Particle-In-Cell approximation to the Vlasov-Poisson with a strong external magnetic field

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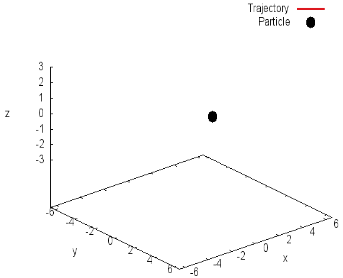
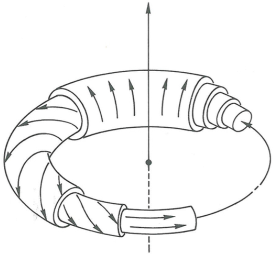
Example of a single particle motion

We consider only one charged particle submitted to an intense electromagnetic field

$$\begin{cases} \varepsilon \dfrac{d\mathbf{x}}{dt} = \mathbf{v}, \\[2ex] \varepsilon \dfrac{d\mathbf{v}}{dt} = \mathbf{E}(t,\mathbf{x}) + \dfrac{1}{\varepsilon}\mathbf{v} \times \mathbf{B}(t,\mathbf{x}), \end{cases}$$

where **E** and **B** are given and non uniform.





Trajectory of a single particle under the effect of an external magnetic field

Outline of the Talk

- Understand how to recover the correct guiding center velocity at the level of the kinetic equation
- Preserve the structure of the Vlasov-Poisson system in phase space and construct numerical scheme on the original problem not on gyrokinetic models.

1. Part I. Modeling and scaling issues
   - Vlasov-Poisson system with a strong external magnetic field
   - 2D problem
   - 3D problem
2. Part II. Strategy for numerical schemes
3. Numerical simulations
   - Vlasov-Poisson system with nonhomogeneous magnetic field

Vlasov-Poisson system with a strong external magnetic field

Assumptions

- Consider the Vlasov-Poisson system $3D \times 3D$
- Consider that the magnetic field is uniform $\mathbf{B}_{\rm ext} = b_{\rm ext}(t, \mathbf{x})\, e_z$, where e_z stands for the unit vector in the z-direction,
- we are interesting by the long time asymptotic of electrons

After a rescaling, it yields

$$\begin{cases} \varepsilon \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla_{\mathbf{x}} f + \left[\mathbf{E} - \dfrac{b_{\rm ext}(t, \mathbf{x})}{\varepsilon} \mathbf{v}^{\perp} \right] \cdot \nabla_{\mathbf{v}} f = 0. \\ \mathbf{E} = -\nabla \phi, \quad -\Delta \phi = \rho - \rho_i. \end{cases}$$

where

$$\rho = \int_{\mathbb{R}^3} f d\mathbf{v}.$$

From the works of Arsenev, or DiPerna-Lions, there exist global in time weak solutions (energy and L^p estimates are uniform with respect to time and ε).
$\rightarrow$ this framework allows us to study the asymptotic $\varepsilon \rightarrow 0$.

In the limit $\varepsilon \to 0$ for the 2D problem

First case : $b_{\text{ext}} = b_0$.

If we are not interesting by the details of the dynamics of electrons, we only want the evolution of the density ρ.

We define

$$\rho^\varepsilon := \int_{\mathbb{R}^2} f^\varepsilon \, d\mathbf{v}, \quad \text{and} \quad \mathbf{J}^\varepsilon := \int_{\mathbb{R}^2} f^\varepsilon \mathbf{v} \, d\mathbf{v},$$

and get

$$\begin{cases} \dfrac{\partial \rho^\varepsilon}{\partial t} + \dfrac{1}{\varepsilon} \operatorname{div}_{\mathbf{x}}(\mathbf{J}^\varepsilon) = 0 \\[2ex] \varepsilon \dfrac{\partial \mathbf{J}^\varepsilon}{\partial t} + \operatorname{div}_{\mathbf{x}} \displaystyle\int_{\mathbb{R}^d} \mathbf{v} \times \mathbf{v} \, f^\varepsilon \, d\mathbf{v} + \mathbf{E} \, \rho^\varepsilon - \dfrac{b_0}{\varepsilon} \mathbf{J}^{\varepsilon\perp} = 0 \end{cases}$$

Then

1. the leading term induces that $f(\mathbf{v}) \simeq F(\|\mathbf{v}\|)$,
2. in the limit $\varepsilon \to 0$

$$\frac{b_0}{\varepsilon} \mathbf{J}^\varepsilon \to - \left(\nabla_{\mathbf{x}} \int_{\mathbb{R}^d} \|\mathbf{v}\|^2 \, F \, d\mathbf{v} + \mathbf{E} \, \rho \right)^\perp$$

In the limit $\varepsilon \to 0$ for the 2D problem

It gives the guiding center system $\mathbf{U} = -\frac{\mathbf{E}^\perp}{b_0} = \frac{\nabla^\perp \phi}{b_0}$ (it is the $\frac{\mathbf{E}\times\mathbf{B}}{\|\mathbf{B}\|^2}$ drift)

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \mathrm{div}_{\mathbf{x}} \mathbf{U} \rho = 0 \\ -\Delta \phi = \rho - \rho_i. \end{cases}$$

This model satisfies some basic properties

1. preservation of energy

$$\frac{d}{dt} \int \|\mathbf{E}(t, \mathbf{x})\|^2 \, d\mathbf{x} = 0.$$

2. incompressible flow $\mathrm{div}_{\mathbf{x}} \mathbf{U} = 0$.
3. preservation of L^p norm of the density and for any continuous function η

$$\frac{d}{dt} \int \eta(\, \rho(t, \mathbf{x})\,) \, d\mathbf{x} = 0.$$

For the Vlasov-Poisson system, this asymptotic limit is justified rigorously by F. Golse and L. Saint-Raymond for weak solutions [1].

[1] Golse-St Raymond, JMPA'00, St Raymond, JMPA'02, Miot preprint 2016

In the limit $\varepsilon \to 0$ for the 2D problem

Second case : $b_{\text{ext}} \equiv b(\mathbf{x})$.

We cannot get an equation on the density ρ but on the distribution function $F \equiv F(t, \mathbf{x}, w)$ with $w = \|\mathbf{v}\|$.

1. From a Hilbert expansion of $f^\varepsilon = f_0 + \varepsilon f_1 + \ldots$
2. Apply a change of coordinate $\mathbf{v} = w\, \mathbf{e}_w(\theta)$.
3. We integrate on the angular velocity $\theta \in [0, 2\pi]$.

We formally get $f_0 = F(\|\mathbf{v}\|)$ such that

$$\frac{\partial F}{\partial t} + \mathbf{U} \cdot \nabla_{\mathbf{x}} F + u_w \frac{\partial F}{\partial w} = 0,$$

where $\mathbf{U}$ corresponds to the drift velocity and $(\mathbf{U}, u_w)$ is given by

$$\mathbf{U} = -\frac{1}{b}\left(\mathbf{E} - \frac{w^2}{2b}\nabla_{\mathbf{x}_\perp} b\right)^\perp, \quad u_w = -\frac{w}{2b^2}\nabla_{\mathbf{x}}^\perp b \cdot \mathbf{E},$$

We get the $\frac{\nabla_{\mathbf{x}}\|\mathbf{B}\| \times \mathbf{B}}{\|\mathbf{B}\|^3}$ drift.

- The drift-velocity results from two drifts,
- Energy structure is preserved and the flow remains incompressible.

Idea of the proof (take $b_{\rm ext}$ to be constant)

Step 1. Consider that $(\mathbf{E}, \mathbf{B})$ are given and smooth. We define

$$F^\varepsilon(w) = \frac{1}{2\pi}\int_0^{2\pi} f^\varepsilon \, d\theta, \quad \mathbf{J}^\varepsilon(w) = \frac{1}{2\pi}\int_0^{2\pi} \mathbf{e}_w(\theta)\, f^\varepsilon \, d\theta.$$

and

$$\Sigma^\varepsilon = \frac{1}{2\pi}\int_0^{2\pi} \left(\mathbf{e}_w(\theta)\otimes \mathbf{e}_w(\theta) - \frac{1}{2}\,\mathrm{Id}\right) f^\varepsilon \, d\theta$$

Step 2. We get the following equation

$$\begin{aligned} \partial_t F^\varepsilon \quad &+ \quad \mathrm{div}_{\mathbf{x}}\left(\frac{w^2}{2}\nabla_{\mathbf{x}}^\perp F^\varepsilon + \frac{w}{2}\mathbf{E}^\perp\, \partial_w F^\varepsilon \right) \\ &+ \quad \frac{1}{w}\partial_w \left(\frac{w^2}{2}\nabla_{\mathbf{x}}^\perp F^\varepsilon \cdot \mathbf{E} + \frac{w}{2}\mathbf{E}^\perp \cdot \mathbf{E}\, \partial_w F^\varepsilon \right) = -R^\varepsilon(\mathbf{J}^\varepsilon, \Sigma^\varepsilon), \end{aligned}$$

where R^ε is given by

$$\begin{aligned} R^\varepsilon \quad &= \quad w\, \mathrm{div}_x \left(\varepsilon\, w\, \partial_t \mathbf{J}^\varepsilon \,+\, w\, \mathrm{div}_{\mathbf{x}}(\Sigma^\varepsilon) \,+\, \partial_w \Sigma^\varepsilon\, \mathbf{E} \,+\, 2\,\Sigma^\varepsilon\, \mathbf{E}\right)^\perp \\ &+ \quad \frac{1}{w}\partial_w \left(w\, \left(\varepsilon\, w\, \partial_t \mathbf{J}^\varepsilon \,+\, w\, \mathrm{div}_{\mathbf{x}}(\Sigma^\varepsilon) \,+\, \partial_w \Sigma^\varepsilon\, \mathbf{E} \,+\, 2\,\Sigma^\varepsilon\, \mathbf{E}\,\right)^\perp \cdot \mathbf{E}\right). \end{aligned}$$

Step 3. Pass to the limit[2].

[2] Vlasov-Poisson system is still open

In the limit $\varepsilon \to 0$ for the 3D problem

Considering the $3D$ Vlasov-Poisson system with an external magnetic field, it is possible to formally derive an asymptotic model for the limit (F, P). Applying the same strategy, we get[3] (for a uniform magnetic field)

$$\begin{cases} \dfrac{\partial F}{\partial t} + \mathbf{E}_{F\perp}^{\perp} \cdot \nabla_{\mathbf{x}_\perp} F + \mathbf{E}_{P\|} \dfrac{\partial F}{\partial v_\|} + v_\| \dfrac{\partial P}{\partial x_\|} + \mathbf{E}_{F\|} \dfrac{\partial P}{\partial v_\|} = 0, \\ v_\| \dfrac{\partial F}{\partial x_\|} + \mathbf{E}_{F\|} \dfrac{\partial F}{\partial v_\|} = 0, \end{cases}$$

where $\mathbf{E} = (\mathbf{E}_{\alpha\perp}, \mathbf{E}_{\alpha\|})$ is the electric field computed from $\int \alpha d\mathbf{v}$.

Properties

- This model preserve the fundamental properties of the Vlasov-Poisson system : energy conservation, divergence free flow, positivity of F.
- Furthermore, we recover the classical drift velocity ($\mathbf{E} \times \mathbf{B}$, $\nabla|\mathbf{B}| \times \mathbf{B}$, etc)
- Adiabatic invariance.

[3] FF and P. Degond, arXiv:0905.2400 (2016)

Numerical approximation: state of the art

Aim. Construct a numerical scheme wich preserves the asymptotic behavior when $\varepsilon \to 0$.

Difficulty

There is no relaxation limit and no relaxation process to a unique equilibrium.

Concerning the Vlasov-Poisson system with an external magnetic field

- Boris' scheme : a semi-implicit and second order scheme conserving energy...
- N. Crouseilles, M. Lemou and F. Méhats, Asymptotic preserving schemes for highly oscillatory Vlasov-Poisson equation, JCP (2013).
- E. Frénod, S.A. Hirstoaga, M. Lutz, E. Sonnendrücker, Long time behaviour of an exponential integrator for a Vlasov-Poisson system with strong magnetic field, CiCP (2015)

Other related works

- E. Hairer, C. Lubich and G. Wanner, Geometric Numerical Integration Structure-Preserving Algorithms for Ordinary Differential Equations
- Ph. Chartier, E. Faou, group IPSO in Rennes

Particle-In-Cell approximation

Let us now consider the Vlasov-Poisson system with an external magnetic field and the corresponding characteristic curves which read

$$\begin{cases} \varepsilon \dfrac{d\mathbf{X}}{dt} = \mathbf{V}, \\ \varepsilon \dfrac{d\mathbf{V}}{dt} = -\dfrac{b_{\mathrm{ext}}(\mathbf{X})}{\varepsilon} \mathbf{V}^{\perp} + \mathbf{E}(t, \mathbf{X}), \\ \mathbf{X}(t^0) = \mathbf{x}^0, \ \mathbf{V}(t^0) = \mathbf{v}^0, \end{cases} \tag{1}$$

where the electric field is computed from the Poisson equation.

The Particle-In-Cell method,

- we consider a set of particles characterized by a weight $(w_{\mathbf{k}})_{\mathbf{k} \in \mathbb{N}}$ and their position in phase space $(\mathbf{x}_{\mathbf{k}}^n, \mathbf{v}_{\mathbf{k}}^n)_{\mathbf{k} \in \mathbb{N}}$ computed by discretizing the Vlasov-Poisson system at time $t^n = n\,\Delta t$.
- the solution f is discretized as follows

$$f_h^{n+1}(\mathbf{x}, \mathbf{v}) := \sum_{\mathbf{k} \in \mathbb{Z}^d} w_{\mathbf{k}}\, \varphi_h(\mathbf{x} - \mathbf{x}_{\mathbf{k}}^n)\, \varphi_h(\mathbf{v} - \mathbf{v}_{\mathbf{k}}^n),$$

Asymptotic limit for the characteristics

For the PIC method, it is crucial to consider that the electric and magnetic fields are smooth in such a way that characteristic curves are well defined.

First case : $b_{\text{ext}} = b_0$.

Let us study the long time behavior of the solution $(\mathbf{X}^\varepsilon, \mathbf{V}^\varepsilon)$ to

$$(\varepsilon \mathbf{X}^\varepsilon)' = \mathbf{V}^\varepsilon, \quad (\varepsilon \mathbf{V}^\varepsilon)' = \mathbf{E}(t, \mathbf{X}^\varepsilon) - \frac{b_0}{\varepsilon} \mathbf{V}^{\varepsilon \perp} \tag{2}$$

Then we combine two equations such that

$$\left(\mathbf{X}^\varepsilon - \varepsilon \frac{\mathbf{V}^{\varepsilon \perp}}{b_0} \right)' = \frac{E^\perp}{b_0}(t, \mathbf{X}^\varepsilon).$$

Passing formally to the limit and thanks to the energy estimate (for smooth electric field), it yields that $\mathbf{X}^\varepsilon \to \mathbf{Y}$ when $\varepsilon \to 0$ and

$$Y' = \frac{E^\perp}{b_0}(t, \mathbf{Y}). \tag{3}$$

It corresponds to the characteristic curves of the guiding center model.

Therefore,

$$\|\rho^\varepsilon - \rho\|_{W^{-1,1}} \le C_t \varepsilon \|f_0\|_{L^1((1+|\mathbf{v}|)d\mathbf{v})}.$$

The simplest scheme : first order Euler semi-implicit

Consider the first order Euler semi-implicit scheme

$$\begin{cases} \varepsilon \dfrac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t} = \mathbf{V}^{n+1}, \\ \varepsilon \dfrac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta t} = -\dfrac{b_{\text{ext}}(\mathbf{X}^n)}{\varepsilon} \mathbf{V}^{n+1\,\perp} + \mathbf{E}(t^n, \mathbf{X}^n), \\ \mathbf{X}^0 = \mathbf{x}^0,\ \mathbf{V}^0 = \mathbf{v}^0 . \end{cases} \tag{4}$$

We want to compare our discrete solution to

$$\frac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = -\frac{\mathbf{E}^{\perp}(t^n, \mathbf{y}^n)}{b_{\text{ext}}(\mathbf{y}^n)}. \tag{5}$$

Proposition

Let us fix $\Delta t > 0$, assume that $(\mathbf{x}^\varepsilon)_\varepsilon$ from (9) is bounded with respect to $\varepsilon > 0$ and

$$(\mathbf{x}^{\varepsilon 0}, \varepsilon \mathbf{v}^{\varepsilon 0}) \to (\mathbf{y}^0, 0), \quad \text{as} \quad \varepsilon \to 0.$$

Then, the limit satisfies (10).

Uniform accuracy result for large time step and small $\varepsilon > 0$

Theorem (FF and L.M. Rodrigues, SINUM 2016)

Consider that the electric field is given $E \in W^{1,\infty}((0,T)\times\mathbb{T}^2)$ *and set* $\lambda := \Delta t/\varepsilon^2$ *and* $\mathcal{R}[\mathbf{W}] = \mathbf{W}^{\perp}/b_{\rm ext}$. *Then,*

$$\mathbf{Z}^n = \varepsilon^{-1}\mathbf{V}^n - \mathcal{R}\left[\mathbf{E}(t^{n-1}, X^{n-1})\right]$$

satisfies

$$\mathbf{Z}^{n+1} = [\mathrm{Id} - \lambda\mathcal{R}]^{-1}(\mathbf{Z}^n - \mathcal{R}[\mathbf{E}(t^n, X^n) - \mathbf{E}(t^{n-1}, X^{n-1})]), \quad n \geq 1.$$

Moreover, there exists $C > 0$ *such that*

$$\|\mathbf{X}^n - \mathbf{Y}^n\| \leq C\,\varepsilon^2 \left[1 + \left\|\frac{1}{\varepsilon}\mathbf{V}^0 - \mathcal{R}[\mathbf{E}(t^0, X^0)]\right\|\right] e^{K_x\, n\Delta t}.$$

Corollary

For the density we have

$$\|\rho_\varepsilon^{n,\Delta t} - \rho^{n,\Delta t}\|_{W^{-1,1}} \leq C_{\mathbf{E},t^n}\,\varepsilon \int f_0\,(1 + \|\mathbf{v}\|)\,d\mathbf{v}.$$

Some comments and improvements

Comments :

1. This method introduces a numerical dissipation such that $(\mathbf{V}^\varepsilon)_\varepsilon \to 0$ and

$$\frac{\mathbf{V}^\varepsilon}{\varepsilon} \longrightarrow \frac{\mathbf{E}^\perp}{b_{\mathrm{ext}}}, \quad \text{as} \quad \varepsilon \to 0.$$

   Therefore $\frac{1}{2}\|V^\varepsilon\|^2$ is not anymore conserved and goes to zero...

2. We do not capture the “grad **B**” drift for non homogeneous magnetic field

$$\frac{1}{2b_{\mathrm{ext}}^2}\|V\|^2\,\nabla^\perp b_{\mathrm{ext}}.$$

   Indeed the numerical dissipation is too strong...

3. We’ve constructed second and third order semi-implicit schemes which preserve asymptotically the order of accuracy, that is, for smooth electro-magnetic fields and when $\varepsilon \to 0$, we get a scheme for the Guiding centre model with the same order of accuracy as for the Vlasov-Poisson system!

Improvements

1. Non homogeneous magnetic field

Second case : $b_{\rm ext} \equiv b(\mathbf{x})$.

We proceed as before

$$(\varepsilon \mathbf{X}^\varepsilon)' = \mathbf{V}^\varepsilon, \quad (\varepsilon \mathbf{V}^\varepsilon)' = \mathbf{E}(t, \mathbf{X}^\varepsilon) - \frac{b(\mathbf{X})}{\varepsilon} \mathbf{V}^{\varepsilon\perp} \tag{6}$$

We want to combine the two equations to remove the most singular term, hence we set $\mathbf{F} = \mathbf{E}/b$ and

Lemma

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{V}^\varepsilon\|^2 &= -\mathbf{V}^\varepsilon \cdot \nabla_{\mathbf{x}} \mathbf{F}^\varepsilon(t, \mathbf{X}^\varepsilon)\, \mathbf{V}^\varepsilon \\ &+ \varepsilon \left[\frac{d}{dt} \left(\mathbf{F}^\varepsilon(t, \mathbf{X}^\varepsilon) \cdot \mathbf{V}^\varepsilon \right) - \mathbf{V}^\varepsilon \cdot \frac{\partial \mathbf{F}^\varepsilon}{\partial t}(t, \mathbf{X}^\varepsilon) \right], \end{aligned} \tag{7}$$

and for $\mathbf{E}^\varepsilon = (E_1^\varepsilon, E_2^\varepsilon)$

$$\left\{ \begin{aligned} &\frac{1}{4} \frac{d}{dt} \left[|v_1^\varepsilon|^2 - |v_2^\varepsilon|^2 \right] = \frac{v_1^\varepsilon\, E_1^\varepsilon - v_2^\varepsilon\, E_2^\varepsilon}{2\,\varepsilon} - \frac{b(\mathbf{X}^\varepsilon)}{\varepsilon^2}\, v_1^\varepsilon\, v_2^\varepsilon, \\ &\frac{d}{dt} \left[v_1^\varepsilon\, v_2^\varepsilon \right] = \frac{v_2^\varepsilon\, E_1^\varepsilon + v_1^\varepsilon\, E_2^\varepsilon}{\varepsilon} - \frac{b(\mathbf{X}^\varepsilon)}{\varepsilon^2} \left[|v_1^\varepsilon|^2 - |v_2^\varepsilon|^2 \right]. \end{aligned} \right. \tag{8}$$

Second case : $b_{\rm ext} \equiv b(\mathbf{x})$.

We set $\mathbf{Z} = \mathbf{V}/b$ and get

$$\mathbf{X}' = \frac{b}{\varepsilon}\mathbf{Z},$$

and get an equation for $\mathbf{Z}$,

$$(\varepsilon \mathbf{Z})' = \varepsilon \frac{\mathbf{V}'}{b} + \varepsilon \left(\frac{1}{b}\right)' \mathbf{V} = \frac{\mathbf{E}}{b} - \frac{b}{\varepsilon}\mathbf{Z}^{\perp} - (\nabla_{\mathbf{x}} b \cdot \mathbf{Z})\,\mathbf{Z}$$

Now, we decompose the last term as

$$\mathbf{Z}\,(\nabla_{\mathbf{x}} b \cdot \mathbf{Z}) = \frac{1}{2}\|\mathbf{Z}\|^2\,\nabla_{\mathbf{x}} b \pm \frac{1}{2}\left[z_1^2 - z_2^2\right]\,\nabla_{\mathbf{x}} b + [z_1\, z_2]\begin{pmatrix} \partial_{x_2} b \\ \partial_{x_1} b \end{pmatrix}.$$

and we get the result by combining the latter equations

$$\left(\mathbf{X} - \varepsilon \mathbf{Z}^{\perp}\right)' = \frac{\mathbf{E}^{\perp}}{b} - \frac{1}{2}\|\mathbf{Z}\|^2 \nabla_{\mathbf{x}}^{\perp} b + O(\varepsilon).$$

Non homogeneous magnetic field

We consider the system

$$\begin{cases} \varepsilon \dfrac{d\mathbf{X}}{dt} = \mathbf{V}, \\[2ex] \varepsilon \dfrac{d\mathbf{V}}{dt} = -\dfrac{b_{\text{ext}}(\mathbf{X})}{\varepsilon} \mathbf{V}^{\perp} + \mathbf{E}(t, \mathbf{X}), \end{cases}$$

also set the kinetic energy $e = \frac{1}{2}\|\mathbf{V}\|^2$ which satisfies

$$\varepsilon \frac{dw}{dt} = \mathbf{E} \cdot \mathbf{V}$$

Finally, we propose

- solve numerically the equation to $(\mathbf{X}, \mathbf{Z})$
- solve the equation to w to get the precise evolution of it (slow scale)
- write the source term on the eqn for $\mathbf{Z}$ as

$$\mathbf{Z}\, \nabla_{\mathbf{x}} b \cdot \mathbf{Z} = \frac{1}{b^2} w \nabla_{\mathbf{x}} b \pm \frac{1}{2} \left[z_1^2 - z_2^2 \right] \nabla_{\mathbf{x}} b + [z_1\ z_2] \begin{pmatrix} \partial_{x_2} b \\ \partial_{x_1} b \end{pmatrix}.$$

Then apply semi-implicit schemes for the equation in $\mathbf{Z}$ and explicit for $(w, \mathbf{X})$ and set

$$\mathbf{V} = \sqrt{2\, w}\, \frac{\mathbf{Z}}{\|\mathbf{Z}\|}.$$

The simplest scheme : first order Euler semi-implicit

Consider the first order Euler semi-implicit scheme

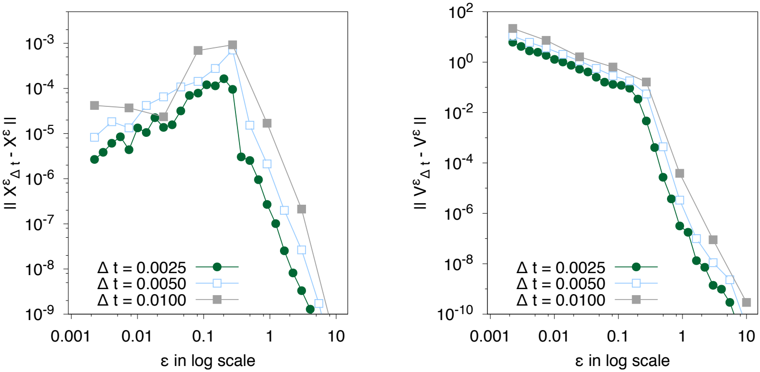
$$\begin{cases} \varepsilon \dfrac{\mathbf{X}^{n+1} - \mathbf{X}^n}{\Delta t} = \mathbf{V}^{n+1}, \\[2ex] \varepsilon \dfrac{\mathbf{V}^{n+1} - \mathbf{V}^n}{\Delta t} = -\dfrac{b_{\text{ext}}(\mathbf{X}^n)}{\varepsilon} \mathbf{V}^{n+1\,\perp} - \phi(e^n, \|\mathbf{V}^n\|^2/2)\, e^n \nabla \log(b(\mathbf{X}^n)) + \mathbf{E}(t^n, \mathbf{X}^n), \\[2ex] \varepsilon \dfrac{e^{n+1} - e^n}{\Delta t} = \mathbf{V}^{n+1} \cdot \mathbf{E}(t^n, \mathbf{X}^n), \\[2ex] \mathbf{X}^0 = \mathbf{x}^0,\ \mathbf{V}^0 = \mathbf{v}^0,\ e^0 = 0.5\, \|\mathbf{v}^0\|^2 . \end{cases} \tag{9}$$

We want to compare our discrete solution to

$$\begin{cases} \dfrac{\mathbf{y}^{n+1} - \mathbf{y}^n}{\Delta t} = -\dfrac{1}{b_{\text{ext}}(\mathbf{y}^n)} \left(\mathbf{E}(t^n, \mathbf{y}^n) - e^n \nabla \log(b(\mathbf{y}^n))\right)^\perp \\[2ex] \dfrac{e^{n+1} - e^n}{\Delta t} = -\dfrac{1}{b_{\text{ext}}(\mathbf{y}^n)}\, e^n \nabla^\perp \log(b(\mathbf{y}^n)) \cdot \mathbf{E}(t^n, \mathbf{y}^n). \end{cases} \tag{10}$$

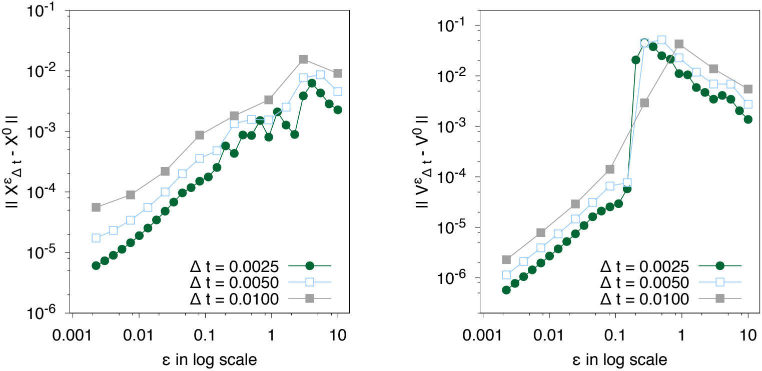
One single particle motion without electric field

Numerical error (a) $\|\mathbf{X}^{\varepsilon}_{\Delta t} - \mathbf{X}^{\varepsilon}\|$, (b) $\|\mathbf{V}^{\varepsilon}_{\Delta t} - \mathbf{V}^{\varepsilon}\|$ obtained with different time steps Δt with the third order scheme

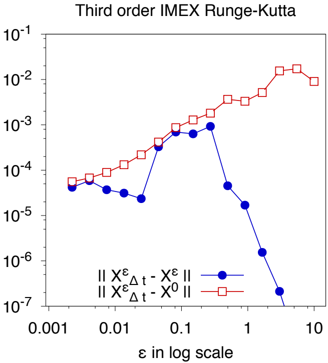


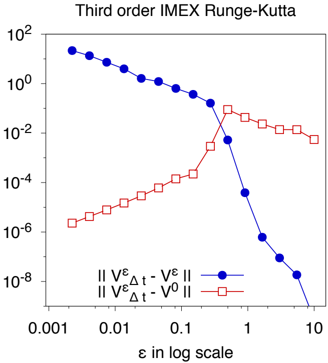
One single particle motion without electric field

Numerical error (a) $\|\mathbf{X}^{\varepsilon}_{\Delta t} - \mathbf{X}^0\|$, (b) $\|\mathbf{V}^{\varepsilon}_{\Delta t} - \mathbf{V}^0\|$ obtained with different time steps Δt with the third order scheme

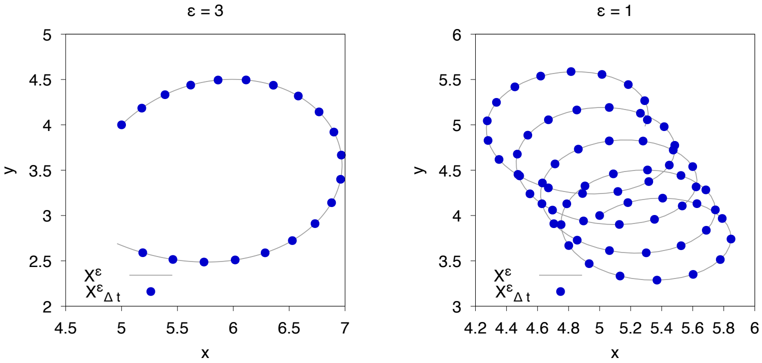


One single particle motion without electric field

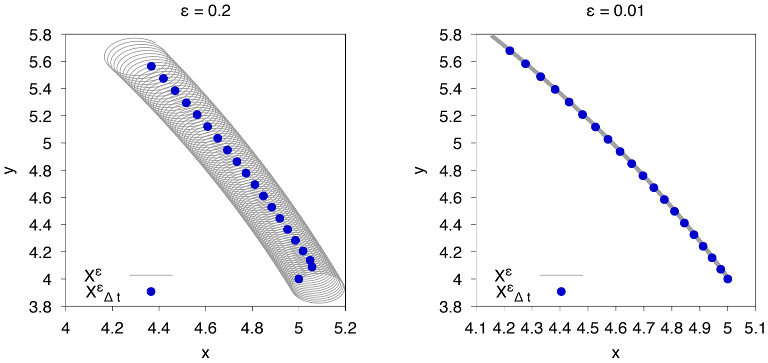




One single particle motion with an electric field



One single particle motion with an electric field



Vlasov-Poisson system with nonhomogeneous magnetic field

We now consider the Vlasov-Poisson system with an external magnetic field set in a disk centred at the origin and of radius $R_0 = 6$, $\Omega = D(0,6)$, with $\varepsilon = 0.05$ and the initial data

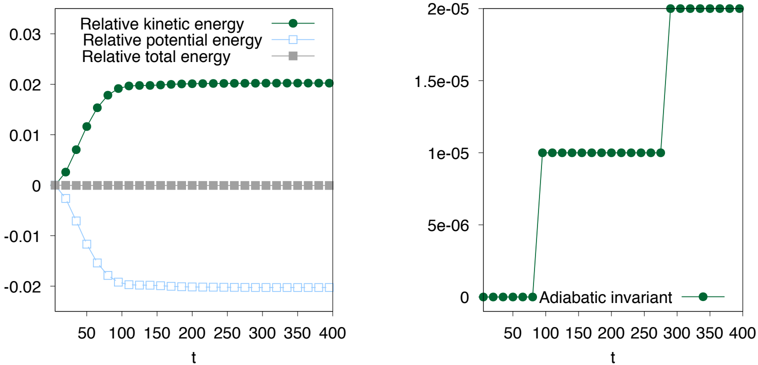
$$f_0(\mathbf{x},\mathbf{v}) = \frac{1}{8\pi^2\, v_{th}^2}\left[\exp\left(-\frac{\|\mathbf{x}-\mathbf{x}_0\|^2}{2}\right) + \exp\left(-\frac{\|\mathbf{x}+\mathbf{x}_0\|^2}{2}\right)\right] \exp\left(-\frac{\|\mathbf{v}\|^2}{2v_{th}^2}\right),$$

with $v_{th} = \sqrt{2}$, $\mathbf{x}_0 = (3/2, -3/2)$.
We perform numerical simulations with

$$b(\mathbf{x}) = \frac{10}{\sqrt{10^2 - \|\mathbf{x}\|^2}},$$

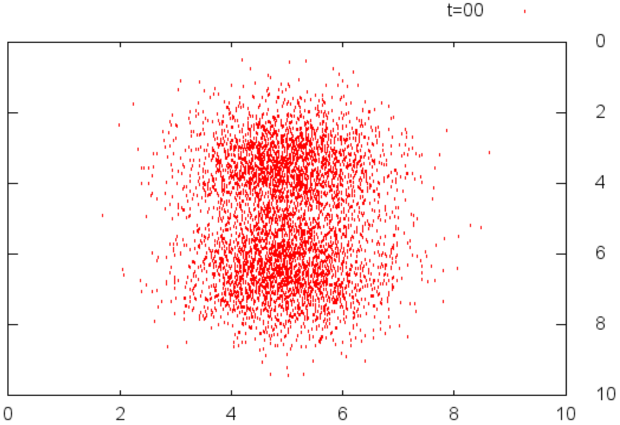
such that it is one at the origin and smaller elsewhere.

Vlasov-Poisson system with nonhomogeneous magnetic field



Vlasov-Poisson system nonhomogeneous magnetic field. Time evolution of total energy and adiabatic invariant obtained with $\Delta t = 0.1$

The Vlasov-Poisson system $\varepsilon = 0.01$



Conclusion

Comments :

- Dominant term is a magnetics field $\frac{1}{\varepsilon}(v \times B) \cdot \nabla_v f$, no more dissipative effects
- We have performed a rigorous analysis on the particle trajectories for a homogeneous magnetic field, but only formal for the non homogeneous case.

Current and future works :

- Applications in plasma physics
  - Treat more complex problems : capture drift due to the gradients of the magnetic field, etc
- Applications to numerical analysis
  - Better understanding of the stability of high order schemes for PIC methods and for the Vlasov-Poisson system.