Multi-homogeneity for global sliding mode design

W. Perruquetti, D. Efimov, E. Bernuau

Ecole Centrale de Lille,
Cité Scientifique, BP 48,
F-59651 Villeneuve d'Ascq Cedex - FRANCE.
tel : +33 3 20 33 54 50 fax : +33 3 20 33 54 18
e-mail : wilfrid.perruquetti@inria.fr

25 August 2011 / Cagliari

Table of Contents

1. Introduction
   - Finite time stability
   - Homogeneity

2. Multi-homogeneity

Finite time stability

$$\dot{x} = -\lceil x \rfloor^a = -\mathrm{sign}(x)|x|^a, x \in \mathbb{R}, \quad (1)$$

for which the solutions are ($a \in]0, 1[$):

$$\phi^x(\tau) = \begin{cases} s(\tau, x) & \text{if} \quad 0 \leq \tau \leq \frac{|x|^{1-a}}{1-a} \\ 0 & \text{if} \quad \tau > \frac{|x|^{1-a}}{1-a} \end{cases}, \quad (2)$$

with $s(\tau, x) = \mathrm{sgn}(x) \left(|x|^{1-a} - \tau(1-a) \right)^{\frac{1}{1-a}}$ and they reach the origin in finite time.

Finite time stability

Observation:

- FTS = infinite eigenvalue assignation for the closed loop system at the origin.
- ∃ a function called *settling time* that performs the time for a solution to reach the equilibrium. The function depends on the initial condition of a solution.
- the right hand side of the ordinary differential equation *can not be locally Lipschitz at the origin*.

Finite time stability

ODE (ordinary differential equation)

$$\dot{x} = f(x) \tag{3}$$

where f is a continuous vector field or DI (differential inclusion)

$$\dot{x} \in F(x) \tag{4}$$

where F is a set valued map (with some additional property upper semi continuous for example).

Finite time stability

finite-time stability (FTS): S + FA (Finite time attractivity)

These systems are assumed to possess unique solutions in forward time

Finite time stability

Definition

A class $\mathcal{K}$ function r belongs to *class $\mathcal{KI}$* if $r \in \mathcal{CL}([0, a])$ and there exists $0 < \epsilon < a$ such that:

$$\int_0^{\epsilon} \frac{dz}{r(z)} < +\infty.$$

From now $\alpha \in]0, 1[$. Let $V : \mathcal{V} \to \mathbb{R}_{\geq 0}$ be a Lyapunov function and r a class $\mathcal{KI}$ function, the first condition is that for all $x \in \mathcal{V}$,

$$\dot{V}(x) \leq -r(V(x)). \qquad (5)$$

Finite time stability

The existence of such a pair (V, r) is still a necessary condition for finite time stability of more general systems (see [2] and [3]). Here, one will see that r can be chosen on a particular form describes as follow.
Let $V : \mathcal{V} \to \mathbb{R}_{\geq 0}$ be a Lyapunov function, the second condition is that for all $x \in \mathcal{V}$,

$$\dot{V}(x) \leq -a(V(x))^{\alpha}, a > 0, \alpha \in]0, 1[. \quad (6)$$

(in fact Bhat $T(x)$ continuous at the origin but in fact what is needed is uniquness outside the origin).

Finite time stability

Theorem (Moulay Perruquetti 2006)

Consider the system (3) with uniqueness of solutions outside the origin, the following properties are equivalent:

(i) *the origin of the system (3) is* **FTS**,

(ii) *there is a Lyapunov function satisfying condition (6),*

(iii) *there is a Lyapunov function and a class $\mathcal{KI}$ function satisfying condition (5).*

Moreover, if V is a Lyapunov function satisfying condition (6) then

$$T(x) \leq \frac{V(x)^{1-\alpha}}{c(1-\alpha)}.$$

1. Introduction
   - Finite time stability
   - Homogeneity
     - First generation
     - Second generation
2. Multi-homogeneity

Homogeneity: first generation ideas

Let us recall (3)

$$\dot{x} = f(x), x \in \mathbb{R}^n, \tag{7}$$

In the 50-60's (Lasalle, Han, etc ...):

Definition

$f : \mathbb{R}^n \to \mathbb{R}^n$ is homogeneous with degree k (or k–homogeneous) iff $\forall \lambda \in \mathbb{R} : f(\lambda x) = \lambda^k f(x)$.

Homogeneity: first generation ideas

$x(t, x_0)$ denotes a solution. For example,

$$\dot{x} = Ax$$

is homogeneous of degree $k = 1$ and we have
$x(t, x_0) = \exp(At)x_0$ thus $x(t, \lambda x_0) = \lambda x(\lambda^{k-1}t, x_0) = \lambda x(t, x_0)$.

$$\dot{x} = -\text{sign}(x)$$

is homogeneous of degree $k = 0$ and we have $(\lambda > 0)$

$$x(t, x_0) = \text{sgn}(x_0)\left(|x_0| - t\right)$$

thus $x(t, \lambda x_0) = \lambda x(\lambda^{k-1}t, x_0) = \lambda x\left(\frac{t}{\lambda}, x_0\right)$.

Homogeneity: first generation ideas

- if $k < 0$ then we will get a discontinuity at the origin,
- if $0 < k < 1$ then the Lipschitz condition is not satisfied (Uniqueness of solutions),
- if $\lambda = -1$ then the function is not real.

In order to avoid such situations, at these times, we add the condition

$$k = \frac{p}{2r+1} > 1,$$

where p and r integers.

Homogeneity: first generation ideas

Example

- $f(x_1, x_2) = \frac{x_1^2+x_2^2}{x_1+x_2}$ is 1–homogeneous but is not linear!
- $f(x_1, x_2) = \frac{x_1^{1/2}+x_2^{1/2}}{x_1+x_2}$ is $-\frac{1}{2}$–homogeneous (not continuous at 0).

Homogeneity: first generation ideas

Proposition

$f : \mathbb{R}^n \to \mathbb{R}^n$ *is* *k–homogeneous* *iff* *each components are k–homogeneous.*

Proposition (Euler)

$f_i : \mathbb{R}^n \to \mathbb{R}$ *is* *k–homogeneous* *iff*

$$\sum_{i=1}^{n} x_i \frac{\partial f_i}{\partial x_i} = k f_i(x), \forall x \in \mathbb{R}^n. \tag{8}$$

Homogeneity: first generation ideas

A useful property is time and state parametrization

$$\begin{aligned} x &= \lambda y \\ s &= \lambda^{k-1} t \end{aligned}$$

leads to

$$\frac{dy}{ds} = f(y) \quad (9)$$

from which we deduce $\phi(t, x_0) = \phi(t, \lambda y_0) = \lambda\phi(s, y_0)$ (using the following notation $y_0 = \lambda^{-1} x_0$)

$$\lambda\phi(\lambda^{k-1} t, x_0) = \phi(t, \lambda x_0) \quad (10)$$

Homogeneity: first generation ideas

Theorem

If the equilibrium of (3) where f is homogeneous, is locally AS then it is GAS.

Homogeneity: first generation ideas

Properties:

1. trajectories are rays from the origin.
2. Thus it is sufficient to study what happens on the unit sphere $\mathbb{S}$:

$$
\begin{aligned}
y &= \frac{x}{\|x\|} = \frac{x}{r}, r^2 = x^T x \\
2r\dot{r} &= \dot{x}^T x + x^T \dot{x} = r(f^T(ry)y + y^T f(ry)) \\
&= 2r^{k+1} y^T f(y) \\
\dot{r} &= r^k y^T f(y) \\
\dot{y} &= \frac{d(x/r)}{dt} = \frac{\dot{x}}{r} - \frac{x\dot{r}}{r^2} = \frac{rf(ry) - ryr^k y^T f(y)}{r^2} \\
&= r^{k-1}\left[f(y) - \left(y^T f(y)\right) y\right]
\end{aligned}
$$

Homogeneity: first generation ideas

Taking a point $y_0 \in \mathbb{S}$ then we have $f(y_0) - \left(y_0^T f(y_0)\right) y_0 = 0$: so for the original system in x the corresponding trajectory is a ray passing through the origin and the point belonging to the sphere. Along this ray we have

$$\begin{aligned} \dot{r} &= ar^k, a = y_0^T f(y_0) \\ r(t) &= \left[r_0^{1-k} + (1-k)at\right]^{\frac{1}{1-k}}, \quad k \neq 1 \\ r(t) &= r_0 \exp(at), \quad \text{if } k = 1 \end{aligned}$$

Homogeneity: first generation ideas

From this one obtains:

Proposition

According to (a, k)

- $k = 1$,

$a < 0$	$\lim_{t\to\infty} r(t) \to 0$	*AS*
$a > 0$	$\lim_{t\to\infty} r(t) \to \infty$	*I*
$a = 0$	$r(t) = r_0$	*S*

- $k > 1$,

$a < 0$	$\lim_{t\to\infty} r(t) \to 0$	*AS*
$a > 0$	$\lim_{t\to\infty} r(t) \to \infty$	*I (finite time blow up)*
$a = 0$	$r(t) = r_0$	*S*

- $k < 1$,

$a < 0$	$\lim_{t\to\infty} r(t) \to 0$	*AS : STF (finite time)*
$a > 0$	$\lim_{t\to\infty} r(t) \to \infty$	*I*
$a = 0$	$r(t) = r_0$	*S*

Homogeneity: first generation ideas

Example

Let us consider

$$\begin{aligned}
\dot{x}_1 &= -\left(\frac{x_1^2 + x_2^2}{x_1 + x_2}\right) x_1 + x_1 x_2 \\
\dot{x}_2 &= -x_1^2 - \left(\frac{x_1^2 + x_2^2}{x_1 + x_2}\right) x_2
\end{aligned}$$

which is 2–homogeneous.

Homogeneity: first generation ideas

Example

On the sphere $(x_1^2 + x_2^2 = 1)$

$$
\begin{aligned}
f(y_0) &= \left(y_0^T f(y_0)\right) y_0 \\
&\begin{cases} -\left(\frac{x_1^2+x_2^2}{x_1+x_2}\right) x_1 + x_1 x_2 = -\frac{\left(x_1^2+x_2^2\right)^2}{x_1+x_2} x_1 \\ -x_1^2 - \left(\frac{x_1^2+x_2^2}{x_1+x_2}\right) x_2 = -\frac{\left(x_1^2+x_2^2\right)^2}{x_1+x_2} x_2 \end{cases} \\
&\begin{cases} -\frac{x_1}{x_1+x_2} + x_1 x_2 = -\frac{x_1}{x_1+x_2} \\ -x_1^2 - \frac{x_2}{x_1+x_2} = -\frac{x_2}{x_1+x_2} \end{cases}
\end{aligned}
$$

Homogeneity: first generation ideas

Example

Which have two equilibriums

$$(-1,1)$$
$$(1,-1)$$

for which $a = x^T f(x) = -\frac{\left(x_1^2+x_2^2\right)^2}{x_1+x_2} = \infty$ ($a = \infty, k > 1$: blow up)

Homogeneity: first generation ideas

Example

Let us consider

$$\begin{aligned}\dot{x}_1 &= -\left(x_1^2+x_2^2\right)x_1 + x_1x_2^2\\ \dot{x}_2 &= -x_1^2x_2 - \left(x_1^2+x_2^2\right)x_2\end{aligned}$$

which is 3–homogeneous. On the sphere $(x_1^2+x_2^2=1)$

$$\begin{aligned}f(y_0) &= \left(y_0^T f(y_0)\right)y_0\\ &\begin{cases}-1x_1+x_1x_2^2 = -\left(x_1^2+x_2^2\right)^2 x_1\\ -x_1^2x_2 - 1x_2 = -\left(x_1^2+x_2^2\right)^2 x_2\end{cases}\end{aligned}$$

Homogeneity: first generation ideas

Example

There is 4 points

$$(\pm 1, 0)$$
$$(0, \pm 1)$$

for which $a = x^T f(x) = -\left(x_1^2 + x_2^2\right)^2 = -1$ ($a < 0, k > 1$: AS)

Homogeneity: first generation ideas

An important Liapunov function characterization of GAS is

Theorem

Let f be homogeneous C^1 function such that $f(0) = 0$ the the two following conditions are equivalent

- *the origin is GAS,*
- *there exist an homogeneous Liapunov function of class C^∞ s.t. V and $-\dot{V}$ are positive definite.*

Homogeneity: first generation ideas

Corollary

Let f^i be C^1 homogeneous vector fields with degree $i \geq k$ and let $f = \sum_{i \geq k} f^i$ such that $f(0) = 0$. If the origin of $\dot{x} = f^k(x)$ is LAS then the origin of $\dot{x} = f(x)$ is GAS.

Homogeneity: second generation ideas

Second generation homogeneity ideas:

Let $\lambda > 0, r_i > 0, i \in \{1, \ldots, n\}$ called weights one can define:

- the *vector of weights* $r = (r_1, \ldots, r_n)^T$,
- the *dilation matrix*

$$\Lambda_r = \mathrm{diag}\{\lambda^{r_i}\}_{i=1}^{n}, \qquad (11)$$

note that $\Lambda_r x = (\lambda^{r_1} x_1, \ldots, \lambda^{r_i} x_i, \ldots, \lambda^{r_n} x_n)^T$.

- let $\mathbf{r}$ denotes the finite product $r_1 r_2 \ldots r_n$ then the *r–homogeneous norm* of $x \in \mathbb{R}^n$ is defined by:

$$n_r(x) = (|x_1|^{\frac{\mathbf{r}}{r_1}} + \ldots + |x_i|^{\frac{\mathbf{r}}{r_i}} + \ldots + |x_n|^{\frac{\mathbf{r}}{r_n}})^{\frac{1}{\mathbf{r}}}. \qquad (12)$$

Homogeneity: second generation ideas

Sufficient condition for ODE (or DI) to be finite time stable:

Lemma

Suppose there exists a Lyapunov function $V(x)$ defined on a neighborhood $\mathcal{U} \subset \mathbb{R}^n$ of the origin of system (3) and some constants $\tau, \gamma > 0$ and $0 < \beta < 1$ such that

$$\frac{d}{dt}V(x)_{|(3)} \leq -\tau V(x)^{\beta} + \gamma V(x), \quad \forall x \in \mathcal{U}\backslash\{0\}.$$

Then the origin of system (3) is **FTS**. *The set $\Omega = \left\{x \in \mathcal{U} : V(x)^{1-\beta} < \frac{\tau}{\gamma}\right\}$ is contained in the domain of attraction of the origin. The settling time satisfies $T(x) \leq \frac{\ln\left(1-\frac{\gamma}{\tau}V(x)^{1-\beta}\right)}{\gamma(\beta-1)}$, $x \in \Omega$.*

Homogeneity: second generation ideas

Definition

A function $h : \mathbb{R}^n \to \mathbb{R}$ is *r–homogeneous* with degree $d_{r,h} \in \mathbb{R}$ if for all $x \in \mathbb{R}^n$ we have (Hermes 90) :

$$\lambda^{-d_{r,h}} h(\Lambda_r x) = h(x). \tag{13}$$

When such a property holds, we write $\deg_r(h) = d_{r,h}$.

Homogeneity: second generation ideas

Such homogeneity notion can be also defined for vector fields, ordinary differential system (3)

Definition

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r–homogeneous with degree $d_{r,f} \in \mathbb{R}$, with $d_{r,f} > -\min_{i\in\{1,\ldots,n\}}(r_i)$ if for all $x \in \mathbb{R}^n$ we have (see Hermes 90) :

$$\lambda^{-d_{r,f}} \Lambda_r^{-1} f(\Lambda_r x) = f(x), \tag{14}$$

which is equivalent to all i-th component f_i being r–homogeneous function of degree $r_i + d_{r,f}$. When such a property holds, we write $\deg_r(f) = d_{r,f}$. The system (3) is r–homogeneous of degree $d_{r,f}$ if the vector field f is homogeneous of degree $d_{r,f}$.

Homogeneity: second generation ideas

Theorem (Rosier)

For the system (3) with r–homogeneous and continuous function f the following properties are equivalent:

- *the system (3) is (locally) asymptotically stable;*
- *there exists a continuously differentiable homogeneous Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ such that for all $x \in \mathbb{R}^n$,*

$$\alpha_1(x) \leq \quad V(x) \quad \leq \alpha_2(x) \tag{15}$$

$$DV(x)f(x) \quad = \quad -\alpha(x) \tag{16}$$

$$\lambda^{-d}V(\Lambda_r x) \quad = \quad V(x), d \geq 0, \tag{17}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha \in \mathcal{K}$.

Homogeneity: second generation ideas

Theorem

Let f be defined on $\mathbb{R}^n$ and be a continuous r–homogeneous vector field with negative degree. If the origin of system (3) is Locally AS and then it is **Globally FTS**.

Multi-homogeneity: definitions

Restricting the set of admissible λ (local homogeneity):

Definition

A function $h : \mathbb{R}^n \to \mathbb{R}$ with $h(0) = 0$ is (r_0, λ_0, h_0)*–homogeneous with degree* $d_{r_0,h_0} \in \mathbb{R}$ with $h_0(0) = 0$ if for all $x \in S_{r_0}$ we have :

$$\lim_{\lambda \to \lambda_0} \left(\lambda^{-d_{r_0,h_0}} h(\Lambda_{r_0} x) - h_0(x) \right) = 0. \tag{18}$$

Remark

In the paper [1] by Andrieu et al. this definition has been introduced for $\lambda_0 = 0$ *and* $\lambda_0 = \infty$ *(the function* h *is called homogeneous in the bi-limit if it is simultaneously* $(r_0, 0, h_0)$*–homogeneous and* $(r_\infty, \infty, h_\infty)$*–homogeneous).*

Multi-homogeneity: definitions

Definition (to be continued)

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is with $f(0) = 0$ is (r_0, λ_0, f_0)*–homogeneous with degree* $d_{r_0,f_0} \in \mathbb{R}$ with $f_0(0) = 0$ and $d_{r_0,f_0} > -\min_{i\in\{1,\ldots,n\}}(r_{0i})$ if for all $x \in S_{r_0}$ we have :

$$\lim_{\lambda\to\lambda_0}\left(\lambda^{-d_{r_0,f_0}}\Lambda_{r_0}^{-1}f(\Lambda_{r_0}x) - f_0(x)\right) = 0, \tag{19}$$

The system (3) is (r_0, λ_0, f_0)*–homogeneous with degree* $d_{r_0,f_0} \in \mathbb{R}$ if the vector field f is (r_0, λ_0, f_0)*–homogeneous with degree* $d_{r_0,f_0} \in \mathbb{R}$. The coefficients $r_{0i} > 0, i \in 1, \ldots, n$ are called the weights, d_{r_0,h_0} (respectively d_{r_0,f_0}) is the degree of homogeneity (it may depend on λ_0) and h_0 (respectively f_0) is the approximating function of h (respectively f) at λ_0.

Multi-homogeneity: definitions

Definition (end)

A set valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ with $F(0) \ni 0$ is (r_0, λ_0, F_0)*–homogeneous with degree* $d_{r_0,F_0} \in \mathbb{R}$ with F_0 being a set valued map such that $F_0(0) = 0$ if for all $x \in S_{r_0}$ we have :

$$\lim_{\lambda \to \lambda_0} \left(\lambda^{-d_{r_0,F_0}} \Lambda_{r_0}^{-1} F(\Lambda_{r_0} x) - F_0(x) \right) = 0. \tag{20}$$

The system (4) is (r_0, λ_0, F_0)*–homogeneous with degree* $d_{r_0,F_0} \in \mathbb{R}$ if the set valued map F is (r_0, λ_0, F_0)*–homogeneous with degree* $d_{r_0,F_0} \in \mathbb{R}$.

Multi-homogeneity: definitions

Definition

The function f (respectively the vector field f, the system (3), the multi-valued function F, the differential inclusion (4)) is homogeneous in the multi-limit if there exist a finite number of triplet $(r_i, \lambda_i, g_i$ (respectively f_i, F_i)) for which the function (respectively the vector field f, the system (3), the multi-valued function F, the differential inclusion (4)) is $(r_i, \lambda_i, g_i$ (respectively f_i, F_i))–locally homogeneous for each index i.

Multi-homogeneity: definitions

Example

Let us consider the following function

$$h^0 : x \mapsto \lfloor x \rceil^{\frac{1}{3}} + \lfloor x \rceil^3, \tag{21}$$

It is easy to see that in this case this function cannot be homogeneous in the classical sense. At the origin: $h_0(x) = \lfloor x \rceil^{\frac{1}{3}}$ is dominating and is homogeneous of degree $d_{r_0,h_0} = 1$ with weight $r_0 = 3$. Indeed, $\forall x \in S_{r_0}$ we have

$$\lim_{\lambda \to 0} \lambda^{-1} \lfloor \lambda^3 x \rceil^{\frac{1}{3}} = \lfloor x \rceil^{\frac{1}{3}} = h_0(x) \tag{22}$$

$$\lim_{\lambda \to 0} \lambda^{-1} \lfloor \lambda^3 x \rceil^{3} = 0 \tag{23}$$

Multi-homogeneity: definitions

Example

At infinity: $h_\infty(x) = \lfloor x \rceil^3$ is dominating and is homogeneous of degree $d_{r_\infty,h_\infty} = 1$ with weight $r_\infty = \frac{1}{3}$. Indeed, $\forall x \in S_{r_\infty}$ we have

$$\lim_{\lambda \to +\infty} \lambda^{-1} \left\lfloor \lambda^{\frac{1}{3}} x \right\rceil^{\frac{1}{3}} = 0 \qquad (24)$$

$$\lim_{\lambda \to +\infty} \lambda^{-1} \left\lfloor \lambda^{\frac{1}{3}} x \right\rceil^{3} = \lfloor x \rceil^3 = h_\infty(x) \qquad (25)$$

$$(26)$$

Finally this function h^0 is $(3, 0, h_0)$–*homogeneous with degree one* and $(\frac{1}{3}, +\infty, h_\infty)$–*homogeneous with degree one*. Clearly this function is also continuous at any point in particular at $x = 1$.

Multi-homogeneity: definitions

Example

Let us consider the following function

$$h^1 : x \mapsto \frac{x^5}{(1+x^2)} + \frac{\lfloor x \rceil^{\frac{1}{3}}}{(1+x^2)}, \tag{27}$$

At the origin: $h_0(x) = \lfloor x \rceil^{\frac{1}{3}}$ is dominating and is homogeneous of degree $d_{r_0,h_0} = 1$ with weight $r_0 = 3$. Indeed, $\forall x \in S_{r_0}$ we have

$$\lim_{\lambda \to 0} \lambda^{-1} \frac{\lfloor \lambda^3 x \rceil^{\frac{1}{3}}}{(1+\lambda^6 x^2)} = \lfloor x \rceil^{\frac{1}{3}} = h_0(x) \tag{28}$$

$$\lim_{\lambda \to 0} \lambda^{-1} \frac{\lambda^{15} x^5}{(1+\lambda^6 x^2)} = 0 \tag{29}$$

Multi-homogeneity: definitions

Example

At infinity: $h_\infty(x) = \lfloor x \rceil^3$ is dominating and is homogeneous of degree $d_{r_\infty,h_\infty} = 1$ with weight $r_\infty = \frac{7}{5}$. Indeed, $\forall x \in S_{r_\infty}$ we have

$$\lim_{\lambda \to +\infty} \lambda^{-1} \frac{\left\lfloor \lambda^{\frac{7}{5}} x \right\rceil^{\frac{1}{3}}}{(1+\lambda^6 x^2)} = 0 \quad (30)$$

$$\lim_{\lambda \to +\infty} \lambda^{-1} \frac{\left\lfloor \lambda^{\frac{7}{5}} x \right\rceil^{5}}{(1+\lambda^6 x^2)} = \lfloor x \rceil^3 = h_\infty(x) \quad (31)$$

Finally this function h^1 is $(3, 0, h_0)$–*homogeneous with degree one* and $(\frac{7}{5}, +\infty, h_\infty)$–*homogeneous with degree one.*

Multi-homogeneity: definitions

Obtained results:

- stability (local homogeneity),
- un-stability (local homogeneity),
- universal formulae for constructing approximating functions,
- oscillation characterization using multi-homogeneity concepts,
- extension for fde,
- etc ...

Let us consider the following

$$\dot{z} = f(z) + \sum_{i=1}^{m} g_i(z) u_i, \quad z \in \Omega, \quad y = h(z), \tag{32}$$

And assume that it can be transformed into the following chain of integrator

$$\begin{aligned} \dot{x}_1 &= a_1 x_1 + x_2 + \phi_1(y,u) \\ \dot{x}_2 &= a_2 x_1 + x_3 + \phi_2(y,u) \\ \ldots &= \ldots \\ \dot{x}_n &= a_n x_1 + \phi_2(y,u) \\ y &= x_1. \end{aligned} \tag{33}$$

Theorem (Perruquetti et al. 2006)

$\exists \varepsilon > 0 : \forall \alpha \in]1-\varepsilon, 1[$, *system (33) admits the following* **GFTO***:*

$$\begin{cases} \dot{\hat{x}}_1 = a_1 y + \phi_1(y,u) + \hat{x}_2 + k_1 \lceil y - \hat{x}_1 \rfloor^{\alpha_1} \\ \dot{\hat{x}}_2 = a_2 y + \phi_2(y,u) + \hat{x}_3 + k_2 \lceil y - \hat{x}_1 \rfloor^{\alpha_2} \\ \vdots \\ \dot{\hat{x}}_n = a_2 y + \phi_2(y,u) + k_n \lceil y - \hat{x}_1 \rfloor^{\alpha_n} \end{cases} \quad (34)$$

where the α_i *are defined by*

$$\alpha_i = i\alpha - (i-1), \quad i = 1, \ldots, n, \quad \alpha \in \left]1 - \frac{1}{n}, 1\right[. \quad (35)$$

The gains are given such that $(A - KC)$ *is Hurwitz.*

Assumptions:

- System is **UO** (Uniformly observable) for any bounded input:

$$\begin{cases} \dot{x}_1 = x_2 + \sum_{j=1}^{m} g_{1,j}(x_1)u_j \\ \dot{x}_2 = x_3 + \sum_{j=1}^{m} g_{2,j}(x_1, x_2)u_j \\ \vdots \\ \dot{x}_{n-1} = x_n + \sum_{j=1}^{m} g_{n-1,j}(x_1, \dots, x_{n-1})u_j \\ \dot{x}_n = \varphi(x) + \sum_{j=1}^{m} g_{n,j}(x)u_j \\ y = x_1 = Cx \end{cases} \tag{36}$$

  (using a change of coordinate) where $C = (1 \quad 0 \cdots 0)$, φ and $g_{i,j}$ $(i = 1, \dots, n, j = 1, \dots, m)$ are analytic functions with $\varphi(0) = 0, g_{ij}(0, \dots, 0) = 0$.
- the functions $g_{i,j}$ and φ are globally Lipschitz with constant l and u is bounded by $u_0 \in \mathbb{R}_+$, that is $\|u\|_\infty \leq u_0$.

Theorem (Shen 2008)

System (36) admits a semi-global observer of the form:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1)u_j + k_1\lceil y - \hat{x}_1\rfloor^{\alpha_1} \\ \dot{\hat{x}}_2 = \hat{x}_3 + \sum_{j=1}^{m} g_{2j}(\hat{x}_1, \hat{x}_2)u_j + k_2\lceil y - \hat{x}_1\rfloor^{\alpha_2} \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + \sum_{j=1}^{m} g_{n,j}(\hat{x})u_j + k_n\lceil y - \hat{x}_1\rfloor^{\alpha_n} \end{cases} \tag{37}$$

where the α_i are given by (35) and the gains are given by

$$K = [k_1, \ldots, k_n]^T = S_\infty^{-1}(\theta)C^T, \tag{38}$$

Theorem (end)

where $S_\infty(\theta)$ is the unique solution of the matrix equation:

$$\begin{cases} \theta S_\infty(\theta) + A^T S_\infty(\theta) + S_\infty(\theta) A - C^T C = 0 \\ S_\infty(\theta) = S_\infty^T(\theta) \end{cases} \tag{39}$$

where $(A)_{i,j} = \delta_{i,j-1}$, $1 \le i, j \le n$, and $C = (1 \quad 0 \ldots 0)$.

Theorem (Menard et al. 2010)

For (36) with a bounded input, there exists $0 < \theta^* < \infty$ *and* $\varepsilon > 0$ *such that for all* $\theta > \theta^*$ *and* $\alpha \in]1-\varepsilon, 1[$, *we have the following* **GFTO**:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \sum_{j=1}^{m} g_{1,j}(\hat{x}_1)u_j + k_1(\lceil e_1 \rfloor^{\alpha_1} + \rho e_1) \\ \dot{\hat{x}}_2 = \hat{x}_3 + \sum_{j=1}^{m} g_{2,j}(\hat{x}_1, \hat{x}_2)u_j + k_2(\lceil e_1 \rfloor^{\alpha_2} + \rho e_1) \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + \sum_{j=1}^{m} g_{n,j}(\hat{x})u_j + k_n(\lceil e_1 \rfloor^{\alpha_n} + \rho e_1) \end{cases}$$

where $e_1 = x_1 - \hat{x}_1$, *the powers* α_i *are defined by (35), the gains* k_i *by (38), and* $\rho = \left(\frac{n^2 \theta^{\frac{2}{3}} S_1 + 1}{2}\right)$, *where*

Theorem (end)

$$S_1 = \max_{1\le i,j\le n} |S_\infty(1)_{i,j}|.|S_\infty^{-1}(1)_{j,1}|. \quad (40)$$

In addition, the settling time $T(e_0)$ *(where* $e_0 = x_0 - \hat{x}_0$*) of the error dynamics is bounded by* $\frac{\ln\left(\frac{4r^2}{V(e_0)}\right)}{\kappa(\theta)} + \frac{\ln\left(1-\frac{b_1}{b_2}\left(4r^2\right)^{1-\overline{\alpha}}\right)}{b_2(\overline{\alpha}-1)}$ *(where all the parameters and the Lyapunov function* V *are given in the proof).*

Key Point is multi-homogeneity at zero and infinity.

Theorem (Perruquetti et al. 2011)

For (36) with a bounded input u, there exists $0 < \theta^* < \infty$ such that for all $\theta > \theta^*$ and $\alpha \in [1 - 1/n, 1[$, (29) is a **GFTO**.

Remark

When $\alpha = 1 - \frac{1}{n}$, $\alpha_i = 1 - \frac{i}{n}$, $\quad i = 1, \ldots, n$ *this is :*$\alpha_n = 0$ *thus*

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \sum_{j=1}^m g_{1,j}(\hat{x}_1)u_j + k_1(\lceil e_1 \rfloor^{1-\frac{1}{n}} + \rho e_1) \\ \dot{\hat{x}}_2 = \hat{x}_3 + \sum_{j=1}^m g_{2,j}(\hat{x}_1, \hat{x}_2)u_j + k_2(\lceil e_1 \rfloor^{1-\frac{2}{n}} + \rho e_1) \\ \vdots \\ \dot{\hat{x}}_n = \varphi(\hat{x}) + \sum_{j=1}^m g_{n,j}(\hat{x})u_j + k_n(\mathrm{sign}(e_1) + \rho e_1) \end{cases}$$

1. Introduction

2. Multi-homogeneity
   - Observation/differentiation
   - Control

In the rest we will consider a perturbed chain of integrator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \vdots \\ \dot{x}_{n-1} = x_n \\ \dot{x}_n = a(x) + b(x)u \\ y = x_1 = Cx \end{cases} \tag{41}$$

Conjecture

Assume that $b(x) \neq 0$, then there exists $0 < \theta^* < \infty$ and $\alpha \in [1 - 1/n, 1[$, such that for all $\theta > \theta^*$ and the following control:

$$\begin{cases} u = \frac{(-a(x)+v(x))}{b(x)} \\ v(x) = \sum_{i=1}^{n} k_i(\lceil x_i \rfloor^{\alpha_i} + \rho x_i) \end{cases}$$

globally *finite-time stabilize the system (41), where the powers α_i are defined by (35), the gains k_i and ρ are given explicitly.*

Bibliography

V. Andrieu, L. Praly, and A. Astolfi, "Homogeneous approximation, recursive observer design, and output feedback," *SIAM J. Control Optimization*, vol. 47, no. 4, pp. 1814–1850, 2008.

V.T. Haimo.
Finite time controllers.
SIAM J. Control and Optimisation, 24(4):804–833, 1986.

E. Moulay and W. Perruquetti.
Finite time stability of non linear systems.
In *IEEE Conference on Decision and Control*, 2003.