From 1rst order to Higher order Sliding modes

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Introduction

A simple stabilization problem: double integrator

How to stabilize ?

$$\boxed{\ddot{x} = u} \qquad (1)$$

F1 car.

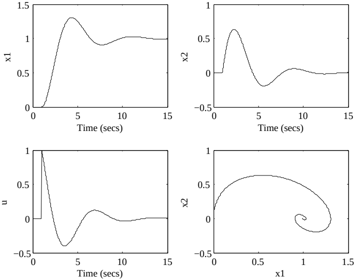
$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \end{aligned} \qquad (2)$$

Introduction

A simple stabilization problem: double integrator

Classical solution: State feedback

State feedback which is $\equiv$ to frequency approach or polynomial approach



Assume that (x_1, x_2) is available.

Stabilization even with a bounded control !!

Sate feedback: stabilization of (2) with $u = -x_1 - \frac{1}{\sqrt{2}} x_2$

Introduction

A simple stabilization problem: double integrator

A variable structure controller

If the speed is not available : Observer (dynamic extension)

An alternative solution with output feedback?

$$u = f(x_1)$$

The simplest function being:

$$u = \alpha x_1$$

(variable α)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \cancel{\alpha} x_1 \end{aligned} \quad (3)$$

Introduction

A simple stabilization problem: double integrator

Strategy 1:

position x available and the signum of $\dot{x}$ (in fact of $x\dot{x}$)
How to play with α ?

$$\ddot{x} + \alpha x = 0$$

$$x_1(t) = x_0 \cos(\sqrt{\alpha} t) + \frac{\dot{x}_0}{\sqrt{\alpha}} \sin(\sqrt{\alpha} t) \quad (4)$$

$$x_2(t) = -x_0 \sqrt{\alpha} \sin(\sqrt{\alpha} t) + \dot{x}_0 \cos(\sqrt{\alpha} t) \quad (5)$$

$$(x_0 \sqrt{\alpha} x_1 + \frac{\dot{x}_0}{\sqrt{\alpha}} x_2)^2 + (\dot{x}_0 x_1 - x_0 x_2)^2 = (x_0^2 \sqrt{\alpha} + \frac{\dot{x}_0^2}{\sqrt{\alpha}})^2 \quad (6)$$

☞ Solutions are ellipsoids

Introduction

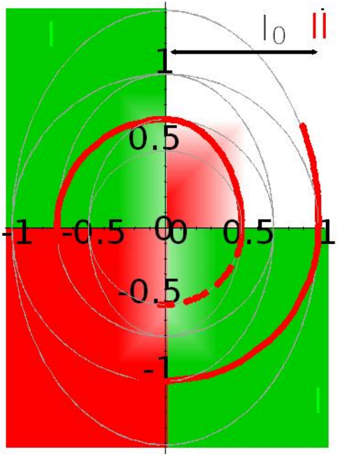
A simple stabilization problem: double integrator

Area I : $x_1 x_2 < 0, \alpha_{\mathrm{I}} = 1$

Area II : $x_1 x_2 > 0, \alpha_{\mathrm{II}} = 2$

After $2k+1$ switching:

$$l_{2k+1} = \frac{\alpha_{\mathrm{I}}}{\alpha_{\mathrm{II}}} l_0$$



Introduction

A simple stabilization problem: double integrator

Strategy 2:

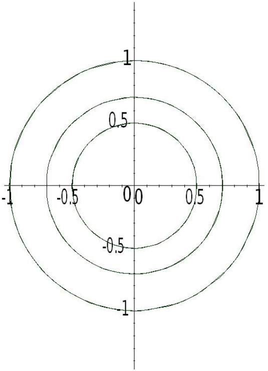
position x and velocity $\dot{x}$ are available

How to play with α ?

☞ $\alpha = 1$

$$\ddot{x} + x = 0$$

☞ Solutions are ellipsoids



Phase portrait

Introduction

A simple stabilization problem: double integrator

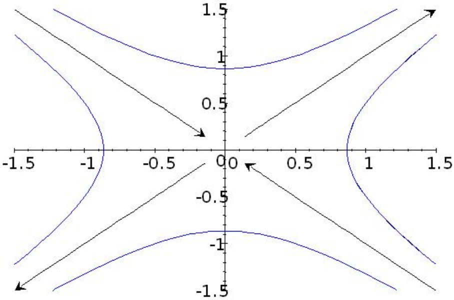
- $\alpha = -1$

$$\ddot{x} - x = 0$$

- Solutions are hyperbolas

$$x(t) = x_0 \cosh(t) + \dot{x}_0 \sinh(t)$$

Phase portrait



One stable and one unstable manifold

Introduction
A simple stabilization problem: double integrator

Area I : $\alpha_{\mathrm{I}} = -1$

$$x_1 < 0 \wedge x_1 + x_2 \geq 0$$

or

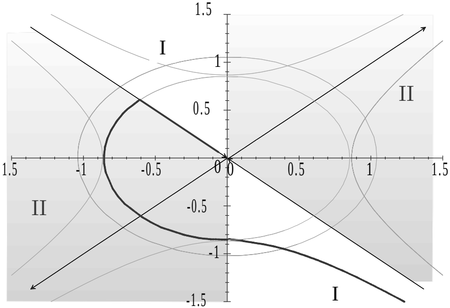
$$x_1 > 0 \wedge x_1 + x_2 \leq 0$$

Area II : $\alpha_{\mathrm{II}} = 1$

$$x_1 \leq 0 \wedge x_1 + x_2 < 0$$

or

$$x_1 \geq 0 \wedge x_1 + x_2 > 0$$



Introduction
Some first questions

Problems:

- Notion of solution,
- Discontinuous Control (damaging the actuators),
- How to find the switching logic ?

Panis (Canada 97)

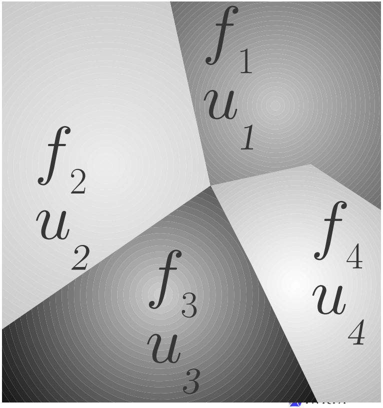
Introduction

Variable Structure System

General Problem formualtion for VSS:

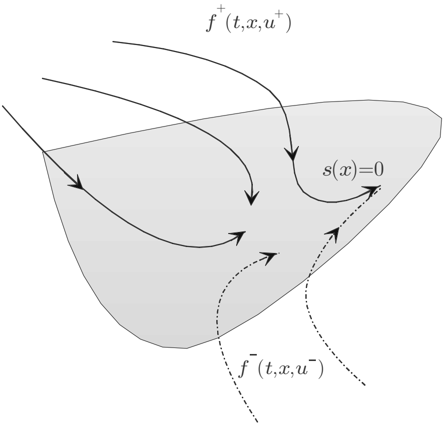
$$\dot{x} = f_i(t, x, u_i)$$

Find the switching logic and the control ?



Introduction
Sliding Mode Control

SMC (1rst order and Higher):



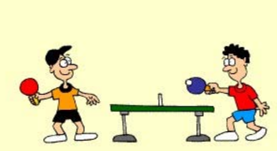
"Slap" principle

Introduction

Sliding Mode Control

Objective

To constrain the trajectories of system $\dot{x} = f(x) + g(x)u$ to reach, in a finite time, and then, to stay onto the sliding surface chosen according to the control objectives



Sliding mode control

$$u = \begin{cases} u^+(s) & \text{if } \operatorname{sign}(s(x)) > 0 \\ u^- s) & \text{if } \operatorname{sign}(s(x)) < 0 \end{cases} \quad \text{with} \quad u^+ \neq u^-$$

A simple sliding mode control design

$$u = u_{eq} + u_{disc}$$

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Sliding Mode Control

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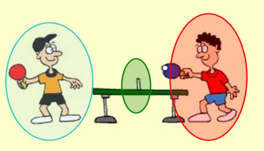

Sliding mode control

$$u = \begin{cases} u^{+}(s) & \text{if } \operatorname{sign}(s(x)) > 0 \\ u^{-}s) & \text{if } \operatorname{sign}(s(x)) < 0 \end{cases} \qquad \text{with} \quad u^{+} \neq u^{-}$$

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A simple sliding mode control design

$$u = u_{eq} + u_{disc}$$

- given by $s = \dot{s} = 0$, (invariance of the sliding surface)
- $u_{disc} = -k\text{sign}(s)$, (convergence in finite time onto the surface)

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Sliding Mode Control

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Sliding mode control

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A simple sliding mode control design

$$u = u_{eq} + u_{disc}$$

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Introduction

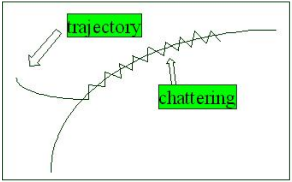
Sliding Mode Control: Advantages vs disadvantages

Advantages:

- System order reduction
- Finite time convergence (adjust time response)
- Robustness w.r.t. parametric uncertainties
  and disturbances

Disadvantages:

- Chattering phenomena (actuator damage)
- Noise sensitivity (??)
- Output feedback (??)



Introduction

Sliding Mode Control: One more time ...

Sliding mode control design:

- hitting phase (or reaching phase), and the
- sliding phase.

stability/attractivity concepts:

- existence of sliding motions is a contraction property (locally),
- shaping procedure: stabilization problem ("tune" the shape of the sliding : in sliding minimum phase system).

Introduction

Sliding Mode Control: One more time ...

$$\begin{cases} \dot{x}_1 = \frac{x_2}{(1+x_2^2)} - 2\frac{x_1 x_2}{1+x_2^2}u \\ \dot{x}_2 = u \end{cases} \tag{7}$$

This system "seems" complex, however, if we set

$$z_1 = x_1(1 + x_2^2)$$
$$z_2 = x_2$$

(note that it defines a global diffeomorphism), then one obtains

Introduction

Sliding Mode Control: One more time ...

$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = u \end{cases} \tag{8}$$

and it becomes obvious that if in sliding mode $z_2 = -z_1$, then z_1 converges asymptotically to zero $(\dot{z}_1 = z_2 = -z_1)$ and thus z_2 also converges. In this step of design (the "sliding phase"), the shape of the sliding manifold arises naturally.

Introduction

Sliding Mode Control: One more time ...

Now, we need to force the system to evolve on the constraint $z_2 = -z_1$. For this, let us define the sliding surface as

$$\mathcal{S} = \{z \in \mathbb{R}^2 : s(z) = 0\} \tag{9}$$

$$s(z) = z_2 + z_1 \tag{10}$$

Then, according to the equivalent control method, we need the control to satisfy

$$u(z) = \begin{cases} u^+(z) \text{ if } s(z) > 0 \\ u^-(z) \text{ if } s(z) < 0 \end{cases}$$

$$\min(u^+(z), u^-(z)) < u_{\text{eq}} = -z_2 < \max[u^+(z), u^-(z)]$$

in order to ensure that a sliding mode exists on $\mathcal{S}$.

Introduction

Sliding Mode Control: One more time ...

This leads to various design controls, for example,

$$u(z) = \begin{cases} -1 \text{ if } s(z) > 0 \\ 1 \text{ if } s(z) < 0 \end{cases}$$

which ensures a finite time convergence to $\mathcal{S}$ as soon as the initial conditions are close enough to the surface and satisfy $|z_2| < 1$.
But, can we provide a better characterization of the initial conditions leading to a sliding mode?

Introduction

Sliding Mode Control: One more time ...

An alternative to this control is

$$u(z) = \begin{cases} -z_2 - 1 \text{ if } s(z) > 0 \\ -z_2 + 1 \text{ if } s(z) < 0 \end{cases} \tag{11}$$

which ensures a finite time convergence to $\mathcal{S}$, whatever the initial conditions. But since the chattering problem remains, can we stabilize the system while reducing the chattering?

Differential Inclusion: Notion of solution

$$\dot{x} = f(x, x), \forall x \in \mathcal{X} \setminus \mathcal{S} \tag{12}$$

where $\mathcal{X}$ is the state manifold (locally diffeomrophic to $\mathbb{R}^n$).

Problem : f is not defined on a manifold of codimension one (if $\mathcal{S} = \{x \in R^n : s(x) = 0\}$ and s is a scalar function) thus Cauchy-Lipschitz and Peano Theorem does not apply (existence (and uniqueness) of solutions).

Notion of solutions on the manifold : extend the vector field f on the manifold $\mathcal{S}$. (Aizerman, Filipov, Utkin, ...)

Differential Inclusion: Notion of solution

Main points of view :

- Real world (system is not discontinuous), just take into account (delays, hysterisis, saturation) in a small vicinity of the sliding manifold $\mathcal{S}_\varepsilon = \{x \in R^n : \|s(x)\| \leq \varepsilon\}$ (ε radius), then use the usual results, then $\varepsilon \to 0$: Sliding mode are then limit of "classical solution". That is Aizerman's point of view
- embed the discontinuous system into a Differential Inclusion (Filipov),
- Equivalent control theory (Utkin).

Differential Inclusion: Notion of solution

Filipov's points of view : replace the ODE with discontinuous right-hand side

$$\dot{x} = f(t,x), \forall x \in \mathcal{X} \setminus \mathcal{S} \subset \mathbb{R}^n$$

with the following differential inclusion

$$\dot{x} \in F(t,x)$$

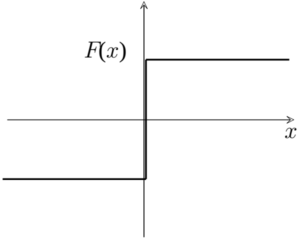
which capture the behaviors of the original system, where

$$F(t,x) = \bigcap_{\varepsilon>0} \bigcap_{\mu(M)=0} \overline{\text{conv}}(f(t, B_\varepsilon(x) - M))$$

Differential Inclusion: Notion of solution

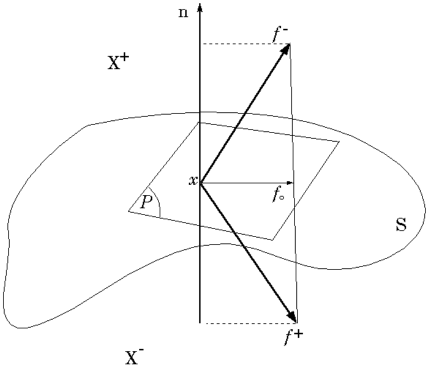
$$\dot{x} = -\text{sign}(x), x \in \mathbb{R}$$

$$F(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x > 0 \\ [-1, 1] & \text{if} \quad x = 0 \end{cases}$$



Signum

Differential Inclusion: Notion of solution



$$x \in \mathcal{S}$$

$$\dot{x} = f_0(t, x)$$

f_0 **should be in** $T_x\mathcal{S}$**.**

$$f_0(t,x) \in \overline{\text{conv}}\{f^+(t,x), f^-(t,x)\} \cap T_x\mathcal{S}$$

Differential Inclusion: Notion of solution

$$
\begin{aligned}
f_0 &= \alpha f^+ + (1-\alpha) f^-, \alpha \in [-1, 1] \\
f_0(t,x) \in T_x\mathcal{S} &\Leftrightarrow \langle ds, f_0 \rangle = 0 \\
&\Leftrightarrow \alpha \langle ds, f^+ \rangle + (1-\alpha)\langle ds, f^- \rangle = 0 \\
\alpha &= \frac{\langle ds, f^- \rangle}{\langle ds, f^- - f^+ \rangle} \\
\dot{x} &= f_0 = \frac{\langle ds, f^- \rangle}{\langle ds, f^- - f^+ \rangle} f^+ - \frac{\langle ds, f^+ \rangle}{\langle ds, f^- - f^+ \rangle} f^-
\end{aligned}
$$

Differential Inclusion: Notion of solution

Utkin's points of view : On the sliding manifold, replace the dynamics by the following ODE (equivalent dynamics)

$$\dot{x} = f_{eq}(t, x, u_e q)$$

where $f_{eq}(t, x, u_e q)$ ensure invariance of the sliding manifold that is

$$f_{eq}(t, x, u_e q) : s(x(t)) = 0, \forall t > 0$$

thus s is identically zero which implies that $\dot{s}$ is also zero

Remark

Filipov and Utkin thechnics are equivalent only for system linear in the control that is $\dot{x} = f(x) + g(x)u$.

Differential Inclusion: Notion of solution

Counter example :

$$\dot{x}_1 = 0.3x_2 + x_1 u \quad (13)$$

$$\dot{x}_2 = -0.7x_1 + 4x_1 u^3 \quad (14)$$

Sliding manifold defined by:

$$s(x) = x_2 + x1$$

Control: $u = -\text{sign}(s(x)x_1)$.

Differential Inclusion: Notion of solution

Sliding mode occurs if $s\dot{s} < 0$ (close to) :

$$\dot{s} = 0.3x_2 + x_1 u - 0.7x_1 + 4x_1 u^3 \quad (15)$$

$$\dot{s} = 0.3x_2 - 0.7x_1 + x_1 u(1 + 4u^2) \quad (16)$$

If $s(x) \simeq 0, x_2 \simeq -x_1$

$$\dot{s} = -x_1 + x_1 u(1 + 4u^2) \quad (17)$$

$$s\dot{s} = -sx_1 - 5|sx_1| < 0 \quad (18)$$

☞ Yes sliding will occur

Differential Inclusion: Notion of solution

Equivalent dynamics (Filipov):

$$f^+(x) = \begin{pmatrix} 0.3x_2 + x_1 \\ 3.3x_1 \end{pmatrix}, f^-(x) = \begin{pmatrix} 0.3x_2 - x_1 \\ -4.7x_1 \end{pmatrix}, \quad (19)$$

$$\alpha = \frac{\begin{pmatrix} 1 & 1 \end{pmatrix} f^-(x)}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} -2x_1 \\ -8x_1 \end{pmatrix}} = \frac{0.3x_2 - 5.7x_1}{-10x_1}$$

When x close to the sliding manifold ($x_2 \simeq -x_1$) we have $\alpha = \frac{-6x_1}{-10x_1} = \frac{6}{10}$ thus the equivalent dynamics is

$$\begin{aligned} \dot{x}_1 &= \alpha(0.3x_2 + x_1) + (1-\alpha)(0.3x_2 - x_1) & (20) \\ &= \frac{6}{10}(0.3x_2 + x_1) + \frac{4}{10}(0.3x_2 - x_1) & (21) \\ &= 0.3x_2 + 0.2x_1 = -0.1x_1 & (22) \end{aligned}$$

☞ AS

Differential Inclusion: Notion of solution

Equivalent dynamics (Utkin):

$$\dot{s} = 0.3x_2 + x_1 u - 0.7x_1 + 4x_1 u^3$$

When x close to the sliding manifold ($x_2 \simeq -x_1$) we have

$$\begin{aligned} \dot{s} &= -x_1 + x_1 u(1 + 4u^2) = 0 && (23) \\ &\Leftrightarrow u(1 + 4u^2) = 1 \vee x_1 = 0 && (24) \end{aligned}$$

$$u(1 + 4u^2) = 1 \Leftrightarrow u = 0.5, u \in \mathbb{R}$$

$$\dot{x}_1 = 0.3x_2 + 0.5x_1$$

When x close to the sliding manifold ($x_2 \simeq -x_1$)

$$\dot{x}_1 = 0.2x_1$$

☞ Unstable

Sliding Mode Control : First order sliding mode

☞ Non Linear affine systems

$$\dot{x} = f(x) + g(x)u \tag{25}$$

☞ Sliding manifold being defined by a C^1 function (same dimension as u)

$$\mathcal{S} = \{x \in R^n : s(x) = 0\} \tag{26}$$

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Sliding Mode Control : First order sliding mode

Attractivity and invariance condition

$$u = \begin{cases} u^+ & \text{if} \quad s(x) > 0 \\ u^- & \text{if} \quad s(x) < 0 \end{cases} \tag{27}$$

- Attractivity condition:

$$s^T \dot{s} < 0 \Leftrightarrow \min(u^+, u^-) < u_{eq} < \max(u^+, u^-) \tag{28}$$

- Invariance condition:

$$\dot{s} = 0 \Leftrightarrow u = u_{eq} \tag{29}$$

Sliding Mode Control : First order sliding mode

Attractivity and invariance condition

Be careful, this condition does not imply that the sliding manifold is reached in finite time. Thus, this condition (for the existence of a sliding mode) should be replaced by a more restrictive condition for example (mu-reachability condition)

$$s^T \dot{s} < -\mu s \tag{30}$$

Show that $V(s) = s^T s$ goes to zero in finite time

Sliding Mode Control : First order sliding mode

Attractivity and invariance condition

Let us consider a linear system

$$\dot{x} = Ax + Bu \tag{31}$$

with a linear sliding surface

$$\mathcal{S} = \{x \in R^n : s(x) = Cx\} \tag{32}$$

$$s^T \dot{s} = s^T C(Ax + Bu) < 0 \tag{33}$$

Equivalent control if CB invertible

$$\dot{s} = 0 \Leftrightarrow u_{eq} = -(CB)^{-1} CAx \tag{34}$$

If $u = -(CB)^{-1} K \text{sign}(s) + u_{eq}, s^T \dot{s} = -\sum_{i=1}^{m} k_i |s_i| < 0$

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Sliding Mode Control : First order sliding mode

Sliding mode equivalent dynamics

$$\dot{s} = \frac{\partial s}{\partial x}(f(x) + g(x)u) = \mathcal{L}_f s + \mathcal{L}_g s u$$

Equivalent control if $\mathcal{L}_g s$ invertible

$$\dot{s} = 0 \Leftrightarrow u_{eq} = -(\mathcal{L}_g s)^{-1}\mathcal{L}_f s \tag{35}$$

Thus the equivalent dynamics are

$$\dot{x} = f(x) + g(x)\left(-(\mathcal{L}_g s)^{-1}\mathcal{L}_f s\right) \tag{36}$$

$$= \left(Id - g(x)\left(-(\mathcal{L}_g s)^{-1}\frac{\partial s}{\partial x}\right)\right) f(x) \tag{37}$$

$\left(Id - g(x)\left(-(\mathcal{L}_g s)^{-1}\frac{\partial s}{\partial x}\right)\right)$ is a projection operator

Sliding Mode Control : First order sliding mode

Sliding mode equivalent dynamics

Let us consider a linear system

$$\dot{x} = Ax + Bu \tag{38}$$

Equivalent control if CB invertible

$$u_{eq} = -(CB)^{-1}CAx \tag{39}$$

$$\dot{x} = (Id - B(CB)^{-1}C)Ax \tag{40}$$

$$= A_{eq}x \tag{41}$$

$$A_{eq} = (Id - B(CB)^{-1}C)A \tag{42}$$

Sliding Mode Control : First order sliding mode

Sliding mode equivalent dynamics

Using a change of coordinates one can obtain $(B_2 \in \mathcal{M}_m(\mathbb{R}))$

$$B = \begin{pmatrix} 0 \\ B_2 \end{pmatrix}, A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}.$$

$$A_{eq} = \begin{pmatrix} A_{11} & A_{12} \\ -C_2^{-1}C_1A_{11} & -C_2^{-1}C_1A_{12} \end{pmatrix} \quad (43)$$

$$= P^{-1}\begin{pmatrix} A_{11} - A_{12}C_2^{-1}C_1 & A_{12} \\ 0 & 0 \end{pmatrix} P \quad (44)$$

☞ A_{eq} has at least m zero eigenvalues and at most $n-m$ non zero ones.

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Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

$$\dot{x} = Ax + Bu + p \quad (45)$$

$$p \in \mathrm{span}(B) \quad (46)$$

☞ (46) is called the matching condition, thus we have $p = Bp^*$.
Put (45) into a controllable canonical form (hereafter $m = 1$)

$$\dot{x}_c = A_c x + B_c(u + p^*) \quad (47)$$

$$A_c = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 0 & 1 \\ -a_{c1} & \dots & \dots & \dots & -a_{cn} \end{pmatrix}, B_c = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (48)$$

Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

☞ Select a linear sliding manifold $\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = 0\}$ where

$$s(x) = x_{cn} + \sum_{i=1}^{n-1} a_i x_{ci}$$

$$\begin{aligned} \dot{s} &= -\sum_{i=1}^{n} a_{ci} x_{ci} + u + p^* + \sum_{i=1}^{n-1} a_i x_{ci+1} && (49) \\ &= \sum_{i=1}^{n} a_{ci}^{\diamond} x_{ci} + u + p^*, a_{ci}^{\diamond} = -a_{ci} + a_{ci-1} && (50) \end{aligned}$$

Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

☞ Control

$$u = -k\text{sign}(s) - \sum_{i=1}^{n} a_{ci}^{\diamond} x_{ci} \quad (51)$$

$$s\dot{s} = -k|s| + |p^*||s|, \quad (52)$$

If the disturbance is bounded $\sup|p^*| < \infty$, then take $k = \mu + \sup|p^*|$

$$s\dot{s} < -\mu|s|$$

Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

☞ Equivalent dynamics

$$\dot{x}_{c1} = x_{c2} \tag{53}$$

$$\vdots = \vdots \tag{54}$$

$$\dot{x}_{cn-2} = x_{cn-1} \tag{55}$$

$$\dot{x}_{cn-1} = x_{cn} = -\sum_{i=1}^{n-1} a_i x_{ci} \tag{56}$$

a_i (Hurwitz) ☞ No influence of the perturbation once the sliding manifold is reached (only the hitting phase is influenced)

Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

Example: double integrator

$$\dot{x}_1 = x_2 \tag{57}$$

$$\dot{x}_2 = u + p, \sup |p| < \infty \tag{58}$$

- ☞ Sliding manifold: $\mathcal{S} = \{x \in \mathbb{R}^n : s(x) = 0\}, s(x) = x_2 + a_1 x_1$
- ☞ Compute Equivalent control (without disturbance $p = 0$)

$$\dot{s} = 0 = u_{eq} + a_1 x_2 \Leftrightarrow u_{eq} = -a_1 x_2$$

Sliding Mode Control : First order sliding mode

Robustness with respect to matched disturbance

Example: double integrator

☞ Control driving the solutions to S in finite time

$$u = u_{eq} + u_{disc}, u_{disc} = -k\mathrm{sign}(s)$$

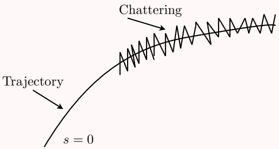
$$s\dot{s} = s(u_{disc} + p) < -\mu|s|, k = \mu + \sup|p|.$$

Sliding Mode Control: First order sliding mode

Advantages

- Insensibility against perturbations (matching perturbations)
- The choice of surface $s(x,t) = 0$ allow to choose *a priori* the closed-loop dynamics

Disadvantages



- Chattering phenomenon
- $s(x,t)$ must have a relative degree equal to 1 wrt. u
- The trajectories are not robust against perturbations during the reaching phase

Sliding Mode Control: First order sliding mode

Advantages

- Insensibility against perturbations (matching perturbations)
- The choice of surface $s(x,t)=0$ allow to choose *a priori* the closed-loop dynamics

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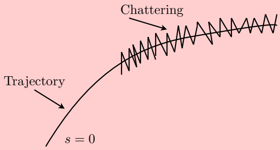


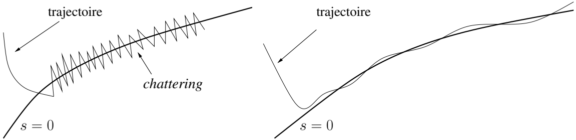
- Chattering phenomenon
- $s(x,t)$ must have a relative degree equal to 1 wrt. u
- The trajectories are not robust against perturbations during the reaching phase

Higher Order Sliding Mode Control

Objective

To constrain the system trajectories to evolve onto the sliding surface:

$$\mathcal{S}_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\}$$



mode glissant d'ordre $r = 1$ mode glissant d'ordre $r > 1$

Higher Order Sliding Mode Control

- Introduced by A. Levant (Ph. D. supervisor Emel'yanov) in 87
- Ideal :

$$\mathcal{S}_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\}$$

- Real :

$$\begin{aligned} |s| &= O(T_s^r) && (59) \\ |\dot{s}| &= O(T_s^{r-1}) && (60) \\ \ldots &= \ldots && (61) \\ \left|s^{(r-1)}\right| &= O(T_s) && (62) \\ T_s &= \text{sampling period} && (63) \end{aligned}$$

With respect to a bounded deterministic Lebesgue-measurable noise (bounded by ε): $|s| = O(\varepsilon^{1/2^{r-1}})$

Higher Order Sliding Mode Control

☞ Advantages:

- Robustness w.r.t. bounded matching perturbation,
- Reduce the of the sliding dynamics up to at most $(n-r)$ (in fact if counting the added integrators exactly $(n-r)$).
- Finite Time convergence to $\mathcal{S}_r$,
- Chattering reduction (sometimes see relative degree of s),
- Higher convergence accuracy.

Higher Order Sliding Mode Control

$$\mathcal{S}_r = \left\{ x \in \mathbb{R}^n : s = \dot{s} = \ldots = s^{(r-1)} = 0 \right\} \tag{64}$$

Let the set $\mathcal{S}_r$ be non-empty and assume that it consists of Filippov's trajectories of the discontinuous dynamic system.

Definition

Any motion (Filipov sense) in the set $\mathcal{S}_r$ is called an r-sliding mode with respect to the constraint function s.

Higher Order Sliding Mode Control

☞ Sliding mode and relative degree

$$\dot{x} = f(t, x, u), s = s(t, x)$$

Theorem (H. Sira-Ramirez 89)

A first order sliding mode exists iff the relative degree of s w.r.t the above defined system is one.

Equivalent dynamics is stable $\Leftrightarrow$ system is minimum phase w.r.t s. Relative degree r strictly greater than one : Only an r-sliding mode algorithm leads to a finite time convergence on the sliding manifold.

Higher Order Sliding Mode Control

☞ Problem : find algorithms ensuring higher order sliding modes. There exist for $r = 1, 2$ and 3 for any $r > 3$ there no satisfactory constructive algorithm (only the structure is proposed and existence is proved for large enough parameters)

☞ Ideal Algorithms:

- The necessary information increase with the order
- Twisting and Super-twisting [Levant]
- Sub-optimal [Bartolini]
- Nested HOSM [Levant]
- Quasi-continuous HOSM [Levant]

☞ Real Algorithms:

- Good approximation for 2nd order
- Drift algorithm [Emel'yanov]
- Discretized version of ideal ones

Higher Order Sliding Mode Control

2-order sliding mode algorithms

$$\ddot{s} = a(t,x) + b(t,x,u)u$$

Hypothesis:

1. For any continuous $u(t)$ s.t. $|u| \leq U_M, U_M > 1$ the solution of the system is well defined for all t.
2. $\exists u_1 \in (0,1)$ s.t. for any continuous function $u(t)$ with $|u(t)| > u_1$, $\exists t_1$, s.t $s(t)u(t) > 0$ for each $t > t_1$. ($u(t) = -\mathrm{sign}[s(t_0)]$, enforces $s = 0$ in finite time)
3. $\exists s_0 > 0, u_0 < 1, \Gamma_m > 0, \Gamma_M > 0$ such that if $|s(t,x)| < s_0$ then

$$0 < \Gamma_m \leq |b(t,x,u)| \leq \Gamma_M \quad , \forall |u| \leq U_M, x \in \mathcal{X} \tag{65}$$

   and the inequality $|u| > u_0$ entails $\dot{s}u > 0$.

4. $\exists A > 0$ s.t within $|s(t,x)| < s_0$ the following inequality holds $\forall t, x \in \mathcal{X}, |u| \leq U_M$

$$|a(t,x)| \leq A \tag{66}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

$\partial r(s) = 1$

$y_1 = s, y_2 = \dot{s}$, after some transient

$|a(t,x)| \leq A, 0 < \Gamma_m \leq b(t,x,u) \leq \Gamma_M, A > 0.$

$$\left\{ \begin{array}{ll} \dot{y}_1 & = y_2 \\ \dot{y}_2 & = a(t,x) + b(t,x,u)\dot{u} \end{array} \right. \quad (67)$$

with $y_2(t)$ unmeasured but with a possibly known sign.

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

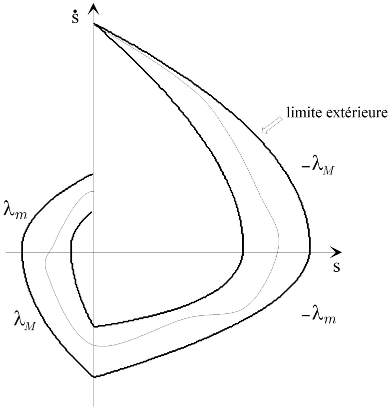
$$\dot{u}(t) = \begin{cases} -u & if \quad |u| > 1, \\ -\lambda_m \text{sign}(y_1) & if \quad y_1 y_2 \leq 0; \; |u| \leq 1, \\ -\lambda_M \text{sign}(y_1) & if \quad y_1 y_2 > 0; \; |u| \leq 1. \end{cases} \tag{68}$$

Sufficient conditions:

$$\begin{array}{l} \lambda_M > \lambda_m \\ \lambda_m > \frac{4\Gamma_M}{s_0} \\ \lambda_m > \frac{A}{\Gamma_m} \\ \Gamma_m \lambda_M - A > \Gamma_M \lambda_m + A. \end{array} \tag{69}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]



Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

$\underline{\partial r(s) = 2}$

$$u(t) = \begin{cases} -\lambda_m \text{sign}(y_1) & if \quad y_1 y_2 \leq 0 \\ -\lambda_M \text{sign}(y_1) & if \quad y_1 y_2 > 0 \end{cases}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

$$\dot{x}_1 = x_2 \tag{70}$$

$$\dot{x}_2 = x_3 \tag{71}$$

$$\dot{x}_3 = x_1 x_2 + u + p(t) \tag{72}$$

$$\sup_{t \in \mathbb{R}} |p(t)| = \pi \tag{73}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Using (70) : for

$$s_1(x) = x_2 + ax_1 \tag{74}$$

we have

$$\dot{s}_1 = x_3 + ax_2, \tag{75}$$

$$\ddot{s}_1 = x_1x_2 + u + p(t) + ax_3 \tag{76}$$

$$\partial r(s_1) = 2, \tag{77}$$

thus if $s_1(x) = \dot{s}_1(x) = 0$ in finite time then the equiv. dynamicS is $\dot{x}_1 = -ax_1$ thus $x_1(t) \to 0, x_2(t) \to 0, x_3(t) \to 0.$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Using (70) + (71) ($\dot{x}_1 = x_2, \dot{x}_2 = x_3$) : for

$$s_2(x) = x_3 + (\omega_n^2 x_1 + 2\zeta\omega_n x_2) \quad (78)$$

we have

$$\dot{s}_2 = x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta\omega_n x_3), \quad (79)$$

$$\partial r(s_2) = 1, \quad (80)$$

thus if $s_2(x) = 0$ in finite time then the equiv. dynamics is $\dot{x}_1 = x_2, \dot{x}_2 = -(\omega_n^2 x_1 + 2\zeta\omega_n x_2)$ thus $x_1(t) \to 0, x_2(t) \to 0, x_3(t) \to 0$.

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Case 1: 1rst order SM using s_2

$$
\begin{aligned}
s_2(x) &= x_3 + (\omega_n^2 x_1 + 2\zeta\omega_n x_2),\\
\dot{s}_2 &= x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta\omega_n x_3),
\end{aligned}
$$

Compute equiv. control (without p):
$u_{eq} = -x_1 x_2 - (\omega_n^2 x_2 + 2\zeta\omega_n x_3)$

$$
\begin{aligned}
u &= u_{eq} + u_{disc},\\
u_{disc} &= -k\text{sign}(s), k > \pi + \mu\\
s\dot{s} &= -k|s| + sp < -\mu|s|
\end{aligned}
$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Case 2: 2nd order SM using s_2 Since $\partial r(s_2) = 1$ we add $\int$:
Chattering removal

$$\dot{x}_1 = x_2 \quad (81)$$
$$\dot{x}_2 = x_3 \quad (82)$$
$$\dot{x}_3 = x_1 x_2 + u + p(t) \quad (83)$$
$$\dot{u} = v \quad (84)$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Thus we have

$$
\begin{aligned}
s_2(x) &= x_3 + (\omega_n^2 x_1 + 2\zeta\omega_n x_2), \\
\dot{s}_2 &= x_1 x_2 + u + p(t) + (\omega_n^2 x_2 + 2\zeta\omega_n x_3), \\
\ddot{s}_2 &= x_1 x_3 + x_2^2 + v + \dot{p} + (\omega_n^2 x_3 + 2\zeta\omega_n (x_1 x_2 + u + p(t))) \\
&= a(x) + v + (\dot{p} + 2\zeta\omega_n p)
\end{aligned}
$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Compute equiv. control (without p): $v_{eq} = -a(x)$

$$\begin{aligned} v &= v_{eq} + v_{disc}, \\ v_{disc} &= TA(s_2) = \begin{cases} -\lambda_m \text{sign}(s_2) & if \;\; s_2 \dot{s}_2 \leq 0 \\ -\lambda_M \text{sign}(s_2) & if \;\; s_2 \dot{s}_2 > 0 \end{cases} \\ u &= \int v \in C^0 \end{aligned}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

Example

Case 3: 2nd order SM using s_1 Since $\partial r(s_1) = 2$ we can directly use TA (Chattering!!)

$$\dot{s}_1 = x_3 + ax_2, \quad (85)$$

$$\ddot{s}_1 = x_1x_2 + u + p(t) + ax_3 \quad (86)$$

Compute equiv. control (without p): $v_{eq} = -x_1x_2 - ax_3$

$$u = u_{eq} + u_{disc},$$

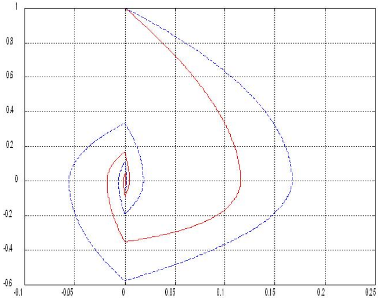
$$u_{disc} = TA(s_1) = \begin{cases} -\lambda_m \text{sign}(s_1) & if \;\; s_1\dot{s}_1 \leq 0 \\ -\lambda_M \text{sign}(s_1) & if \;\; s_1\dot{s}_1 > 0 \end{cases}$$

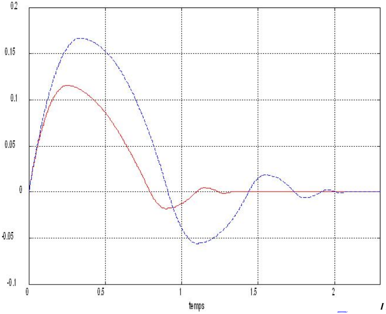
Higher Order Sliding Mode Control

2-order sliding mode algorithms: Twisting Algorithm (TA) [Levant]

☞ Convergence acceleration TA + pole placement (at the same location !!)

$$u = -\alpha^2 s - 2\alpha\dot{s} + \begin{cases} -\lambda_m \text{sign}(s) & if \quad s\dot{s} \leq 0 \\ -\lambda_M \text{sign}(s) & if \quad s\dot{s} > 0 \end{cases}$$





Higher Order Sliding Mode Control

2-order sliding mode algorithms: sub-optimal [Bartolini et al.]

$y_1 = s, y_2 = \dot{s}$, after some transient
$|a(t,x)| \leq A, 0 < \Gamma_m \leq b(t,x,u) \leq \Gamma_M, A > 0.$

$$\begin{cases} \dot{y}_1 &= y_2 \\ \dot{y}_2 &= a(t,x) + b(t,x,u)u. \end{cases} \tag{87}$$

(y_1, y_2) Trajectories are confined within limit parabolic arcs.
Control:

$$\begin{array}{l} v(t) = -\alpha(t)\lambda_M \text{sign}(y_1(t) - \frac{1}{2}y_{1_M}), \\ \alpha(t) = \begin{cases} \alpha^* & if \;\; [y_1(t) - \frac{1}{2}y_{1_M}][y_{1_M} - y_1(t)] > 0 \\ 1 & if \;\; [y_1(t) - \frac{1}{2}y_{1_M}][y_{1_M} - y_1(t)] \leq 0 \end{cases}, \end{array} \tag{88}$$

where y_{1_M} is the last maximum of $y_1(t)$, i.e. the last value of y_1 for t s.t. $y_2 = \dot{y}_1 = 0$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: sub-optimal [Bartolini et al.]

Sufficient conditions:

$$\begin{aligned} &\alpha^* \in (0,1] \cap (0, \tfrac{3\Gamma_m}{\Gamma_M}), \\ &\lambda_M > \max\left(\tfrac{\Phi}{\alpha^*\Gamma_m}, \tfrac{4\Phi}{3\Gamma_m - \alpha^*\Gamma_M}\right). \end{aligned} \tag{89}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

The control is given by:

$$\begin{aligned} u(t) &= u_1(t) + u_2(t) \\ \dot{u}_1(t) &= \begin{cases} -u & if \quad |u| > 1 \\ -W\text{sign}(y_1) & if \quad |u| \leq 1 \end{cases} \\ u_2(t) &= \begin{cases} -\lambda |s_0|^\rho \text{sign}(y_1) & if \quad |y_1| > s_0 \\ -\lambda |y_1|^\rho \text{sign}(y_1) & if \quad |y_1| \leq s_0 \end{cases} \end{aligned} \tag{90}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

Sufficient conditions :

$$\begin{array}{l} W > \frac{\Phi}{\Gamma_m} \\ \lambda^2 \geq \frac{4A}{\Gamma_m^2} \frac{\Gamma_M(W+A)}{\Gamma_m(W-A)} \\ 0 < \rho \leq \frac{1}{2} \end{array} \tag{91}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Super twisting Algorithm (STA) [Levant]

Simplified version if b does not depend on control, u does not need to be bounded and $s_0 = \infty$:

$$\begin{aligned} u &= -\lambda |s|^{\rho} \operatorname{sign}(y_1) + u_1, \\ \dot{u}_1 &= -W \operatorname{sign}(y_1). \end{aligned}$$

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Drift Algorithm (DA) [Emelyanov]

Control (s relative degree is 1):

$$\dot{u} = \begin{cases} -u & if \;\; |u| > 1 \\ -\lambda_m \mathrm{sign}(\Delta y_{1_i}) & if \;\; y_1 \Delta y_{1_i} \leq 0; \; |u| \leq 1 \\ -\lambda_M \mathrm{sign}(\Delta y_{1_i}) & if \;\; y_1 \Delta y_{1_i} > 0; \; |u| \leq 1 \end{cases} \tag{92}$$

where $\lambda_m > 0, \lambda_M > 0$ are proper positive constants such that $\lambda_m < \lambda_M$ and $\frac{\lambda_M}{\lambda_m}$ is sufficiently large, and $\Delta y_{1_i} = y_1(t_i) - y_1(t_i - \tau)$, $t \in [t_i, t_{i+1})$.

Higher Order Sliding Mode Control

2-order sliding mode algorithms: Drift Algorithm (DA) [Emelyanov]

Similar controller (when s is relative degree 2) :

$$\dot{u} = \begin{cases} -\lambda_m \text{sign}(\Delta y_{1_i}) & if \;\; y_1 \Delta y_{1_i} \leq 0 \\ -\lambda_M \text{sign}(\Delta y_{1_i}) & if \;\; y_1 \Delta y_{1_i} > 0 \end{cases}$$

Higher Order Sliding Mode Control

r-order sliding mode algorithms: Hommogeneous SM [Levant]

Let p least common multiple of $1, 2, \ldots, r$

$$s^{(r)} \in [-C, C] + [K_m, K_M]u \quad (93)$$

$$\varphi_{0,r} = s \quad (94)$$

$$N_{1,r} = |s|^{\frac{r-1}{r}} \quad (95)$$

$$\varphi_{i,r} = s^{(i)} + \beta_i N_{i,r} \text{sign}(\varphi_{i-1,r}) \quad (96)$$

$$N_{i,r} = \left(|s|^{\frac{p}{r}}| + \ldots + |s^{(i-1)}|^{\frac{p}{r-i+1}}|\right)^{\frac{r-i}{p}} \quad (97)$$

$$u = -\lambda \text{sign}(\varphi_{r-1,r}(s, \dot{s}, \ldots, s^{(r)})) \quad (98)$$

β_i hard to find but can be set in advance and λ should be large enough !

Higher Order Sliding Mode Control

r-order sliding mode algorithms: Quasi continuous Hommogeneous SM [Levant]

$$s^{(r)} \in [-C, C] + [K_m, K_M]u$$

$$\varphi_{0,r} = s \qquad N_{0,r} = |s| \qquad \Psi_{0,r} = \frac{\varphi_{0,r}}{N_{0,r}} = \text{sign}(s)$$

$$\varphi_{i,r} = s^{(i)} + \beta_i N_{i-1,r}^{\frac{r-i}{r-i+1}} \Psi_{i-1,r} \qquad N_{i,r} = |s^{(i)}| + \beta_i N_{i-1,r}^{\frac{r-i}{r-i+1}} \Psi_{i-1,r} \qquad \Psi_{i,r} = \frac{\varphi_{i,r}}{N_{i,r}}$$

$$u = -\lambda \text{sign}(\Psi_{r-1,r}(s, \dot{s}, \ldots, s^{(r)}))$$

β_i hard to find but can be set in advance and λ should be large enough !

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Higher Order Sliding Mode Control

$$\dot{x} = f(x) \tag{99}$$

where f is a continuous vector field or differential inclusion

$$\dot{x} \in F(x) \tag{100}$$

where F is set valued map.

Higher Order Sliding Mode Control

Sufficient condition for ODE (or DI) to be finite time stable:

Lemma

Suppose there exists a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin of system (99) and some constants $\tau, \gamma > 0$ and $0 < \beta < 1$ such that

$$\frac{d}{dt}V(x)_{|(99)} \leq -\tau V(x)^{\beta} + \gamma V(x), \quad \forall x \in \mathcal{U}\backslash\{0\}.$$

Then the origin of system (99) is FTS. The set $\Omega = \left\{x \in \mathcal{U} : V(x)^{1-\beta} < \frac{\tau}{\gamma}\right\}$ is contained in the domain of attraction of the origin. The settling time satisfies $T(x) \leq \frac{\ln\left(1-\frac{\gamma}{\tau}V(x)^{1-\beta}\right)}{\gamma(\beta-1)}$, $x \in \Omega$.

Higher Order Sliding Mode Control

Let $\lambda > 0, r_i > 0, i \in \{1, \ldots, n\}$ called weights one can define:

- the *vector of weights* $r = (r_1, \ldots, r_n)^T$,
- the *dilation matrix*

$$\Lambda_r = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^{n}, \tag{101}$$

  note that $\Lambda_r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_i} x_i, \ldots, \lambda^{r_n} x_n)^T$.

- let $\mathbf{r}$ denotes the finite product $r_1 r_2 \ldots r_n$ then the r*–homogeneous norm* of $x \in \mathbb{R}^n$ is defined by:

$$n_r(x) = (|x_1|^{\frac{\mathbf{r}}{r_1}} + \ldots + |x_i|^{\frac{\mathbf{r}}{r_i}} + \ldots + |x_n|^{\frac{\mathbf{r}}{r_n}})^{\frac{1}{\mathbf{r}}}. \tag{102}$$

Higher Order Sliding Mode Control

Definition

A function $h : \mathbb{R}^n \to \mathbb{R}$ is *r–homogeneous with degree $d_{r,h} \in \mathbb{R}$* if for all $x \in \mathbb{R}^n$ we have (Hermes 90) :

$$\lambda^{-d_{r,h}} h(\Lambda_r x) = h(x). \tag{103}$$

When such a property holds, we write $\deg_r(h) = d_{r,h}$.

Higher Order Sliding Mode Control

Let us note that for any positive real number λ:

$$\lambda^{-1} n_r(\Lambda_r x) = n_r(x), \tag{104}$$

this is $\deg_r(n_r) = 1$. Let us introduce the following compact set

$$S_r = \{x \in \mathbb{R}^n : n_r(x) = 1\}, \tag{105}$$

Higher Order Sliding Mode Control

Remark

In fact in stead of dealing with S_r *one can take any closed curve properly chosen diffeomorphic to* $\mathbb{S}^{n-1}$.

Higher Order Sliding Mode Control

Such homogeneity notion can be also defined for vector fields, ordinary differential system (99)

Definition

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r–homogeneous with degree $d_{r,f} \in \mathbb{R}$, with $d_{r,f} > -\min_{i\in\{1,\ldots,n\}}(r_i)$ if for all $x \in \mathbb{R}^n$ we have (see Hermes 90) :

$$\lambda^{-d_{r,f}} \Lambda_r^{-1} f(\Lambda_r x) = f(x), \tag{106}$$

which is equivalent to all i-th component f_i being r–homogeneous function of degree $r_i + d_{r,f}$. When such a property holds, we write $\deg_r(f) = d_{r,f}$. The system (99) is r–homogeneous of degree $d_{r,f}$ if the vector field f is homogeneous of degree $d_{r,f}$.

Higher Order Sliding Mode Control

Theorem

If the system (99) is locally AS and r–homogeneous with negative degree then it is FTS.

Higher Order Sliding Mode Control

$$\dot{x} = f(x) + g(x)u, s(t,x)$$

with $\partial r(s) = cte = \rho \in \mathbb{N}^+$. HOSM $\Leftrightarrow$ FTS for the following system :

$$\begin{cases} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ & \vdots & \\ \dot{z}_{\rho-1} & = & z_\rho \\ \dot{z}_\rho & = & a(x,t) + b(x,t)u \end{cases} \tag{107}$$

$$\begin{cases} z = [z_1, z_2, \ldots, z_{\rho-1}, z_\rho]^T & = & \left[s, \dot{s}, \ldots, s^{(\rho-2)}, s^{(\rho-1)}\right]^T \\ a(x,t) & = & L_f^\rho s(x,t) \\ b(x,t) & = & L_g L_f^{\rho-1} s(x,t) \end{cases}$$

Higher Order Sliding Mode Control

a, b have known nominal part denoted by $\overline{a}, \overline{b}$ their unknown part being described by δ_a, δ_b, this is:

$$\begin{cases} a &= \overline{a} + \delta_a \\ b &= \overline{b} + \delta_b \end{cases}$$

Higher Order Sliding Mode Control

Assumptions: Assume that the nominal part $\bar{b}$ is invertible.
Using:

$$u = \bar{b}^{-1} (w - \bar{a}) \tag{108}$$

where $w \in \mathbb{R}$ is the knew input (107) leads to:

$$\begin{cases} \dot{z}_1 & = & z_2 \\ \dot{z}_2 & = & z_3 \\ & \vdots & \\ \dot{z}_{\rho-1} & = & z_\rho \\ \dot{z}_\rho & = & \vartheta(x,t) + (1 + \zeta(x,t))\, w \end{cases} \tag{109}$$

where ϑ, ζ are given by:

$$\begin{cases} \vartheta & = & \delta_a - \delta_b \bar{b}^{-1} \bar{a} \\ \zeta & = & \delta_b \bar{b}^{-1} \end{cases}$$

Higher Order Sliding Mode Control

Assumption: $\vartheta(x,t), \zeta(x,t)$ bounded: $\exists a(x) > 0$ and $\exists 0 < b \leq 1$ s.t.:

$$\begin{cases} |\vartheta(x,t)| \leq a(x) \\ |\zeta(x,t)| \leq 1 - b \end{cases} \tag{110}$$

Idea: FTS the unperturbed chain of integrator using homogeneity

Higher Order Sliding Mode Control

$$\begin{cases} \dot{z}_1 & = & z_2 \\ & \vdots & \\ \dot{z}_\rho & = & w \end{cases} \tag{111}$$

Higher Order Sliding Mode Control

Theorem (Bhat 2005)

Let $k_1, \ldots, k_\rho$ *positives ctes s.t.* $p^\rho + k_\rho p^{\rho-1} + \ldots + k_2 p + k_1$ *is Hurwitz. Then* $\exists \epsilon \in (0,1)$ *s.t* $\forall \nu \in (1-\epsilon, 1)$*, (111) is FTS by:*

$$w(z) = -k_1 \text{sign}(z_1)|z_1|^{\nu_1} - \ldots - k_\rho \text{sign}(z_\rho)|z_\rho|^{\nu_\rho} \qquad (112)$$

where $\nu_1, \ldots, \nu_\rho$ *are given by:*

$$\nu_{i-1} = \frac{\nu_i \nu_{i+1}}{2\nu_{i+1} - \nu_i}, \quad i = 2, \ldots, \rho, \qquad (113)$$

with $\nu_\rho = \nu$ *et* $\nu_{\rho+1} = 1$.

Table of Contents

Integral Sliding Mode Control

Objective

To remove the reaching phase
☞ To guarantee the robustness properties against perturbations in the model from the initial time instance

Philosophy

☞ To choose the sliding variable such that the system trajectories are already on the sliding surface at the initial time instance

Integral Sliding Mode Control

Objective

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Integral Sliding Mode Control

- $w_{nom}(z)$ FTS the unperturbed system(111).
- $w_{disc}(z)$ is built to cope with $\vartheta(x,t)$ and $\zeta(x,t)$ for (109).
- Leading to a $\rho-$ order sliding mode for $s(x,t)$.

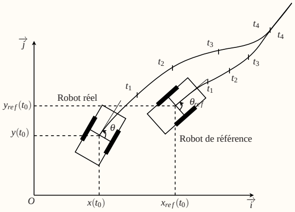
Table of Contents

Problem setup

Reference trajectory

$$\begin{bmatrix} \dot{x}_{ref} \\ \dot{y}_{ref} \\ \dot{\theta}_{ref} \end{bmatrix} = \begin{bmatrix} \cos\theta_{ref} & 0 \\ \sin\theta_{ref} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_{ref} \\ w_{ref} \end{bmatrix}$$

Objective



Individual tracking of the optimal planned trajectory for each robot i

☞ To stabilize the tracking errors:

$$\begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix} = \begin{bmatrix} x - x_{ref} \\ y - y_{ref} \\ \theta - \theta_{ref} \end{bmatrix}$$

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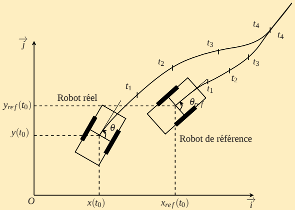

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Problem setup

Difficulties

☞ Presence of perturbations and parametric uncertainties in the model:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix} + p(q,t)$$

Algo. 1

Assumptions

- Perturbations satisfy the matching condition
- Perturbations are bounded by known positive functions
- Reference velocities are continuous and bounded
- No stop point

Algo. 1

The tracking errors asymptotically converge toward zero under:

$$u = u_{nom} + u_{disc}$$

Continuous term u_{nom} [Jiang et al., 2001]

u_{nom} stabilize the tracking errors without perturbation

$$u_{nom} = \begin{bmatrix} v_{ref}\cos e_3 + \mu_3 \tanh e_1 \\ w_{ref} + \frac{\mu_1 v_{ref} e_2}{1+e_1^2+e_2^2}\frac{\sin e_3}{e_3} + \mu_2 \tanh e_3 \end{bmatrix}$$

with

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_\theta \end{bmatrix}$$

Algo. 1

The tracking errors asymptotically converge toward zero under:

$$u = u_{nom} + u_{disc}$$

Discontinuous term u_{disc}

u_{disc} reject the effect of the perturbation from the initial time instance

$$u_{disc} = \begin{bmatrix} -G_1(e)sign(\sigma_1) \\ -G_2(e)sign(-e_2\sigma_1 + \sigma_2) \end{bmatrix}$$

with $\sigma = [\sigma_1, \sigma_2]^T$ given by:

$$\sigma = \sigma_0(e) + e_{aux}$$

- $\sigma_0(e) = [-e_1, -e_3]^T$: linear combinaison of state
- integral part $\begin{cases} \dot{e}_{aux} = \begin{bmatrix} v_{ref}\cos e_3 \\ w_{ref} \end{bmatrix} - \begin{bmatrix} 1 & -e_2 \\ 0 & 1 \end{bmatrix} u_{nom}(e) \\ e_{aux} = -\sigma_0(e(0)) \end{cases}$

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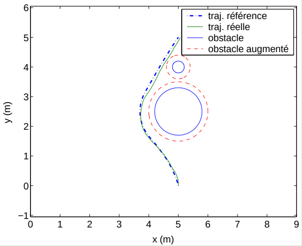
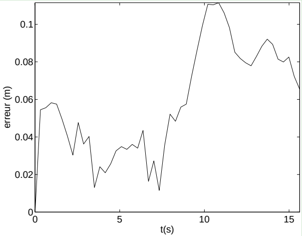
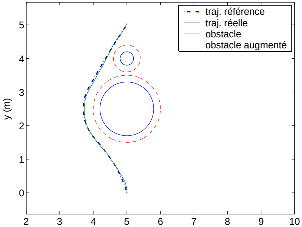
Experimental results: Algo. 1

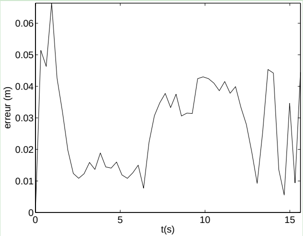
Single nominal control





ISMC





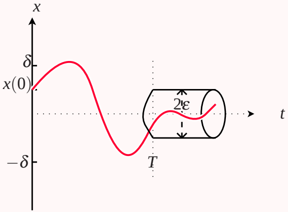
Algo. 1

Limitations

- conservative assumptions
- discontinuities on velocities
- perturbations must satisfy the matching condition

Solution

☞ Practical stabilization using second order ISMC



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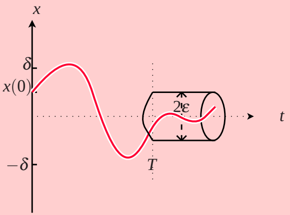
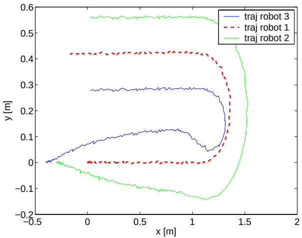
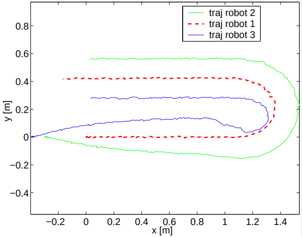


Experimental results

ISM of Order 1



ISM of Order 2



Experimental results

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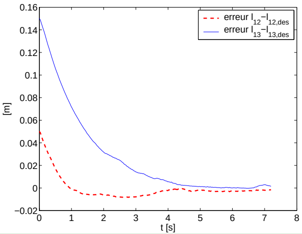
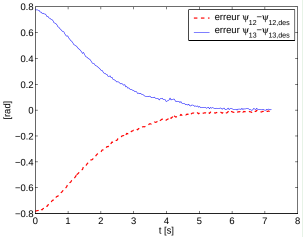
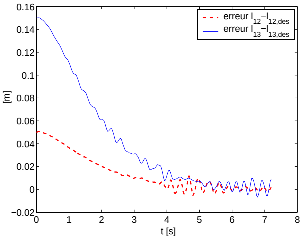
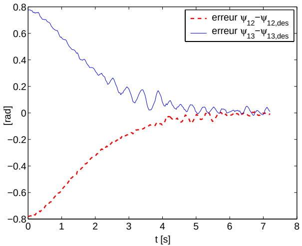




ISM of Order 2





Video

Video

Video with 3 miabot

Video

Video with 7 miabots

Video