Estimation problems: Algebraic point of view

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EESTEC Workshop: Robotics and Intelligent Systems
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☞ Papers can be found at `http://hal.inria.fr/`.

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Introduction to the main ideas

Parametric estimation and its applications

☞ Parametric estimation :

$$y = F(x, \Theta) + n, \quad (1)$$

- observed signal y, "true" signal x and Θ (parameters),
- n is a noise corrupting the observation.

☞ Find a "good" $\Theta_{\approx}$ (Existing results : proba. knowledge of n).

☞ Alien new standpoint of algebraic flavour:

- differential algebra which plays with respect to differential equations a similar rôle to commutative algebra with respect to algebraic equations;
- module theory (linear algebra over rings which are not necessarily commutative);
- operational calculus which was a most classical tool among control and mechanical engineers.

Introduction to the main ideas

Parametric estimation and its applications

☞ In automatic control :

- identifiability and identification of uncertain parameters in the system equations, including delays, (L or NL and even for closed loop systems);
- estimation of state variables, which are not measured (even for closed loop systems);
- fault diagnosis and isolation;
- observer-based chaotic synchronization.

☞ Signal, image and video processing : noise removal, i.e. estimation

- signal modelling, demodulation, restoration, (blind) equalisation, etc,
- Data compression, Decoding for error correcting codes
- Detection of abrupt change, . . .

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A first simple example: parameter estimation

$$\boxed{\dot{y}(t) = ay(t) + u(t) + \gamma_0.} \tag{2}$$

where a is an unknown parameter to be identified and γ_0 is an unknown, constant perturbation.
Using operational calculus and $y_0 = y(0)$:

$$s\widehat{y}(s) = a\widehat{y}(s) + \widehat{u}(s) + y_0 + \frac{\gamma_0}{s}.$$

☞ Eliminate the term γ_0 : use operator $D_s \times s$:

$$\frac{d}{ds}\left[s\left\{s\widehat{y}(s) = a\widehat{y}(s) + \widehat{u}(s) + y_0 + \frac{\gamma_0}{s}\right\}\right]$$

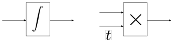
$$\Rightarrow 2s\widehat{y}(s) + s^2\widehat{y}'(s) = a\left(s\widehat{y}'(s) + \widehat{y}(s)\right) + s\widehat{u}'(s) + \widehat{u}(s) + y_0.$$

A first simple example: parameter estimation

☞ Estimation of parameter a : Assume $y_0 = 0$ (if not use D_s^2 to eliminate y_0), for any $\nu > 0$,

$$s^{-\nu}\left[2s\widehat{y}(s) + s^2\widehat{y}'(s)\right] = s^{-\nu}\left[a(s\widehat{y}'(s) + \widehat{y}(s)) + s\widehat{u}'(s) + \widehat{u}(s)\right].$$

$$a = \frac{2\int_0^T d\lambda \int_0^\lambda y(t)dt - \int_0^T ty(t)dt + \int_0^T d\lambda \int_0^\lambda tu(t)dt - \int_0^T d\lambda \int_0^\lambda d\sigma \int_0^\sigma u(t)dt}{\int_0^T d\lambda \int_0^\lambda d\sigma \int_0^\sigma y(t)dt - \int_0^T d\lambda \int_0^\lambda ty(t)dt}, (\nu = 3). \quad (3)$$

- two kind of operations 

- $T > 0$ can be very small $\Rightarrow$ fast estimation.
- ν number of iterative integrals $\Rightarrow$ filtering (mean processing) : one can also use low pass filter $s \to (1 + \tau s)$.
- including a noise (fast fluctuating signal), of zero mean $\dot{y}(t) = ay(t) + u(t) + \gamma_0 + n(t)$ (filtering)

A first simple example: parameter estimation

This example, even simple, clearly demonstrated how ALIEN's techniques proceed:

- they are algebraic: operations on s-functions;
- they are non-asymptotic: parameter a is obtained from (3) in finite time;
- they are deterministic: no knowledge of the statistical properties of the noise n is required.

A simple example: numerical derivation

Let us recall

$$\mathcal{L}^{-1}\left(\frac{1}{s^m}\frac{d^n X(s)}{ds^n}\right) = \frac{(-1)^n t^{m+n}}{(m-1)!}\int_0^1 w^{m-1,n}(\tau)x(t\tau)d\tau, \quad m\geq 1, \; n\in\mathbb{N} \tag{4}$$

where

$$w^{m,n}(t) = (1-t)^m t^n \tag{5}$$

☞ Normalized

☞ The noise passing through the filter is amplified by t^{m+n}

A simple example: numerical derivation

☞ Estimate $x^{(2)}(0)$ through the truncated series of order 2:

$$\mathcal{R}: \qquad X = \frac{x(0)}{s} + \frac{x^{(1)}(0)}{s^2} + \frac{x^{(2)}(0)}{s^3}$$

Idea: kill undesired terms (blue) except the one to estimate (red)

Step 1 $\times s^2$: $s^2 X = s x(0) + x^{(1)}(0) + \frac{x^{(2)}(0)}{s}$

Step 2 $\frac{d^2}{ds^2}$: $2X + 4s\frac{dX}{ds} + s^2\frac{d^2X}{ds^2} = \frac{2}{s^3}x^{(2)}(0)$

Step 3 $\times \frac{1}{s^3}$: $\frac{2}{s^3}X + \frac{4}{s^2}\frac{dX}{ds} + \frac{1}{s}\frac{d^2X}{ds^2} = \frac{2}{s^6}x^{(2)}(0)$

Step 4 Go back to the time domain (use of $\mathcal{L}^{-1}$ (4)):

$$\frac{2t^5}{5!}x^{(2)}(0) = t^3\int_0^1 \left(2w^{2,0}(\tau) - 4w^{1,1}(\tau) + w^{0,2}(\tau)\right) y(t\tau)d\tau$$

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A simple example: numerical derivation

Let us mention that finally we have applied to the relation $\mathcal{R}$, the operator

$$\Pi = \frac{1}{s^3}\,\frac{d^2}{ds^2}\,s^2$$

where

- $\Pi \in \mathbb{R}(s)\left[\frac{d}{ds}\right]$
- $\Pi = \frac{1}{s}\frac{d^2}{ds^2} + \frac{4}{s^2}\frac{d}{ds} + \frac{2}{s^3}$

Preliminary remarks

Let $f(s)$ be a polynomial in the variable s. By Leibniz's rule:

$$\left(\frac{d}{ds}s\right)(f) = \frac{d}{ds}(sf) = f + s\frac{df}{ds} \quad \text{and} \quad \left(s\,\frac{d}{ds}\right)(f) = s\frac{df}{ds}$$

So

$$\left(\frac{d}{ds}s\ - s\ \frac{d}{ds}\right)(f) = f \Longrightarrow \frac{d}{ds}s\ - s\ \frac{d}{ds} = 1$$

Or using the commutator notation:

$$\left[\frac{d}{ds}, s\right] = \frac{d}{ds}s - s\frac{d}{ds} = 1$$

Set $p := \frac{d}{ds}$ and $q := s \times \cdot$, therefore

$$[p, q] = pq - qp = 1$$

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Preliminary remarks

Back to the first example

In the first example we have used

$$\Pi = \frac{1}{s^3}\ \frac{d^2}{ds^2}\ s^2$$

Here we look at p^2q^2. Since $pq = qp + 1$ we have

$$\begin{aligned} p^2q^2 &= p(qp+1)q = pqpq + pq = (qp+1)(qp+1) + (qp+1) \\ &= qpqp + 3qp + 2 = q(qp+1)p + 3qp + 2 = q^2p^2 + 4qp + 2 \end{aligned}$$

Thus we find again, since $p := \frac{d}{ds},\ q := s \times \cdot$:

$$\Pi = \frac{1}{s}\frac{d^2}{ds^2} + \frac{4}{s^2}\frac{d}{ds} + \frac{2}{s^3}$$

Preliminary remarks

Back to the first example

Now let us note that $\frac{d}{ds}\ s^2\ \frac{d}{ds}s$ reads as

$$pq^2pq = q^3p^2 + 4q^2p + 2q$$

which means that these two operators $\frac{1}{s^3}\ \frac{d^2}{ds^2}\ s^2$ and $\frac{1}{s^3}\ \frac{d}{ds}\ s^2\ \frac{d}{ds}s$ are the same, they can be written as

$$\frac{1}{s}\frac{d^2}{ds^2} + \frac{4}{s^2}\frac{d}{ds} + \frac{2}{s^3}$$

☞ Use a "canonical form"

Preliminary remarks
Algebras

Definition

An $\mathbb{R}$-algebra A is a $\mathbb{R}$-vector space equipped with a product $\cdot : A \times A \to A$ satisfying

$$\begin{aligned} a \cdot (b + c) &= a \cdot b + a \cdot c \\ (a + b) \cdot c &= a \cdot c + b \cdot c \\ a \cdot 1 &= 1 \cdot a = a \\ \lambda(a \cdot b) &= (\lambda a) \cdot b = a \cdot (\lambda b) \end{aligned}$$

for all a, b, $c \in A$, $\lambda \in \mathbb{R}$

Preliminary remarks
Weyl Algebra

☞ Let us note that A $:= \mathbb{R}[s]\left[\frac{d}{ds}\right]$ has a Weyl Algebra structure (non commutative since $\left[\frac{d}{ds}, s\right] = 1$)

☞ Thus a canonical basis of A is $\left\{ s^i \frac{d^j}{ds^j} \middle| (i,j) \in \mathbb{N} \right\}$

☞ Any $F \in$ A can be rewritten into its *canonical form*

$$F = \sum_{i,j} \lambda_{ij} s^i \frac{d^j}{ds^j}, \;\; \lambda_{ij} \in \mathbb{R} \tag{6}$$

A preliminary remark
Weyl Algebra

☞ One can associate to the Weyl Algebra A an algebra B defined as the differential operators on $\frac{d}{ds}$ with coefficients in $\mathbb{R}(s)$

$$\mathsf{B} := \mathbb{R}(s)\left[\frac{d}{ds}\right]$$

☞ Any $F \in \mathsf{B}$ can be rewritten into its *canonical form*

$$F = \sum_{i,j} \lambda_{ij} g_i(s) \frac{d^j}{ds^j}, \quad \text{with} \quad g_i(s) \in \mathbb{R}(s) \tag{7}$$

Preliminary remarks
Ideal

Definition
Let $(A, +, \cdot)$ be a ring. A left ideal $\mathcal{I}$ of A is a non-empty subset such that $(I, +)$ is a subgroup of $(A, +)$ and

$$\text{for all } a \in A, x \in \mathcal{I}, \text{ then } ax \in \mathcal{I}$$

For example, even integers form an ideal of $\mathbb{Z}$

Definition
An integral domain A is a commutative ring such that for any two elements a and b in A, $ab = 0 \Rightarrow a = 0$ or $b = 0$

For instance, the ring of $n \times n$-matrices is not an integral domain

Preliminary remarks

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Similar definition for a right ideal For example, even integers form an ideal of $\mathbb{Z}$

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Preliminary remarks
PID

Definition

A principal ideal domain A is an integral domain where any ideal is principal (i.e. it can be generated by a single element)

For example, $\mathbb{Z}$ is an principal ideal domain

Preliminary remarks
Generator

Proposition

$$\mathsf{B} = \mathbb{R}(s)\left[\frac{d}{ds}\right]$$

is a principal left domain. So any left ideal can be generated by a single element of B

Preliminary remarks
Some vocabulary

☞

$$h = \frac{P}{Q} \in \mathbb{R}(s)$$

is in a *(strict) finite-integral form* if $h \in \mathbb{R}\left[\frac{1}{s}\right]$ (resp. $h \in \frac{1}{s}\mathbb{R}\left[\frac{1}{s}\right]$)

☞ A differential operator

$$F = \sum_{i=0}^{\ell} F_i(s)\frac{d^i}{ds^i} \in \mathsf{B}$$

inherits one of the above properties if all $F_i(s)$ have this property

Table of Contents

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

- $x(t)$ a smooth signal on an interval $I \subset \mathbb{R}_+$,

$$x(t) = \sum_{k \geq 0} x^{(k)}(0) \frac{t^k}{k!}$$

Taylor series at $t = 0$

- $y(t)$ signal observation with noise $\varpi(t)$,

$$y(t) = x(t) + \varpi(t)$$

- Goal: To estimate $x^{(n)}(0)$ from $y(t)$

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

$$x_N(t) = \sum_{k=0}^{N} x^{(k)}(0)\frac{t^k}{k!}$$ truncated Taylor series ($N \geq n$)

$\Downarrow$ Laplace transform

$$X_N(s) = \sum_{i=0}^{N} \frac{x^{(i)}(0)}{s^{i+1}}$$ the operational analog of $x_N(t)$ on I

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

☞ Unknown parameters $\Theta = \{\theta_1, \ldots, \theta_m\}$ (here the derivatives $x^{(i)}(0)$) are divided into $\Theta_{\text{est}} = \{x^{(n)}(0)\}$ and $\Theta_{\overline{\text{est}}} = \Theta \setminus \Theta_{\text{est}}$ (here the other derivatives)

☞ numerical differentiation = Parameter estimation !! = find a good approximation of $\Theta_{\text{est}} = \{x^{(n)}(0)\}$ using

$$y = x(\Theta) + \varpi, \quad (8)$$

where y is the real measured noisy signal, x (the "true" signal) depends implicitly on the parameters Θ and ϖ is the additive noise

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

☞ Algebraic extensions

$\mathbb{R}_\Theta := \mathbb{R}(\Theta)$, $\mathbb{R}_{\Theta_{\text{est}}} := \mathbb{R}(\Theta_{\text{est}})$ and $\mathbb{R}_{\Theta_{\overline{\text{est}}}} := \mathbb{R}(\Theta_{\overline{\text{est}}})$

☞ Then

$$X_N(s) = \sum_{i=0}^{N} \frac{x^{(i)}(0)}{s^{i+1}}$$

reads as

$$\mathcal{R}\left(s, X(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right) : \quad P(X(s)) + Q + \overline{Q} = 0 \tag{9}$$

where $P \in \mathbb{R}_{\Theta_{\text{est}}}[s]\left[\frac{d}{ds}\right]$, $Q \in \mathbb{R}_{\Theta_{\text{est}}}[s]$ and $\overline{Q} \in \mathbb{R}_{\Theta_{\overline{\text{est}}}}[s]$,

$$P = s^{N+1}, Q = -s^{N-n}x^{(n)}(0), \overline{Q} = -\sum_{i \neq n}^{N} s^{N-i}x^{(i)}(0)$$

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

☞ It might not be obvious to built an annihilator in this form since

$$Q = -s^{N-n}x^{(n)}(0), \overline{Q} = -\sum_{i \neq n}^{N} s^{N-i}x^{(i)}(0)$$

☞ Yes we can (!!) : a minimal annihilator is of order N

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

Example $N = n = 2$

Using the notation $\mathcal{R} : P(X(s)) + Q + \bar{Q}$:

- if $P = 1$, $Q = -\frac{x^{(2)}(0)}{s^3}$, $\bar{Q} = -\frac{x(0)}{s} - \frac{x'(0)}{s^2}$, we obtain

$$\Pi = \frac{2}{s^3} + \frac{4}{s^2}\frac{d}{ds} + \frac{1}{s}\frac{d^2}{ds^2}$$

- if $P = s^3$, $Q = -x^{(2)}(0)$, $\bar{Q} = -s^2 x(0) - s x'(0)$, we obtain for example

$$\Pi = \frac{2}{s^6} - \frac{2}{s^5}\frac{d}{ds} + \frac{1}{s^4}\frac{d^2}{ds^2}$$

Dual core Non-A

Once upon a time... ALIEN numerical differentiation

Example $N = n = 2$

Indeed for a general

$$\Pi = g_0(s) + g_1(s)\frac{d}{ds} + g_2(s)\frac{d^2}{ds^2} \text{ with } g_k(s) = \sum_i \frac{a_{ki}}{s^i},\ k = 0, 1, 2,$$

we have $\Pi(\bar{Q}) = 0$ giving some algebraic relations on the a_{ki}. For instance we can find $g_0 = \frac{2}{s^6}$, $g_1 = -\frac{2}{s^5}$ and $g_2 = \frac{1}{s^4}$, so $\Pi(P(X(s)) + \bar{Q} + Q) = 0$ with $P = s^3$, $Q = -x^{(2)}(0)$, $\bar{Q} = -s^2x(0) - x'(0)$ gives

$$\Pi = \frac{2}{s^6} - \frac{2}{s^5}\frac{d}{ds} + \frac{1}{s^4}\frac{d^2}{ds^2}$$

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Once upon a time... ALIEN numerical differentiation

Example $N = n = 2$

Moreover with this annihilator

$$\Pi(s^3X) = \left(\frac{2}{s^3} - \frac{2}{s^5}\frac{d}{ds}s^3 + \frac{1}{s^4}\frac{d^2}{ds^2}s^3\right)(X)$$

$$= \left(\frac{2}{s^3} + \frac{4}{s^2}\frac{d}{ds} + \frac{1}{s}\frac{d^2}{ds^2}\right)(X) = \Pi_{P=1}(X)$$

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Once upon a time... ALIEN numerical differentiation

Lemma

For all $\Pi \in \mathbb{R}(s)\left[\frac{d}{ds}\right]$ *with* $g_0 = 0$, $P \in \mathbb{R}[s]\left[\frac{d}{ds}\right]$ *and any* X *we have*

$$\Pi(PX) = \Pi(P)X + P\Pi(X)$$

Thus $\Pi(s^3X) = \Pi(s^3)X + s^3\Pi(X)$

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Dual core Non-A

Parameter estimation for sinusoidal biased signal

For a sinusoidal biased noisy signal, to estimate the triplet (amplitude, phase, frequency) is of practical importance for most engineers:

- signal demodulation in communication,
- voltage control of boost converter in power electronics,
- circadian rhythm in biology,
- ...
- modal identification for a flexible beam which plays a central role in AFM : the amplitude of the observed signal is influenced by the interaction forces between the atoms and the 'pointe' (leading to an image of the observed atoms)

Dual core Non-A

Parameter estimation for sinusoidal biased signal

This generic problem consists in estimating the parameters α, ϕ and $\omega > 0$ for the signal

$$x = \alpha \sin(\omega t + \phi) + \beta \tag{10}$$

using a biased noisy measure (from sensor):

$$y = \alpha \sin(\omega t + \phi) + \beta + \varpi, \tag{11}$$

where β is an unknown constant bias and ϖ is the noise.

Dual core Non-A

Parameter estimation for sinusoidal biased signal

Some other technics:

- Least Square method,
- EKF,
- non linear adaptive observers
- ...

None of these technics give the triplet within a sufficiently small time window (fraction of the signal period) and in a robust manner using noisy measures. The only quite satisfactory solutions were provided by Hebert and Dayan (unbiased case).

☞ Here using some knowledge about minimal annihilators, we'll obtain a less noise sensitive solution...

Dual core Non-A

Parameter estimation for sinusoidal biased signal

☞ Unknown parameters $\Theta = \{\theta_1, \dots, \theta_m\}$ classified into Θ_{est} and $\Theta_{\overline{\text{est}}} = \Theta \setminus \Theta_{\text{est}}$ (here the biais)

☞ Parameter estimation = find a good approximation of Θ_{est} using [1]

$$y = x(\Theta) + \varpi, \tag{12}$$

where y is the real measured noisy signal, x (the "true" signal) depends implicitly on the parameters Θ and ϖ is the additive noise

[1] $y = \varpi_1 x + \varpi_2$, where ϖ_1 (respec. ϖ_2) is multiplicative noise de with unitary mean (resp. additive noise) is encompassed in our setting noticing that $\varpi = (\varpi_1 - 1)x + \varpi_2$ have the same mean as ϖ_2.

Dual core Non-A

Parameter estimation for sinusoidal biased signal

☞ $x(\Theta)$ is supposed to be the output (generated) by a LTI EDO $\sum_{i=0}^{n} a_i x^{(i)} = 0$, which in the operational domain reads as

$$\sum_{i=0}^{n} a_i s^i X(s) + \sum_{i=0}^{n} a_i \left(\sum_{j=0}^{i-1} s^{i-1-j} x^{(j)}(0) \right) = 0. \qquad (13)$$

where s is the Laplace variable, $X(s)$ is the Laplace transform of the signal x.

☞ Mikusinski point of view is more general and it encompass in a same setting Distribution and Laplace transform.

Dual core Non-A

Parameter estimation for sinusoidal biased signal

☞ Algebraic extension

$\mathbb{R}_\Theta := \mathbb{R}(\Theta)$, $\mathbb{R}_{\Theta_{\text{est}}} := \mathbb{R}(\Theta_{\text{est}})$ and $\mathbb{R}_{\Theta_{\overline{\text{est}}}} := \mathbb{R}(\Theta_{\overline{\text{est}}})$.

☞ Then (13) reads as

$$\mathcal{R}\left(s, X(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right): \quad P(X(s)) + Q + \overline{Q} = 0 \tag{14}$$

where $P \in \mathbb{R}_{\Theta_{\text{est}}}[s]\left[\frac{d}{ds}\right]$, $Q \in \mathbb{R}_{\Theta_{\text{est}}}[s]$ and $\overline{Q} \in \mathbb{R}_{\Theta_{\overline{\text{est}}}}[s]$

Dual core Non-A

Parameter estimation for sinusoidal biased signal

☞ Since $\mathsf{A}_\Theta := \mathbb{R}_\Theta[s]\left[\frac{d}{ds}\right]$ has a Weyl Algebra structure $\Rightarrow$ Basis!

☞ $F \in \mathsf{B}_\Theta := \mathbb{R}_\Theta(s)\left[\frac{d}{ds}\right]$ can be rewritten into its *canonical form*

$$F = \sum_{i,j} \lambda_{i,j} g_i(s) \frac{d^j}{ds^j}, \quad \text{with} \quad g_i(s) \in \mathbb{R}_\Theta(s) \tag{15}$$

Dual core Non-A

Parameter estimation for sinusoidal biased signal

Let M_S be the B-torsion module generated by $S = \{x_i \mid i \in \mathcal{I}\}$.
$\forall i \in \mathcal{I}$, x_i is torsion, i.e. $\mathrm{Ann}_{\mathsf{B}}(x_i) := \{F \in \mathsf{B} \mid F \cdot x_i = 0\} \neq 0$.

☞ $\mathrm{Ann}_{\mathsf{B}}(M_S)$ is a left principal ideal of B (thus generated by a unique element see proposition 1).

☞ Any $\pi \in \mathrm{Ann}_{\mathsf{B}}(M_S) \subset \mathsf{B}$ is called an *S–annihilator*.

☞ $\mathrm{Ann}_{\mathsf{B}}(M_S)$ contains a proper annihilator which can be chosen in finite-integral form.

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

☞ Using such setting it is clear that a parameter estimation problem : find a family $\{\Pi_i\}_{i=1}^{r}$ of annihilators $\Pi_i \in \mathrm{Ann}_{\mathsf{B}}(M_{\overline{Q}})$ so that:
this family of S–annihilators applied to $\mathcal{R}$ (37) gives a set equations in the Θ_{est} (with proper elements to be able to come back into the time domain).

☞ Note that $\Pi_i \in \mathrm{Ann}_{\mathsf{B}}(M_{\overline{Q}})$ means $\Pi_i(\overline{Q}) = 0$: we kill the effect of $\overline{Q}$.

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

☞ Look at (4) (inverse Laplace formula): find annihilator in finite-integral form.

☞ These annihilator should be with minimal degree in $\frac{d}{ds}$ (minimal order) in order to reduce the noise effect.

☞ The obtained system should be well balanced.

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

☞ signal (33) satisfies the following ODE

$$\ddot{x} + \omega^2 (x - \beta) = 0 \tag{16}$$

☞ $\Theta_{\text{est}} = \{\theta_1 := \omega^2, \theta_2 := -\alpha \sin(\phi) = -x(0) + \beta, \theta_3 := -\alpha\omega \cos(\phi) = -\dot{x}(0)\}$

☞ $\Theta_{\overline{\text{est}}} = \{\theta_4 := -\beta\}$.

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

☞ Equation (16) in the operational domain can be rewritten into (37)

$$\mathcal{R}\left(s, X(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right): \quad P(X(s)) + Q + \overline{Q} = 0 \qquad (17)$$

with $P = s(s^2 + \theta_1)$, $Q = s^2\theta_2 + s\theta_3, \overline{Q} = (s^2 + \theta_1)\theta_4$

☞ Thus we are looking for annihilators $\Pi \in \mathbb{R}(s)\left[\frac{d}{ds}\right]$ such that $\Pi\left(\overline{Q}\right) = 0$

☞ Annihilators $\frac{1}{s^4}\frac{d^3}{ds^3}, \frac{1}{s^5}\frac{d^4}{ds^4}s$ and many others work

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

☞ We consider annihilator of the form

$$\Pi = \sum_{i=0}^{m} \frac{a_{3,i}}{s^i}\frac{d^3}{ds^3} + \sum_{i=0}^{m} \frac{a_{2,i}}{s^i}\frac{d^2}{ds^2} + \sum_{i=0}^{m} \frac{a_{1,i}}{s^i}\frac{d}{ds} + \sum_{i=0}^{m} \frac{a_{0,i}}{s^i}.$$

☞ Since $\deg(\overline{Q}) = 2$ the minimal annihilator is of degree 2 w.r.t $\frac{d}{ds}$ (we say of order 2); we get

$$\Pi_{\min} = \sum_{i=1}^{m} \frac{a_i}{s^i}\left(s\frac{d^2}{ds^2} - \frac{d}{ds}\right)$$

leading the a family of algebraically dependent relations.

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

In order to linearly identify the two parameters θ_1, θ_3 we need a 3 order annihilator of the form

$$\Pi = \sum_{i=0}^{m} \frac{b_i}{s^i} \frac{d^3}{ds^3} + \sum_{i=1}^{m} \frac{a_i}{s^i} \left(s \frac{d^2}{ds^2} - \frac{d}{ds} \right),$$

leading to the following linear system

$$B = (A_1 A_2) \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix}, \qquad (18)$$

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

Where

$$A_1 = \sum_{i=1}^{m} \frac{1}{s^i} \left(b_i \mathcal{O}_3 + a_i \mathcal{O}_1 \right), A_2 = - \sum_{i=1}^{m} \frac{a_i}{s^i}$$

$$B = \sum_{i=1}^{m} \frac{1}{s^i} \left(b_i \mathcal{O}_4 + a_i \mathcal{O}_2 \right),$$

$$\mathcal{O}_1 = s^2 \frac{d^2 X(s)}{ds^2} + s \frac{dX(s)}{ds} - X(s), \tag{19}$$

$$\mathcal{O}_2 = s^4 \frac{d^2 X(s)}{ds^2} + 5s^3 \frac{dX(s)}{ds} + 3s^2 X(s), \tag{20}$$

$$\mathcal{O}_3 = s \frac{d^3 X(s)}{ds^3} + 3 \frac{d^2 X(s)}{ds^2}, \tag{21}$$

$$\mathcal{O}_4 = s^3 \frac{d^3 X(s)}{ds^3} + 9s^2 \frac{d^2 X(s)}{ds^2} + 18s \frac{dX(s)}{ds} + 6X(s) \tag{22}$$

Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal

Using a good choice of a_i, b_i leads to a well balanced system

$$\begin{pmatrix} \frac{1}{s^5}\mathcal{O}_1 & -\frac{1}{s^5} \\ \frac{1}{s^4}\mathcal{O}_3 & 0 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{s^5}\mathcal{O}_2 \\ \frac{1}{s^4}\mathcal{O}_4 \end{pmatrix}$$

Using (4)

$$\theta_1 = \frac{1}{t^2}\frac{\int_0^1(-w^{0,3}(\tau)+9w^{1,2}(\tau)-\frac{1}{2}w^{2,1}(\tau)+w^{3,0}(\tau))x(t\tau)d\tau}{\int_0^1(\frac{-1}{2}w^{2,3}(\tau)+\frac{1}{2}w^{3,2}(\tau))x(t\tau)d\tau}, \quad (23)$$

$$\theta_3 = \frac{5!}{t^3}\Big(\int_0^1(w^{0,2}(\tau)-5w^{1,1}(\tau)+\frac{3}{2}w^{2,0}(\tau))x(t\tau)d\tau - \theta_1\int_0^1(\frac{1}{2}w^{2,2}(\tau)-w^{3,1}(\tau)-\frac{1}{4!}w^{4,0}(\tau))x(t\tau)d\tau\Big) \quad (24)$$

Dual core Non-A

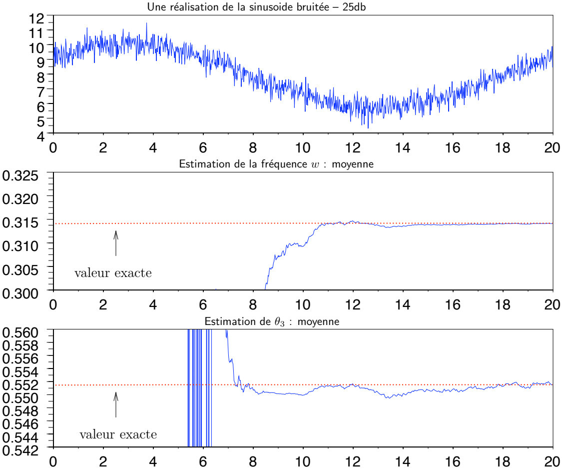
Parameter estimation of a triplet for sinusoidal biased signal

Using the annihilator (θ_1 is already known)
$\Pi_3 = -\frac{1}{2}\left(\frac{1}{s^4} + \frac{\theta_1}{s^6}\right)\frac{d}{ds} + \frac{1}{s^5}$ gives θ_2:

$$\theta_2 = \frac{1}{10}\frac{1}{t^2\theta_1}\Big(20\theta_3 t - t^3\theta_1\theta_3 + \int_0^1 \left(-t^4\theta_1^2(w^{5,0} + 5w^{4,1}) + 40t^2\theta_1(3w^{2,1} - w^{3,0}) - 120(w^{1,0} - w^{0,1})\right) x(t\tau)d\tau\Big)$$

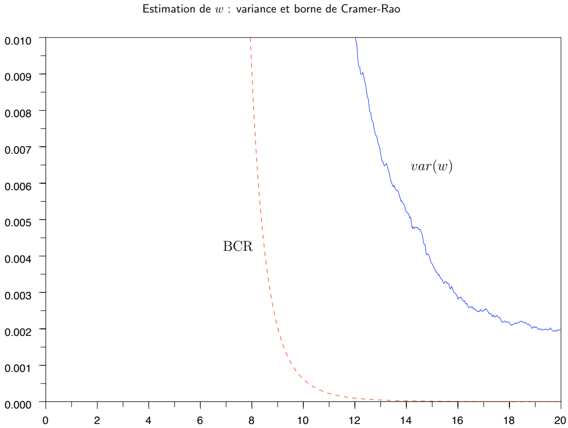
Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal



Dual core Non-A

Parameter estimation of a triplet for sinusoidal biased signal



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Position estimation from accelerometer measurements?

In nature, we can find a lot of quasi-periodic movements.
☞ The key for periodic motion is:

$$\ddot{x} = -\omega^2 x$$

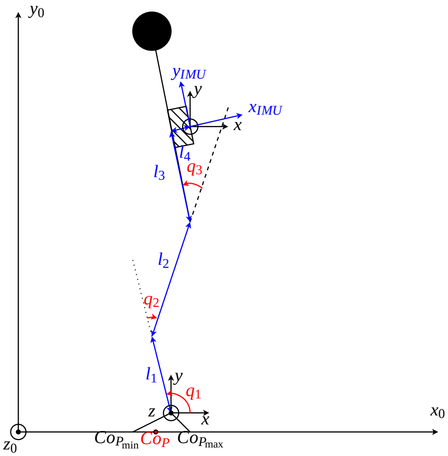
Similarly, for quasi-periodic motion, since

$$x(t) = \sum \sin(.)$$

Squat?

Video

Knowing the human posture is important for the squat



Knowing the human posture is important for the squat

The joint positions are $q = (q_1, q_2, q_3)^T$.
The geometric model is given by:

$$\begin{aligned} x &= l_1 \mathrm{c}_1 + l_2 \mathrm{c}_{12} + l_3 \mathrm{c}_{123} + l_4 \mathrm{s}_{123} \\ y &= l_1 \mathrm{s}_1 + l_2 \mathrm{s}_{12} + l_3 \mathrm{s}_{123} - l_4 \mathrm{c}_{123} \end{aligned} \tag{25}$$

where l_1, l_2, l_3 are assumed to be unknown and

$$\mathrm{c}_{i\ldots j} = \cos(q_i + \ldots + q_j), \mathrm{s}_{i\ldots j} = \sin(q_i + \ldots + q_j).$$

Knowing the human posture is important for the squat

Squat = quasi-periodic motion (repetition of such movement during the exercise).

☞ Hypothesis (realistic): on a moving small time window, each angle is approximated by a line

$$\begin{aligned} q_1(u) &= \gamma_1(u) = \omega_1 u + \phi_1, \\ (q_1 + q_2)(u) &= \gamma_2(u) = \omega_2 u + \phi_2, \\ (q_1 + q_2 + q_3)(u) &= \gamma_3(u) = \omega_3 u + \phi_3, \end{aligned} \tag{26}$$

for $u \in [t - T, t]$, where t is the current time and T the time window length.

Knowing the human posture is important for the squat

To make it short, we use the complex notation:

$$\begin{aligned} p(t) &= x(t) + iy(t) \\ &= l_1 \exp(i\gamma_1) + l_2 \exp(i\gamma_2) + (l_3 - il_4)\exp(i\gamma_3) \\ &= l_1 \exp(i\,(\omega_1 t + \phi_1)) + l_2 \exp(i(\omega_2 t + \phi_2)) \\ &\quad +(l_3 - il_4)\exp(i(\omega_3 t + \phi_3)) \end{aligned} \tag{27}$$

Taking time derivative of (27), one obtains:

$$\begin{aligned} \frac{d^{n+2}p}{dt^{n+2}} &= (i\omega_1)^{n+2} l_1 \exp(i\gamma_1) + (i\omega_2)^{n+2} l_2 \exp(i\gamma_2) \\ &\quad +(i\omega_3)^{n+2}(l_3 - il_4)\exp(i\gamma_3). \end{aligned} \tag{28}$$

Knowing the human posture is important for the squat

Using the measured acceleration

$$m_a = \ddot{p} + \varpi,$$

where ϖ is the noise and p is the position in the sagital plane written as $p = x + iy$ (see (27)), we would like to obtain an estimation of the parameters involved in the joint position description.

Knowing the human posture is important for the squat

$a_{cc} = \ddot{p}$ satisfies (use (28) with $n = 0, 1, 2$):

$$a_{cc}^{(3)} + \theta_3 a_{cc}^{(2)} + \theta_2 a_{cc}^{(1)} + \theta_1 a_{cc} = 0, \tag{29}$$

where $\theta_i, i = 1, 2, 3$ are given by

$$\theta_1 = i\omega_1\omega_2\omega_3 \tag{30}$$
$$\theta_2 = -(\omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3) \tag{31}$$
$$\theta_3 = -i\,(\omega_1 + \omega_2 + \omega_3) \tag{32}$$

Remark that the frequencies are the roots of the characteristic polynomial.

Riche de Prony is back

Parameter estimation for a sinusoidal biased signal

The generic problem consists to estimate the parameters α_k, ϕ_k and $\omega_k > 0$, $k = 1, \ldots, n$ for the signal

$$x = \sum_{i=1}^{n} \alpha_k \exp(i\omega_k t + \phi_k) \tag{33}$$

using a biased noisy measure (from sensor):

$$y = \sum_{i=1}^{n} \alpha_k \exp(i\omega_k t + \phi_k) + \beta + \varpi, \tag{34}$$

where β is an unknown constant bias and ϖ is the noise.

Riche de Prony (1795)

G. M. Riche de Prony, Essai expérimental et analytique : sur les lois de la dilatabilité de fluides élastiques et sur celles de la force expansive de la vapeur de l'eau et de la vapeur de l'alcool à différentes températures, *Journal de l'école polytechnique*, vol. 1, no. 22, pp. 24–76, 1795.

Riche de Prony is back

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- voltage control of boost converter in power electronics,
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Riche de Prony is back

Parameter estimation for a sinusoidal biased signal

There are some other technics such as

- Least Square method,
- EKF,
- non linear adaptive observers

None of these technics give the triplet within a sufficiently small time window (fraction of the signal period) and in a robust manner using noisy measures. The only quite satisfactory solutions were provided by Hebert and Dayan (unbiased case), see also GRESTI and Sysid (accepted).

☞ Here using some knowledge about minimal annihilators, we'll obtain a less noise sensitive solution...

United color of Non-A

☞ Unknown parameters $\Theta = \{\theta_1, \ldots, \theta_m\}$ classified into Θ_{est} ($r = \text{card}(\Theta_{est})$) and $\Theta_{\overline{\text{est}}} = \Theta \setminus \Theta_{\text{est}}$

☞ $x(\Theta)$ is supposed to be the output (generated) by a LTI EDO $\sum_{i=0}^{n} a_i x^{(i)} = \sum_{i=0}^{m} b_i u^{(i)}$, which in the operational domain reads as

$$\sum_{i=0}^{n} a_i s^i X(s) + \sum_{i=0}^{n} a_i \left(\sum_{j=0}^{i-1} s^{i-1-j} x^{(j)}(0) \right) = \sum_{i=0}^{m} b_i s^i U(s). \quad (35)$$

where s is the Laplace variable, $X(s)$ is the Laplace transform of the signal x ($U(s)$ of u).

United color of Non-A

☞ Estimation = find a good approximation of Θ_{est} using

$$y = x(\Theta) + \varpi, \tag{36}$$

☞ Algebraic extension
$\mathbb{R}_\Theta := \mathbb{R}(\Theta)$, $\mathbb{R}_{\Theta_{\text{est}}} := \mathbb{R}(\Theta_{\text{est}})$ and $\mathbb{R}_{\Theta_{\overline{\text{est}}}} := \mathbb{R}(\Theta_{\overline{\text{est}}})$.
☞ Then (13) reads as

$$\mathcal{R}\left(s, X(s), \Theta_{\text{est}}, \Theta_{\overline{\text{est}}}\right): \quad P(X(s)) + R(U(s)) + Q + \overline{Q} = 0 \tag{37}$$

where P and $R \in \mathbb{R}_{\Theta_{\text{est}}}[s]\left[\frac{d}{ds}\right]$, $Q \in \mathbb{R}_{\Theta_{\text{est}}}[s]$ and $\overline{Q} \in \mathbb{R}_{\Theta_{\overline{\text{est}}}}[s]$

ALIEN INRIA

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☞ Annihilator = any $\Pi \in \mathbb{R}_{\Theta_{\text{est}}}[s]\left[\frac{d}{ds}\right]$ such that $\Pi(\overline{Q}) = 0$ (most of the time independent of Θ_{est} but not mandatory !!

☞ Find a set of annihilators $\{\Pi\}_{i=1}^{r}$ such that when applying this family to $\mathcal{R}$ one obtains a set of equations which is solvable and whose solution gives use Θ_{est} (linear or not).

☞ Moreover such solution have to be compatible with formulae (4) so that in the time domain we only have integral of the measured signal (and time functions).

United color of Non-A

To go further:

$$\Pi(PQ) = \Pi(P)Q + \Pi(Q)P$$

(if Π such that $g_0 = 0$) if $Q = Q_1 + Q_2, \Pi_1(Q_1) = \Pi_2(Q_2) = 0$ then $\Pi_1\Pi_2(Q) = 0$

☞ Next step find annihilator in a systematic way ...

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A time delay example

☞ Identification (ODE) : a lot of contributions

☞ Few results for FDE (delay FDE for example): least-square, adaptive, or high-gain algorithms implies low convergence (about 100 times the delay)

☞ But for control we need to know the delay (observer, state feedback, . . .)

Introduction to the main ideas

A second simple example, with delay

$$\dot{y}(t) + ay(t) = y(0)\delta + \gamma_0 H + bu(t - \tau). \quad (38)$$

where a, b are known, γ_0 is a constant perturbation and τ is the parameter to be identified. Consider also a step input $u = u_0 H$. Distributional-like notation (δ Dirac)

$$\ddot{y} + a\dot{y} = \varphi_0 + \gamma_0\,\delta + b\,u_0\,\delta_\tau, \quad (39)$$

where δ_τ: delayed Dirac and $\varphi_0 = (\dot{y}(0) + ay(0))\,\delta + y(0)\,\delta^{(1)}$ (initial conditions).

A time delay example

Schwartz theorem, multiplication by[2] $\alpha(t) = t^3 - \tau\, t^2$:

$$\begin{aligned} t^3\,[\ddot{y} + a\dot{y}] &= \tau\, t^2\,[\ddot{y} + a\dot{y}], \\ bu_0\, t^3 \delta_\tau &= bu_0\, \tau\, t^2 \delta_\tau. \end{aligned}$$

τ available from $k \geq 1$ successive integrations (operator H):

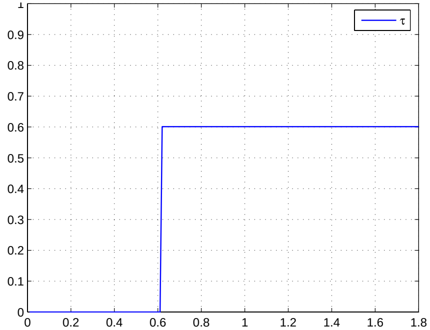
$$\tau = \frac{H^k(w_0 + a\,w_3)}{H^k(w_1 + a\,w_2)}, \qquad t > \tau, \tag{40}$$

where the w_i are defined, using the notation $z_i = t^i\, y$, by:

$$\begin{aligned} w_0 &= -t^3\, y^{(2)} = -6\, z_1 + 6\, z_2^{(1)} - z_3^{(2)}, \\ w_1 &= -t^2\, y^{(2)} = -2\, z_0 + 4\, z_1^{(1)} - z_2^{(2)}, \\ w_2 &= -t^2\, y^{(1)} = 2\, z_1 - z_2^{(1)}, \\ w_3 &= -t^3\, y^{(1)} = 3\, z_2 - z_3^{(1)}. \end{aligned}$$

[2] $\alpha(0) = \alpha'(0) = \alpha(\tau) = 0$

A time delay example



Delay τ identification from algorithm (40)

Numerical simulation with $k = 2$ integrations and $a = 2$, $b = 1$, $\tau = 0.6$, $y(0) = 0.3$, $\gamma_0 = 2$, $u_0 = 1$. Due to the non identifiability over $(0, \tau)$, the delay τ is set to zero until the numerator or denominator in the right hand side of (40) reaches a significant nonzero value.

A time delay example

It relies on the measurement of y and on the knowledge of a. If a is also unknown, the same approach can be utilized for a simultaneous identification of a and τ.

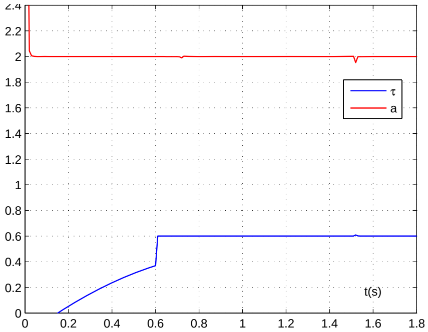
$$\tau(H^k w_1) + a\,\tau(H^k w_2) - a\,(H^k w_3) = H^k w_0, \tag{41}$$

and a linear system with unknown parameters $(\tau, a\,\tau, a)$ is obtained by using different integration orders:

$$\begin{pmatrix} H^2 w_1 & H^2 w_2 & H^2 w_3 \\ H^3 w_1 & H^3 w_2 & H^3 w_3 \\ H^4 w_1 & H^4 w_2 & H^4 w_3 \end{pmatrix} \begin{pmatrix} \widehat{\tau} \\ \widehat{a\tau} \\ -\widehat{a} \end{pmatrix} = \begin{pmatrix} H^2 w_0 \\ H^3 w_0 \\ H^4 w_0 \end{pmatrix}.$$

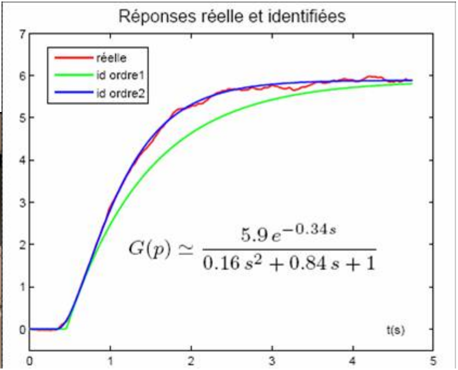
A time delay example

For identifiability reasons, the obtained linear system may be not consistent for $t < \tau$.



Simultaneous identification of a and τ from algorithm (41)

A time delay example



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Estimation: Switching signal estimation

Overview

☞ Assume from now on that all the subsystems models are known and that any pair is strongly distinguishable.

☞ Let us consider a switching system defined by a finite collection of input/output behaviors driven by LODE satisfying the above given assumptions. As soon as the system is not at rest, for the given control, the measured output can be used to determine which subsystem is active.

☞ From now we want to obtain effective real-time algorithm to determine the current "i".

Estimation: Switching signal estimation

Overview

☞ If one is able to construct in real time the following quantities

$$r_i(t) = \mathfrak{a}_i\left(\frac{d}{dt}\right)y_i - \mathfrak{b}_i\left(\frac{d}{dt}\right)u,$$

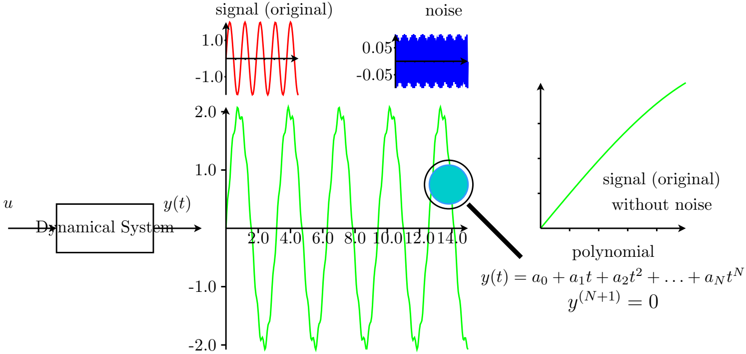
it is clear that the current "i" is such that $r_i(t) = 0$ on a sub-set of $\mathbb{R}$ with non zero measure.

☞ The problem is thus reduced to the real-time computation of time derivative of the output and input despite the noise.

Problem formulation: examples and remarks

All we want to know is in the output signal...

☞ Our point of view



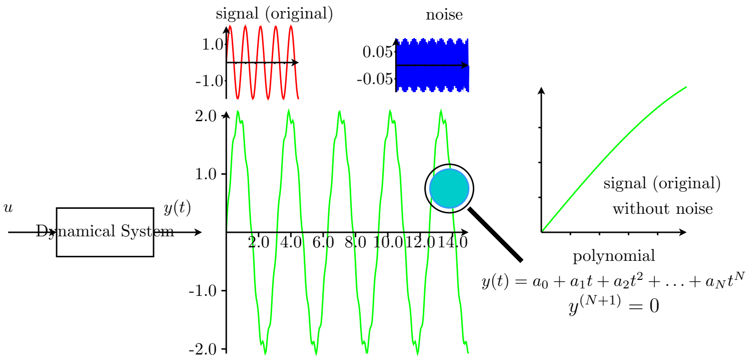
$$y(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_N t^N$$
$$y^{(N+1)} = 0$$

Obtained model relies on real-time estimations of derivatives for noisy signals

Problem formulation: examples and remarks

All we want to know is in the output signal...

☞ Our point of view



Obtained model relies on real-time estimations of derivatives for noisy signals

Estimation

Algorithm

☞ On line:

1. using Alien technics (see before) compute $y, \dot{y}, \ldots, y^{(ky_{\max})}; u, \dot{u}, \ldots, u^{(ku_{\max})}$,
2. check if $r_i(t)$ is zero for some time interval then the corresponding active subsystem is the "i-th",
3. deduce the continuous state estimate using 1.

Example

Let us consider the following switching system
$\dot{x} = A_i x + B_i u, y = C_i x$ where

$$i = 1 : \dot{y} + y = u; \qquad \begin{cases} \dot{x}_1 = -x_1 + u \\ y = x_1 \end{cases}$$

$$i = 2 : \ddot{y} + \dot{y} + y = \dot{u} + u; \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 + u \\ y = x_1 + x_2 \end{cases}$$

$$i = 3 : 2\dot{y} + y = 2u \qquad \begin{cases} \dot{x}_1 = -\frac{1}{2}x_1 + u \\ y = x_1 \end{cases}$$

$$i = 4 : \ddot{y} + \dot{y} + 2y = \dot{u} + u; \qquad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2x_1 - x_2 + u \\ y = x_1 + x_2 \end{cases}$$

Example

☞ For the first order systems, in that follows, x_2 is enforced to zero. Moreover, the output continuity is ensured between two systems whereas initial condition of derivative output is randomly chosen in $[-0.5, +0.5]$.

Residuals associated to previous systems are

$$\begin{aligned}
i = 1 &: r_i = [\dot{y}]_e + [y]_e - u \\
i = 2 &: r_i = [\ddot{y}]_e + [\dot{y}]_e + [y]_e - [\dot{u}]_e - u \\
i = 3 &: r_i = 2[\dot{y}]_e + [y]_e - 2u \\
i = 4 &: r_i = [\ddot{y}]_e + [\dot{y}]_e + 2[y]_e - [\dot{u}]_e - u
\end{aligned}$$

where $[\bullet]_e$ is the estimation of $\bullet$ and to $[y]_e$ corresponds the y denoised signal.

☞ Without noise, output derivatives are estimated according to the well known Euler's method.

Example

Free noise results: constant input

Systems 1 and 2 are indistinguishable for $y_0 = u_0 = 1$, i.e. $r_1 = r_2 = 0$.

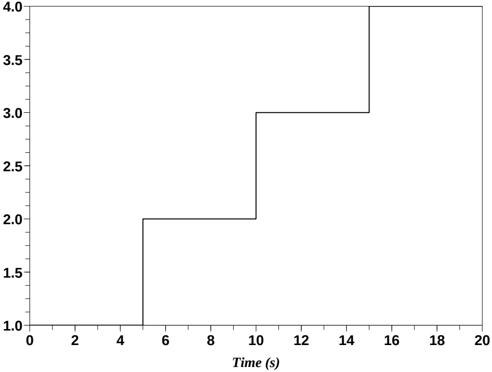


Figure: Switching signal σ

Example

Free noise results: constant input

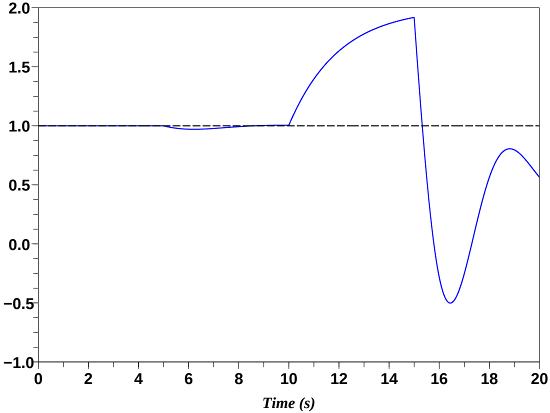


Figure: Output (–); input (- -)

Example

Free noise results: constant input

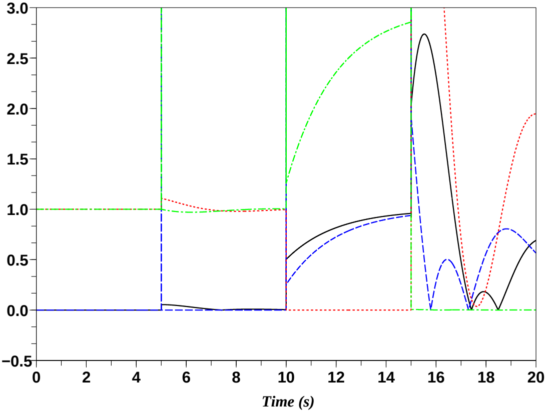


Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

Example

In the next figure, system distinguishability is easy and ensures a very good state estimation.

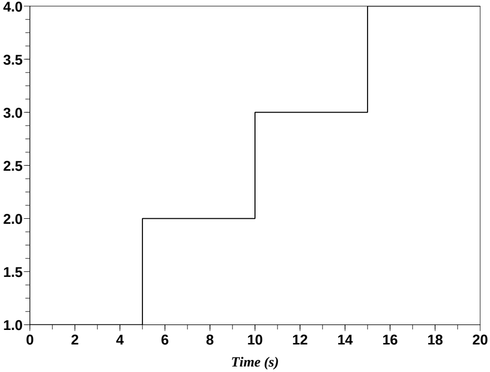


Figure: Switching signal σ

Example

Free noise results: sinusoidal input

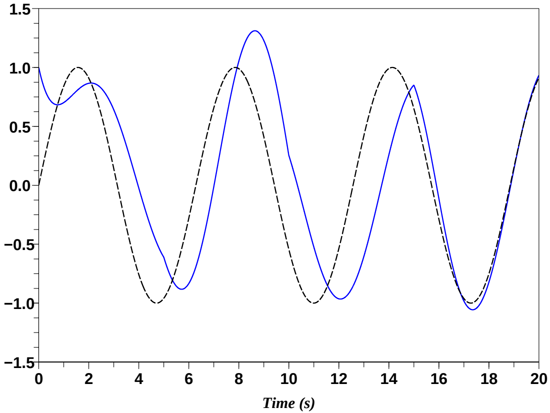


Figure: Output (–); input (- -)

Example

Free noise results: sinusoidal input

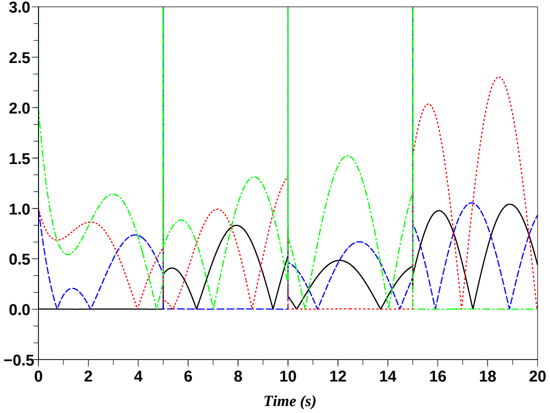


Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

Example

Free noise results: sinusoidal input

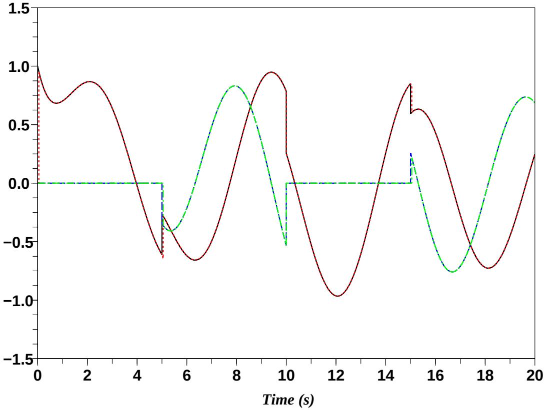


Figure: State : x_1 (–); x_2 (- -); $[x_1]_e$ (. .); $[x_2]_e$ (- .)

Example

☞In noisy case (additive output noise $N(0, 0.01)$), Euler's method is not available.

☞apply recent results on derivative estimation (see [?]) in order to evaluate residuals. They are approximately null when the associated system is active and becomes non zero in other case. However, to take the decision, that is to say to know what is the active system, is not easy (see figure **??**-(c)). Here, the mean of each residual is calculated along a sliding window. Thus at the smallest mean of residual is associated the active system. According to this logic, states are estimated.

Example

Noised results: sinusoidal input

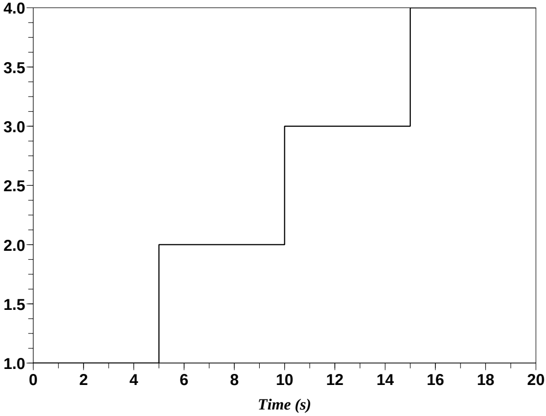


Figure: Switching signal σ

Example
Noised results: sinusoidal input

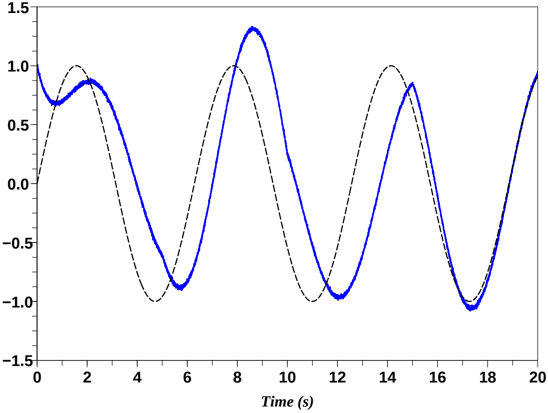


Figure: Output (–); input (- -)

Example

Noised results: sinusoidal input

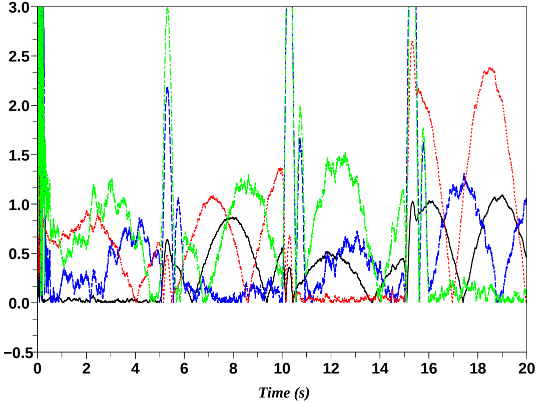


Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

Example

Noised results: sinusoidal input

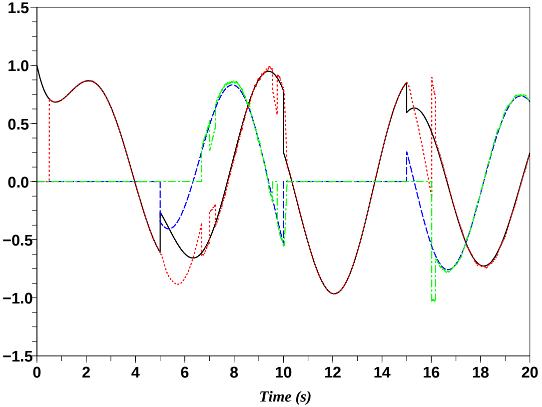


Figure: State : x_1 (–); x_2 (- -); $[x_1]_e$ (. .); $[x_2]_e$ (- .)

Example

In the previous figures, rather than to estimate output derivative in real time, a small constant and known delay is allowed for estimations (see [?] for more details). In this case, in exactly the same simulation context than previously, decision according to residuals is easier.

Example

Noised results: sinusoidal input and delayed estimations

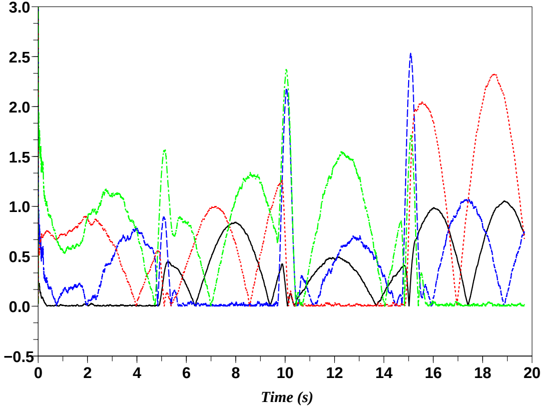


Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

Example

Noised results: sinusoidal input and delayed estimations

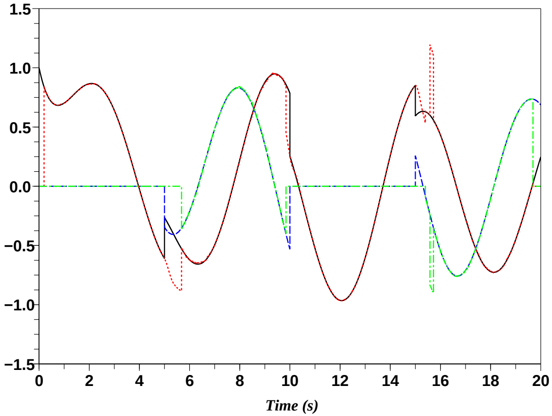


Figure: State : x_1 (–); x_2 (- -); $[x_1]_e$ (. .); $[x_2]_e$ (- .)

Example

Noised results: sinusoidal input and filtered estimations

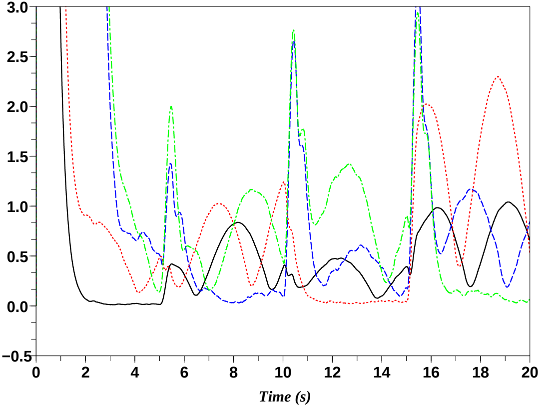


Figure: Residuals : $|r_1|$ (–); $|r_2|$ (- -); $|r_3|$ (. .); $|r_4|$ (- .)

Example

Noised results: sinusoidal input and filtered estimations

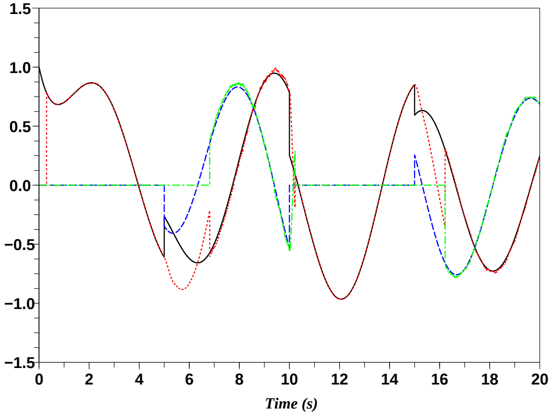


Figure: State : x_1 (–); x_2 (- -); $[x_1]_e$ (. .); $[x_2]_e$ (- .)

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Applicative fields

...in control

- *Collaborative robotics*: The cooperating devices (fleet of drones, mobile robots or UAV) have to fulfill a common objective, subject to environment perturbations and using a limited number of sensors.

Video

Applicative fields
...in control

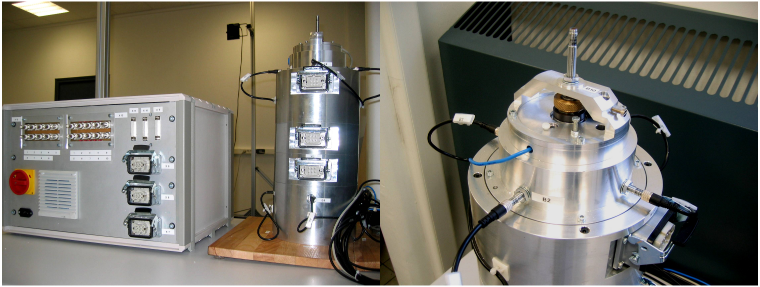
Video

Applicative fields
...in control

Video

Applicative fields
...in control

- *Magnetic levitation* (eliminating Coulomb friction): magnetic shaft benchmark for identification, state reconstruction and output feedback



Magnetic bearing benchmark (micro-meter).

Applicative fields
...in control

- *Friction*: two benchmarks linear drive actuating a cart-pendulum, and a stepper motor.

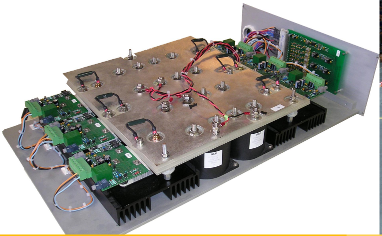
Video

Applicative fields
...in control

Video

Applicative fields
...in control

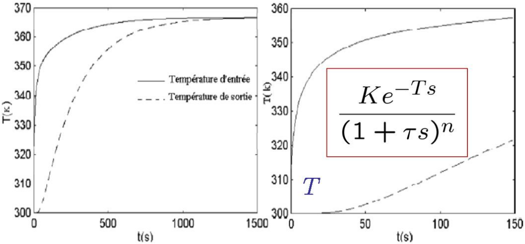
- *Multi-cell chopper* (switching component, hybrid system): observer-based control algorithm
- *Machine tools* (cooperation with ENSAM Lille): high-speed CNC machines PDE flatness-based control combine with closed-loop identification.



Applicative fields

...in control

- *Process engineering*: (chemical engineering, food industry...) can be approached efficiently by a simple linear model with input delay the objective is to design control and parameter closed-loop identification.



Steam generator.

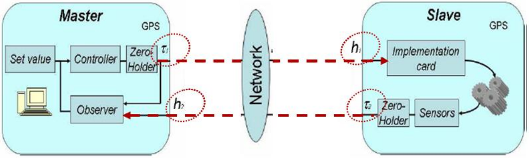
Applicative fields
...in control

- *Aeronautics* (cooperation with Flight Analysis laboratory of the DCSD of ONERA in Lille): delay-based description was introduced so to represent the effects of the penetration of the aircraft through the gust. Combining this description with a fast identification algorithm constitutes a track for the aerodynamic coefficients identification.

Video

Applicative fields
...in control

- *Networked control*: Communication networks (ethernet, wifi, internet, CAN...) have a huge impact on the flexibility and integration of control systems (remote control, wireless sensors, collaborative systems, embedded systems...). However, a network unavoidably introduces time delays in the control loops, which may put the stability and safety performances at risk. Benchmark with computer clock synchronized by GPS is available in Lille.



Networked control.

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Applicative fields

Applications to signal, image and video processing

- *compression of audio signals*,
- *demodulation and its theoretical background*,
- *compression, edge and motion detection of image and video signals.*

Video

Video

Applicative fields

...in Signal, image, and video processing

- *Multi-user detection* In the direct-sequence *code-division multiple access* (DS-CDMA) system: several users with its own *signature* use 1 channel (usually algorithm complexity grows with the number of users which seems not be the case with our technics).
- *Direction-of-arrival estimation* The problem of estimating the direction-of-arrival of multiple sources incident on a uniform array is equivalent to the estimation of some delays.
- *Turbo-codes* error control code, *turbo-equalization*: It seems that turbo-decoding might benefit from our new understanding of estimation.

Applicative fields

...in Signal, image, and video processing

- *Watermarking* a type of cryptography where a hidden message has to be inserted in an image or a video. Our approach to image and video processing has already given promising preliminary results in this field.
- *Cryptography* Pecora and Carroll (1991): synchronized two identical chaotic systems) Nijmeijer and Mareels (1997): the chaotic system synchronization problem has been intimately related to the design of a nonlinear state observer for the chaotic encoding system. Our technics should also be useful in new encryption algorithms that require fast estimation of the state variables and the masked message.