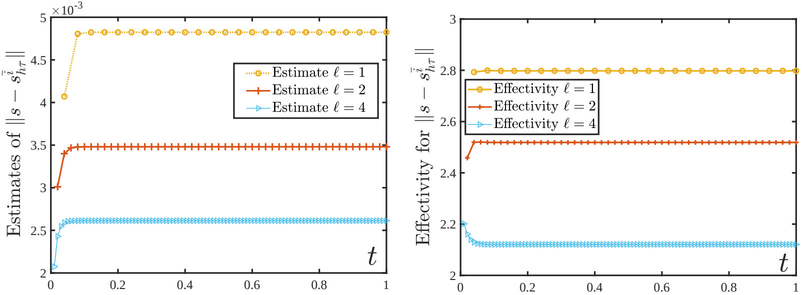
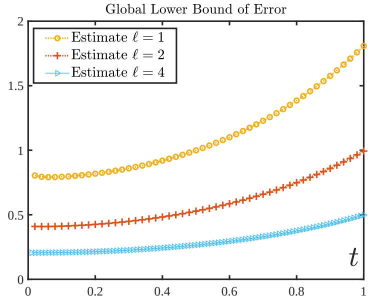
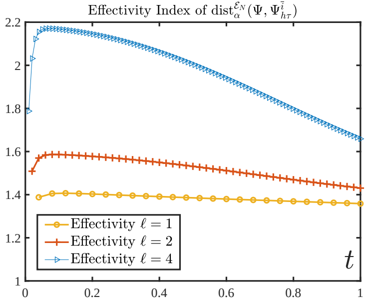


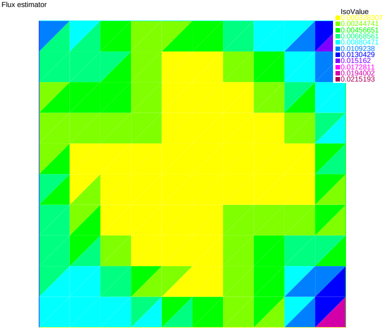
Efficiency measure

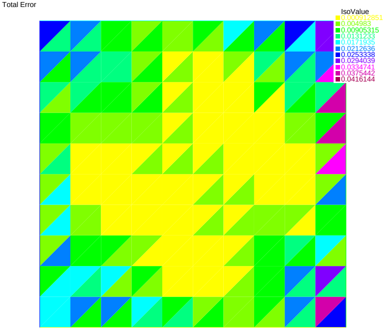
Effectivity Index= $\mathrm{dist}_{\alpha}^{\mathcal{E}_N}(\Psi, \Psi_{h\tau}^{\bar{i}})$/ Estimator



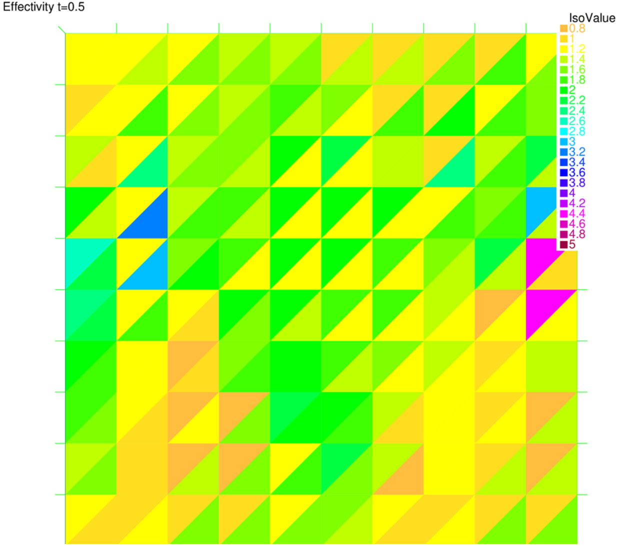


4 Local efficiency





4 Local efficiency



4 Test 2: Degenerate case

Solution

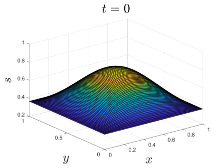
$$\Psi_{\text{exact}}(x, y, t) = 12(1 + t^2)xy(1 - x)(1 - y)$$

$$\theta(\Psi) = \begin{cases} \exp(\Psi - 1) & \text{if } \Psi < 1 \\ 1 & \text{if } \Psi \geq 1 \end{cases}$$

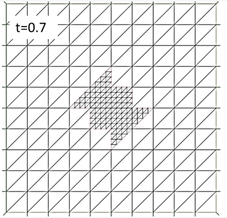
$$k(s) = \begin{cases} s & \text{if } s < 1 \\ 1 & \text{if } s \geq 1 \end{cases}$$

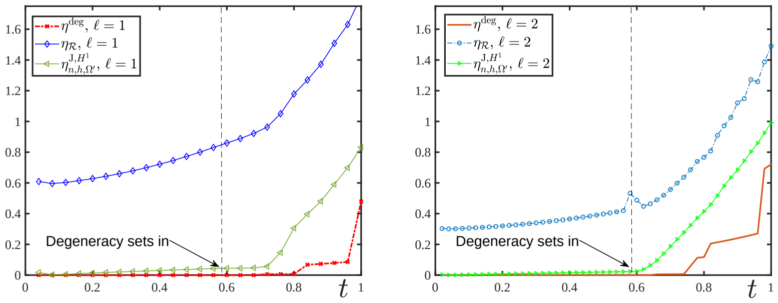
$\bar{\boldsymbol{K}} = \mathbb{I}$, $\boldsymbol{g} = -\boldsymbol{e}_x$

$f(x, y, t)$ set accordingly



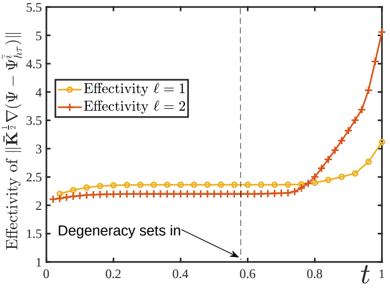
Degenerate domains



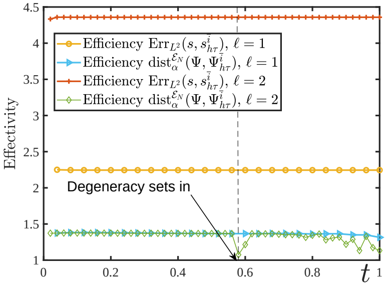


η^{deg}, $\ell = 1$
$\eta_{\mathcal{R}}$, $\ell = 1$
$\eta^{\mathrm{J},H^1}_{n,h,\Omega'}$, $\ell = 1$
Degeneracy sets in
t
η^{deg}, $\ell = 2$
$\eta_{\mathcal{R}}$, $\ell = 2$
$\eta^{\mathrm{J},H^1}_{n,h,\Omega'}$, $\ell = 2$
Degeneracy sets in
t

4 Estimates



Effectivity of $\|\bar{\mathbf{K}}^{\frac{1}{2}}\nabla(\Psi - \Psi^{\bar{i}}_{h\tau})\|$
Effectivity $\ell = 1$
Effectivity $\ell = 2$
Degeneracy sets in
t



Effectivity
Efficiency $\mathrm{Err}_{L^2}(s, s^{\bar{i}}_{h\tau})$, $\ell = 1$
Efficiency $\mathrm{dist}^{\mathcal{E}_N}_{\alpha}(\Psi, \Psi^{\bar{i}}_{h\tau})$, $\ell = 1$
Efficiency $\mathrm{Err}_{L^2}(s, s^{\bar{i}}_{h\tau})$, $\ell = 2$
Efficiency $\mathrm{dist}^{\mathcal{E}_N}_{\alpha}(\Psi, \Psi^{\bar{i}}_{h\tau})$, $\ell = 2$
Degeneracy sets in
t

4 Outlook

- **Adaptive linearisation**: Linearisation would be stopped when $\eta_{n,\Omega}^{\mathrm{lin},j} \ll \eta_{n,h,\Omega}^{\mathrm{F}}$ **(to be implemented)**

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- Derive a posteriori estimates for multiphase problems

4 Outlook

- **Adaptive linearisation**: Linearisation would be stopped when $\eta_{n,\Omega}^{\mathrm{lin},j} \ll \eta_{n,h,\Omega}^{\mathrm{F}}$ **(to be implemented)**
- Derive a posteriori estimates for multiphase problems
- Include heterogeneous media (changing definitions of κ and S)

4 Thank you for your time