Axioms of Adaptivity (AoA) in Lecture 3 (sufficient for optimal convergence rates)

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mini-course at inria as prof. invité UPE

Contents L3: Proofs in PMP

Poisson model problem (PMP) in 2D
lowest-order conforming FEM w.r.t. triangles in shape regular triangulations lead to $\mathcal{R} = \mathcal{T} \setminus \widehat{\mathcal{T}}$, $\widehat{\Lambda_3} = 0$

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Jump Control

Λ_1 Comes from Discrete Jump Control

Given $g \in P_k(\mathcal{T})$ for $\mathcal{T} \in \mathbb{T}$, set

$$[g]_E = \begin{cases} (g|_{T_+})|_E - (g|_{T_-})|_E & \text{for } E \in \mathcal{E}(\Omega) \text{ with } E = \partial T_+ \cap \partial T_-, \\ g|_E & \text{for } E \in \mathcal{E}(\partial\Omega) \cap \mathcal{E}(K). \end{cases}$$

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Lemma (discrete jump control)

For all $k \in \mathbb{N}_0$ there exists $0 < \Lambda_1 < \infty$ s.t., for all $g \in P_k(\mathcal{T})$ and $\mathcal{T} \in \mathbb{T}$,

$$\sqrt{\sum_{K \in \mathcal{T}} |K|^{1/2} \sum_{E \in \mathcal{E}(K)} ||[g]_E||^2_{L^2(E)}} \le \Lambda_1 ||g||_{L^2(\Omega)}.$$

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Proof with discrete trace inequality on $E \in \mathcal{E}(K)$ for $K \in \mathcal{T}$

$$|K|^{1/4}\, ||g|_K||_{L^2(E)} \leq C_{\mathsf{dtr}}\, ||g||_{L^2(K)}.$$

Compute Λ_1 in Proof of Discrete Jump Control

The contributions to LHS of interior edge $E = \partial T_+ \cap \partial T_-$ with edge-patch $\omega_E := \mathrm{int}(T_+ \cup T_-)$ read

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\begin{aligned}
&(|T_+|^{1/2} + |T_-|^{1/2}) ||[g]_E||^2_{L^2(E)} \\
&\leq (|T_+|^{1/2} + |T_-|^{1/2}) \left(||g|_{T_+}||_{L^2(E)} + ||g|_{T_-}||_{L^2(E)} \right)^2 \\
&\leq C^2_{\mathsf{dtr}} (|T_+|^{1/2} + |T_-|^{1/2}) \left(|T_+|^{-1/4} ||g||_{L^2(T_+)} + |T_-|^{-1/4} ||g||_{L^2(T_-)} \right)^2 \\
&\leq C^2_{\mathsf{dtr}} \underbrace{(|T_+|^{1/2} + |T_-|^{1/2})(|T_+|^{-1/2} + |T_-|^{-1/2})}_{\leq C^2_{\mathsf{sr}}} ||g||^2_{L^2(\omega_E)} \\
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The same final result holds for boundary edge $E = \partial T_+ \cap \partial \Omega$ with $\omega_E := \text{int}(T_+)$. The sum of all those edges proves the discrete jump control lemma with

$$\Lambda_1 := \sqrt{3}\, C_{\mathsf{dtr}} C_{\mathsf{sr}}. \quad \square$$

Proof of (A1)

Proof of (A1) with $\delta(\mathcal{T},\hat{\mathcal{T}}) = ||\widehat{P} - P||_{L^2(\Omega)}$

Recall that $\hat{\mathcal{T}}$ is an admissible refinement of $\mathcal{T}$ with respective discrete flux approximations $\widehat{P} := \nabla \widehat{u}_h \in P_0(\hat{\mathcal{T}};\mathbb{R}^2)$ and $P := \nabla u_h \in P_0(\mathcal{T};\mathbb{R}^2)$.

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Given any $T \in \mathcal{T} \cap \hat{\mathcal{T}}$, set

$$\eta(T) := \sqrt{\alpha_T^2 + \beta_T^2} \quad \text{and} \quad \widehat{\eta}(T) := \sqrt{\alpha_T^2 + \widehat{\beta_T}^2}$$

for $\alpha_T := |T|^{1/2}\,||f||_{L^2(T)}$ and

$$\beta_T^2 := |T|^{1/2} \sum_{E\in\mathcal{E}(T)} ||[P]_E||^2_{L^2(E)} \quad \text{resp.} \quad \widehat{\beta_T}^2 := |T|^{1/2} \sum_{E\in\mathcal{E}(T)} ||[\widehat{P}]_E||^2_{L^2(E)}$$

Convention for conforming P_1 FEM: Jumps on boundary edges vanish.

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Convention for conforming P_1 FEM: Jumps on boundary edges vanish.

Then, $\eta(\mathcal{T}\cap\hat{\mathcal{T}}) := \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} \eta^2(T)}$ and $\widehat{\eta}(\mathcal{T}\cap\hat{\mathcal{T}}) := \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} \widehat{\eta^2}(T)}$ are Euclid norms of vectors in $\mathbb{R}^J$ for $J := 2\,|\mathcal{T}\cap\hat{\mathcal{T}}|$.

Proof of (A1) with $\delta(\mathcal{T},\hat{\mathcal{T}}) = ||\widehat{P} - P||_{L^2(\Omega)}$

The reversed triangle inequality in $\mathbb{R}^J$ bounds the LHS in (A1), namely $|\eta(\hat{\mathcal{T}},\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T},\mathcal{T}\cap\hat{\mathcal{T}})| = |\widehat{\eta}(\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T}\cap\hat{\mathcal{T}})|$, from above by

$$\sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} |\widehat{\eta}(T) - \eta(T)|^2} = \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} \underbrace{\left|\sqrt{\alpha_T^2 + \widehat{\beta_T}^2} - \sqrt{\alpha_T^2 + \beta_T^2}\right|^2}_{\le|\widehat{\beta_T}-\beta_T|^2 \text{ (triangle ineq. in } \mathbb{R}^2)}}$$

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The reversed triangle inequality in $\mathbb{R}^3$ and $L^2(E)$ show

$$|\widehat{\beta_T} - \beta_T| = |T|^{1/4}\left|\sqrt{\sum_{E\in\mathcal{E}(T)} ||[\widehat{P}]_E||^2_{L^2(E)}} - \sqrt{\sum_{E\in\mathcal{E}(T)} ||[P]_E||^2_{L^2(E)}}\right|$$

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$$\leq |T|^{1/4}\sqrt{\sum_{E\in\mathcal{E}(T)} ||[\widehat{P}-P]_E||^2_{L^2(E)}}.$$ Altogether,

$$|\widehat{\eta}(\mathcal{T}\cap\hat{\mathcal{T}}) - \eta(\mathcal{T}\cap\hat{\mathcal{T}})| \leq \sqrt{\sum_{T\in\mathcal{T}\cap\hat{\mathcal{T}}} |T|^{1/2} \sum_{E\in\mathcal{E}(T)} ||[\widehat{P}-P]_E||^2_{L^2(E)}}$$

Proof of (A1) with $\delta(\mathcal{T},\hat{\mathcal{T}}) = ||\widehat{P}-P||_{L^2(\Omega)}$

Recall

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and apply the discrete jump control lemma for each component of the piecewise polynomial vector field $\widehat{P} - P \in P_0(\hat{\mathcal{T}};\mathbb{R}^2)$.

This concludes the proof of (A1). $\square$

Proof of (A2)

Proof of (A2) with $\varrho_2 = 2^{-1/4}$ and $\Lambda_2 = \Lambda_1$

Recall that $\hat{\mathcal{T}}$ is an admissible refinement of $\mathcal{T}$ with respective discrete fluxes $\widehat{P} \in P_0(\hat{\mathcal{T}};\mathbb{R}^2)$ and $P \in P_0(\mathcal{T};\mathbb{R}^2)$ as before.

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LHS in (A2) reads

$$\widehat{\eta}(\hat{\mathcal{T}} \setminus \mathcal{T}) = \sqrt{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\alpha_T^2 + \widehat{\beta}_T^2)} \quad \text{(triangle ineq. in } \ell^2\text{)}$$

$$\leq \sqrt{\underbrace{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\alpha_T^2 + \beta_T^2)}_{(i)}} + \sqrt{\underbrace{\sum_{K\in\mathcal{T}\setminus\hat{\mathcal{T}}} \sum_{T\in\hat{\mathcal{T}}(K)} (\widehat{\beta}_T - \beta_T)^2}_{(ii)}}.$$

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Observe $[P]_F = 0$ for $F \in \widehat{\mathcal{E}}(\text{int}(K))$ and $|T| \leq |K|/2$ for $T \in \hat{\mathcal{T}}(K)$.

Proof of (A2) with $\varrho_2 = 2^{-1/4}$ and $\Lambda_2 = \Lambda_1$

Since $[P]_F = 0$ for $F \in \hat{\mathcal{E}}(\mathrm{int}(K))$ and $|T| \leq |K|/2$ for $T \in \hat{\mathcal{T}}(K)$ and $K \in \mathcal{T} \setminus \hat{\mathcal{T}}$,

$$(i) := \sum_{T \in \hat{\mathcal{T}}(K)} (\alpha_T^2 + \beta_T^2) \leq \frac{|K|}{2} ||f||_{L^2(K)}^2 + \frac{|K|^{1/2}}{\sqrt{2}} \sum_{E \in \mathcal{E}(K)} ||[P]_E||_{L^2(E)}^2.$$

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Reversed triangle inequalities in the second term prove

$$|\widehat{\beta}_T - \beta_T| = |T|^{1/4} \left| \sqrt{\sum_{F \in \mathcal{E}(T)} ||[\widehat{P}]_F||^2_{L^2(F)}} - \sqrt{\sum_{F \in \mathcal{E}(T)} ||[P]_F||^2_{L^2(F)}} \right|$$

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This and the discrete jump control lemma conclude the proof. □

Quasiinterpolation

Discrete Quasiinterpolation

Notation $\|\bullet\| := ||\bullet||_{L^2(\Omega)}$ and $|||\bullet||| := \|\nabla\bullet\| := |\bullet|_{H^1(\Omega)}$

Theorem (approximation and stability). $\exists 0 < C = C(\min\angle\mathbb{T}) < \infty$
$\forall\mathcal{T}\in\mathbb{T}\;\forall\hat{\mathcal{T}}\in\mathbb{T}(\mathcal{T}) \quad \forall\hat{V}\in S_0^1(\hat{\mathcal{T}})\;\exists V\in S_0^1(\mathcal{T})$

$$V=\hat{V} \text{ on } \hat{\mathcal{T}}\cap\mathcal{T} \quad \text{and} \quad ||h_{\mathcal{T}}^{-1}(\hat{V}-V)||+|||V|||\le C\,|||\hat{V}|||.$$

Discrete Quasiinterpolation

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$$V = \hat{V} \text{ on } \hat{\mathcal{T}}\cap\mathcal{T} \quad\text{and}\quad ||h_{\mathcal{T}}^{-1}(\hat{V}-V)|| + |||V||| \le C\,|||\hat{V}|||.$$

Proof. Define $V\in S_0^1(\mathcal{T})$ by linear interpolation of nodal values

$$V(z) := \begin{cases} \hat{V}(z) & \text{if } z\in\mathcal{N}(\Omega)\cap\mathcal{N}(T) \text{ for some } T\in\mathcal{T}\cap\hat{\mathcal{T}} \\ \int_{\omega_z}\hat{V}\,dx/|\omega_z| & \text{if } z\in\mathcal{N}(\Omega) \text{ and } \mathcal{T}(z)\cap\hat{\mathcal{T}}(z)=\emptyset \\ 0 & \text{if } z\in\mathcal{N}(\partial\Omega) \end{cases}$$

Since V and $\hat{V}$ are continuous at any vertex of any $T\in\mathcal{T}\cap\hat{\mathcal{T}}$, the first case applies in the definition of $V(z)=\hat{V}(z)$ for all $z\in\mathcal{N}(T)$.

This proves $V=\hat{V}$ on $T\in\mathcal{T}\cap\hat{\mathcal{T}}$. □

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Lemma A. There exists $C(z) \approx \text{diam}(\omega_z)$ with

$$||\hat{V} - V(z)||_{L^2(\omega_z)} \leq C(z)\,||\nabla \hat{V}||_{L^2(\omega_z)}.$$

Proof4Case II: $z \in \mathcal{N}(\Omega)$ and $\mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset$ with $V(z) = \int_{\omega_z} \hat{V}\,dx/|\omega_z|$. Then, the assertion is a Poincare inequality with $C(z) = C_P(\omega_z)$. □

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Proof4Case II: $z \in \mathcal{N}(\Omega)$ and $\mathcal{T}(z) \cap \hat{\mathcal{T}}(z) = \emptyset$ with $V(z) = \int_{\omega_z} \hat{V}\, dx / |\omega_z|$. Then, the assertion is a Poincare inequality with $C(z) = C_P(\omega_z)$. $\square$
Proof4Case III: $z \in \mathcal{N}(\partial\Omega)$ and $V(z) = 0$. Since $\hat{V} - V$ vanishes along the two edges along $\partial\Omega$ of the open boundary patch ω_z with vertex z. Hence the assertion is indeed a Friedrichs inequality with $C(z) = C_F(\omega_z)$. $\square$

Given any node $z \in \mathcal{N}$ in the coarse triangulation, let $\omega_z = \mathrm{int}(\cup \mathcal{T}(z))$ denotes its patch of all triangles $\mathcal{T}(z)$ in $\mathcal{T}$ with vertex z.

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Proof4Case I: $\exists T \in \mathcal{T}(z) \cap \hat{\mathcal{T}}(z)$ for $z \in \mathcal{N}(\Omega)$ and $V = \hat{V}$ on T. This leads to homogenous Dirichlet boundary conditions on the two edges of the open patch $\omega_z \setminus T$ with vertex z and $\hat{V} - V$ allows for a Friedrichs inequality (on the open patch as in Case III for a patch on the boundary)

$$||\hat{V} - V||_{L^2(\omega_z)} \leq C_F(\omega_z \setminus T)\, ||\nabla(\hat{V} - V)||_{L^2(\omega_z)}$$

However, this is not the claim! The idea is to realize that LHS$=||w||_{L^2(\omega_z)}$ for $w := \hat{V} - \hat{V}(z)$, which is affine on T and vanishes at vertex z.

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$$||w||_{L^2(T)} \leq C_{dF}(T)\, ||\nabla w||_{L^2(T)} \leq C_{dF}(T)\, ||\nabla w||_{L^2(\omega_z)}$$

E.g. the integral mean $w_T := \int_T w\, dx/|T|$ of $w := \hat{V} - \hat{V}(z)$ on T satisfies

$$|w_T|^2\, |T| \leq C_{dF}(T)^2\, ||\nabla w||^2_{L^2(\omega_z)}$$

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Compare with integral mean $\overline{w} := \displaystyle\int_{\omega_z} w\,dx/|\omega_z|$ and compute

$$\begin{aligned}|\overline{w} - w_T|^2\,|T| &= |T|^{-1}\,|\int_T (\overline{w} - w)dx|^2 \leq ||w - \overline{w}||^2_{L^2(T)}\\ &\leq ||w - \overline{w}||^2_{L^2(\omega_z)} \leq C_P(\omega_z)^2\,||\nabla w||^2_{L^2(\omega_z)}\end{aligned}$$

Consequently, $\quad |\overline{w} - w_T|^2\,|\omega_z| \leq \underbrace{|\omega_z|/|T|}_{\leq C_{sr}}\,C_P(\omega_z)^2\,||\nabla w||^2_{L^2(\omega_z)}$

The orthogonality of 1 and $w - \overline{w}$ in $L^2(\omega_z)$ is followed by Poincare's and geometric-arithmetic mean inequality to verify

$$\begin{aligned}||w||^2_{L^2(\omega_z)} &= |\overline{w}|^2\,|\omega_z| + ||w-\overline{w}||^2_{L^2(\omega_z)}\\ &\le 2|\overline{w}-w_T|^2\,|\omega_z| + 2|w_T|^2\,|\omega_z| + C_P(\omega_z)^2\,||\nabla w||^2_{L^2(\omega_z)}\end{aligned}$$

The above estimates for $|w_T|^2\,|T|$ and $|\overline{w}-w_T|^2\,|T|$ lead to

$$||w||^2_{L^2(\omega_z)} \le \underbrace{\Big(2|\omega_z|/|T|\,(C_{dF}(T)+C_P(\omega_z)^2)+C_P(\omega_z)^2\Big)}_{=:C(z)^2}\,||\nabla w||^2_{L^2(\omega_z)} \quad \square$$

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W.r.t. triangulation $\mathcal{T}$ and nodal basis functions φ_1, φ_2, φ_3 in $S^1(\mathcal{T})$, let $T = \text{conv}\{P_1, P_2, P_3\} \in \mathcal{T}$ and $\Omega_T := \omega_1 \cup \omega_2 \cup \omega_3$ for $\omega_j := \{\varphi_j > 0\}$

Lemma B. There exists $C(T) \approx h_T$ with

$$||\hat{V} - V||_{L^2(T)} \le C(T)\,||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

Proof of Lemma B. N.B. $V = \sum_{j=1}^3 V(P_j)\,\varphi_j$ and $1 = \sum_{j=1}^3 \varphi_j$ on T
Hence

$$\begin{aligned}
||\hat{V} - V||^2_{L^2(T)} &= \int_T |\sum_{j=1}^3 (\hat{V} - V(P_j))\,\varphi_j|^2\,dx \\
&\le \int_T (\sum_{j=1}^3 |\hat{V} - V(P_j)|^2)\underbrace{(\sum_{k=1}^3 \varphi_k^2)}_{\le 1}\,dx \quad \text{(CS in } \mathbb{R}^3\text{)} \\
&\le \sum_{j=1}^3 ||\hat{V} - V(P_j)||^2_{L^2(T)} \\
&\le \sum_{j=1}^3 C(P_j)^2 ||\nabla \hat{V}||^2_{L^2(\omega_j)} \quad \text{(Lemma A)} \\
&\le \underbrace{(\sum_{j=1}^3 C(P_j)^2)}_{C^2(T)}\ ||\nabla \hat{V}||^2_{L^2(\Omega_T)} \qquad \square
\end{aligned}$$

Lemma C. There exists $C > 0$ (which solely depends on $\min \angle \mathbb{T}$) with

$$||\nabla V||_{L^2(T)} \leq C \, ||\nabla \hat{V}||_{L^2(\Omega_T)}.$$

Proof. N.B. $\nabla V = \sum_{j=1}^{3} V(P_j) \, \nabla \varphi_j$ and $0 = \sum_{j=1}^{3} \nabla \varphi_j$ on T

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Hence

$$\begin{aligned}
||\nabla V||^2_{L^2(T)} &= \int_T |\sum_{j=1}^{3} (\hat{V} - V(P_j))\nabla\varphi_j|^2\, dx \\
&\leq \int_T (\sum_{j=1}^{3} |\hat{V} - V(P_j)|^2)(\underbrace{\sum_{k=1}^{3} |\nabla\varphi_k|^2}_{\leq C(\min\angle T)^2/h_T^2})\, dx \quad \text{(CS in } \mathbb{R}^6\text{)} \\
&\leq C(\min\angle T)^2\, h_T^{-2} \sum_{j=1}^{3} \int_T |\hat{V} - V(P_j)|^2\, dx \\
&\leq \ldots \text{(as before)} \ldots \\
&\leq \underbrace{C(\min\angle T)^2\, h_T^{-2} C^2(T)}_{=:C^2} ||\nabla \hat{V}||^2_{L^2(\Omega_T)} \qquad \square
\end{aligned}$$

Finish of proof of theorem: $||h_{\mathcal{T}}^{-1}(\hat{V} - V)||_{L^2(\Omega)} + |||V||| \lesssim |||\hat{V}|||$.

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Lemma B and C show for some generic constant $C > 0$ hidden in the notation $\lesssim$ and any $T \in \mathcal{T}$ that

$$||h_T^{-1}(\hat{V} - V)||^2_{L^2(T)} + ||\nabla V||^2_{L^2(T)} \lesssim ||\nabla \hat{V}||^2_{L^2(\Omega_T)}$$

Notation $\Omega_T := \cup_{z \in \mathcal{N}(T)} \omega_x$ is the interior of the set of T plus one layer of triangles of $\mathcal{T}$ around.

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The sum over all those inequalities for $T \in \mathcal{T}$ concludes the proof because the overlap of $(\Omega_T)_{T \in \mathcal{T}}$ is bounded by generic constant $C(\min \angle \mathbb{T})$. $\square$

Proof of (A3)

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Given discrete solution U (resp. $\hat{U}$) of CFEM in PMP w.r.t. $\mathcal{T}$ (resp. refinement $\hat{\mathcal{T}}$), set $\hat{e} := \hat{U} - U \in S_0^1(\hat{\mathcal{T}})$ with quasiinterpolant $e \in S_0^1(\mathcal{T})$ as above. Then, $v := \hat{e} - e$ satisfies

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 = a(\hat{e}, v) = \underbrace{F(v) - a(U, v)}_{\mathrm{Res}(v)}$$

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$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 = a(\hat{e}, v) = \underbrace{F(v) - a(U, v)}_{\mathrm{Res}(v)}$$

A piecewise integration by parts with a careful algebra with the jump terms for appropriate signs shows

$$\begin{aligned} -a(U, v) &= -\sum_{E \in \mathcal{E}(\Omega)} \int_E v\,[\partial U/\partial \nu_E]_E\, ds \\ &\le \sqrt{\sum_{E \in \mathcal{E}(\Omega)} |E|^{-1} ||v||^2_{L^2(E)}} \sqrt{\sum_{E \in \mathcal{E}(\Omega)} |E|\, ||[\partial U/\partial \nu_E]_E||^2_{L^2(E)}} \end{aligned}$$

Recall trace inequality

$$|E|^{-1} ||v||^2_{L^2(E)} \le C_{tr}(h^{-2}_{\omega_E} ||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

Finish of Proof of (A3)

to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}(\Omega)} (h^{-2}_{\omega_E}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

$$\lesssim ||h^{-1}_{\mathcal{T}} v||^2_{L^2(\Omega)} + |||v|||^2 \lesssim |||\hat{e}|||^2$$

with the approximation and stability of the quasiinterpolation.

Finish of Proof of (A3)

to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}(\Omega)} (h_{\omega_E}^{-2}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$
$$\lesssim ||h_{\mathcal{T}}^{-1} v||^2_{L^2(\Omega)} + |||v|||^2 \lesssim |||\hat{e}|||^2$$

with the approximation and stability of the quasiinterpolation.

A weighted Cauchy inequality followed by approximation property of quasi-interpolation show

$$F(v) \le ||h_{\mathcal{T}} f||_{L^2(\Omega)}\, ||h_{\mathcal{T}}^{-1} v||_{L^2(\Omega)} \le C ||h_{\mathcal{T}} f||_{L^2(\Omega)}\, |||\hat{e}|||$$

Finish of Proof of (A3)

to estimate

$$\sum_{E\in\mathcal{E}(\Omega)} |E|^{-1}||v||^2_{L^2(E)} \lesssim \sum_{E\in\mathcal{E}(\Omega)} (h^{-2}_{\omega_E}||v||^2_{L^2(\omega_E)} + ||\nabla v||^2_{L^2(\omega_E)})$$

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$$F(v) \le ||h_{\mathcal{T}} f||_{L^2(\Omega)} \, ||h_{\mathcal{T}}^{-1} v||_{L^2(\Omega)} \le C ||h_{\mathcal{T}} f||_{L^2(\Omega)} \, |||\hat{e}|||$$

All this plus shape-regularity (e.g. $|T| \approx h_T^2 \approx h_E^2$) lead to reliability

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) = |||\hat{e}|||^2 \le \Lambda_3 \, \eta(\mathcal{T}) |||\hat{e}|||$$

The extra fact $v = 0$ on $\mathcal{T} \cap \hat{\mathcal{T}}$ and a careful inspection on disappearing integrals in the revisited analysis prove the asserted upper bound in (A3),

$$\delta(\mathcal{T}, \hat{\mathcal{T}}) \le \Lambda_3 \, \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) \qquad \square$$

Proof of (A4)

(A4) follows from (A3) for CFEM with $\Lambda_4 = \Lambda_3^2$

The pairwise Galerkin orthogonality in the CFEM allows for the (modified) LHS in (A4) the representation

$$\sum_{k=\ell}^{\ell+m} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) = \delta^2(\mathcal{T}_\ell, \mathcal{T}_{\ell+m+1})$$

for $m \in \mathbb{N}_0$. (A3) shows that this is bounded from above by $\Lambda_3^2 \eta_\ell^2$. Since $m \in \mathbb{N}_0$ is arbitrary, this implies

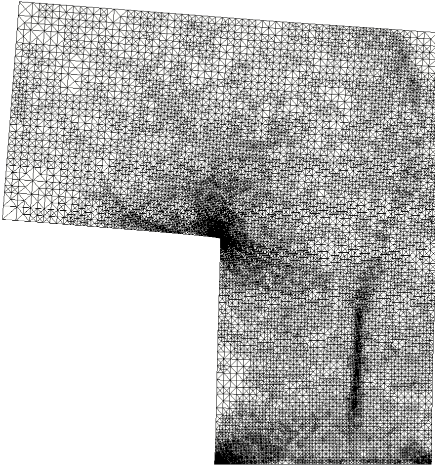
$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) = \lim_{m\to\infty} \sum_{k=\ell}^{\ell+m} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_3^2 \eta_\ell^2. \quad \square$$

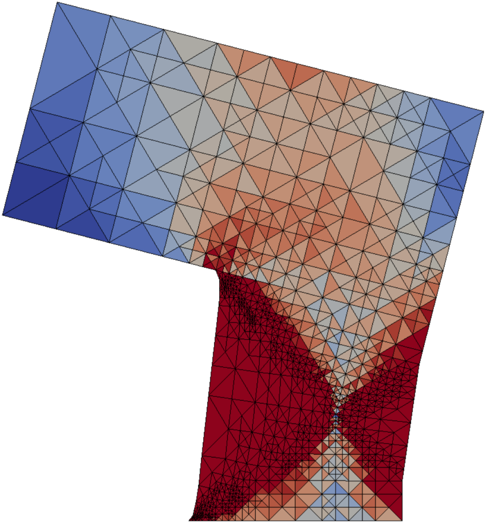
Outlook at Applications

Elastoplasticity

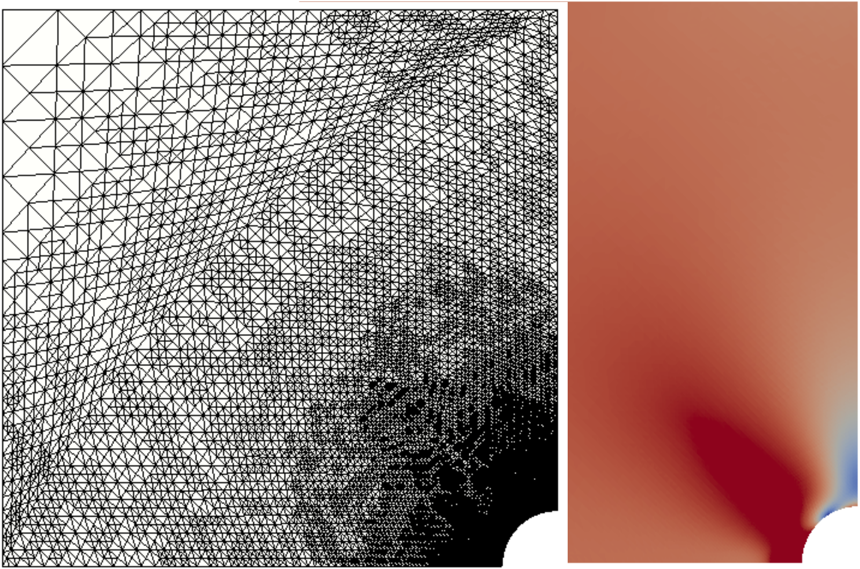
An Optimal Adaptive FEM for Elastoplasticity

[Carstensen-Schröder-Wiedemann: An optimal adaptive FEM for elastoplasticity, Numer. Math. (2015)]

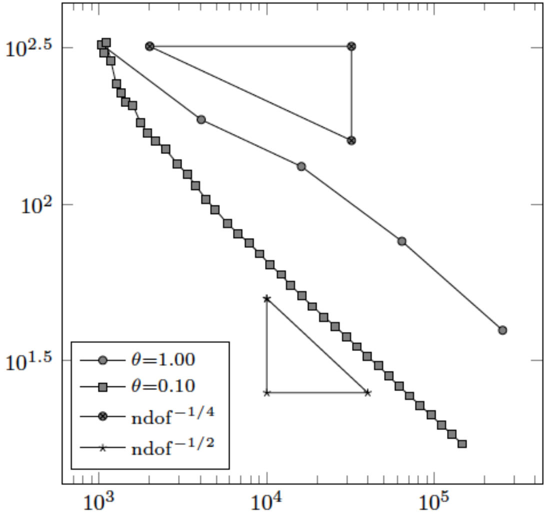




Computational Benchmark in Elastoplasticity



Convergence History in Computational Benchmark



Affine Obstacle Problem

An Optimal Adaptive FEM for an Obstacle Problem

Reference: An optimal Adaptive FEN for an obstacle problem.
Carstensen-Hu Jun, CMAM [Online Since 13/06/2015]

Given RHS $F \in H^{-1}(\Omega)$ (dual to $H_0^1(\Omega)$ w.r.t. energy scalar product a) and affine obstacle $\chi \in P_1(\Omega)$ s.t.

$$K \ := \ \{v \in H_0^1(\Omega) : \ \chi \leq v \quad \text{a.e. in } \Omega\} \neq \emptyset,$$

the obstacle problem allows for a unique weak solution $u \in K$ to

$$F(v-u) \leq a(u, v-u) \quad \textit{for all v } \in K.$$

An Optimal Adaptive FEM for an Obstacle Problem

Reference: An optimal Adaptive FEN for an obstacle problem.
Carstensen-Hu Jun, CMAM [Online Since 13/06/2015]

Conforming discretization leads to discrete solution u_ℓ and a posteriori error control via

$$\eta_E^2 := h_E \, ||[\nabla u_\ell]_E \cdot \nu_E||_{L^2(E)}^2 + \mathsf{Osc}^2(f, \omega_E)$$

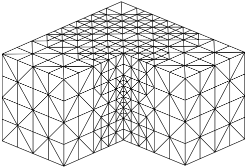
for any interior edge E.

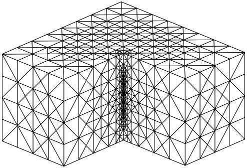
Theorem (Carstensen-Hu 2015). AFEM leads to optimal convergence rates.

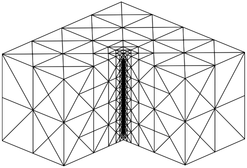
Eigenvalue Problems

Eigenvalue Problem

$$-\Delta u = \lambda u \text{ in } \Omega \quad \text{and} \quad u = 0 \text{ on } \partial\Omega$$







Optimal Computational Complexity - 3D [Carstensen-Gedicke (2013) SINUM]