Uniformly accurate Particle-In-Cell method for Vlasov-Poisson with strong magnetic field

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Outline

- Two-scale method for oscillatory equations
- Long-time Vlasov-Poisson with strong magnetic field
  - Particle-in-cell method
  - Re-formulation of characteristics
  - Uniformly accurate scheme and results
- Conclusions and future work

Two-scale method for oscillatory equations

A classical oscillatory model problem:

$$\partial_t u(t) = F(t, t/\varepsilon, u(t)), \quad t > 0,$$
$$u(0) = u_0.$$

- $0 < \varepsilon \leq 1$ given parameter. $F(t, \tau, u)$: smooth and 2π-periodic in τ.
- $u(t)$: highly oscillatory in t as $0 < \varepsilon \ll 1$.
- Including:
  - Classical Hamiltonian system
  - Dispersive and wave equation: nonlinear Klein-Gordon in nonrelativistic limit; oscillatory nonlinear Schrödinger equation
- Widely studied:
  - Classical discretization: Δt restricted by ε for accuracy.
  - Approximation in limit regime: time-averaging
  - Asymptotic preserving: order reduction

Two-scale method for oscillatory equations

Recent trend: uniformly accurate (UA) numerical method for all $0 < \varepsilon \leq 1$.

Two-scale method (*Ph. Chartier etc. 2015*):

$$\partial_t U(t,\tau) + \frac{1}{\varepsilon}\partial_\tau U(t,\tau) = F(t,\tau,U(t,\tau)), \quad t > 0,\ \tau \in \mathbb{T},$$
$$U(0,\tau) = U_0(\tau).$$

- τ: fast time variable; independent of t.
- $U(t,\tau)$ periodic in τ.
- $U_0(\tau)$: prescribed only at $U_0(0) = u_0$.
- $u(t) = U(t, t/\varepsilon)$.

Freedom to choose $U_0(\tau)$ such that: $\partial_t^k U(t,\tau) = O(1),\ \varepsilon \to 0$, for $k = 0, 1, 2, 3...$

Two-scale method for oscillatory equations

Chapman-Enskog expansion:

$$U(t,\tau) = \underline{U}(t) + h(t,\tau), \quad t \geq 0,\ \tau \in \mathbb{T}$$

- $\underline{U}(t) := \Pi U(t,\tau) = \frac{1}{2\pi}\int_0^{2\pi} U(t,\tau)d\tau$
  - macro-part, $O(1)$
  - independent of τ
- $h(t,\tau) := U(t,\tau) - \underline{U}(t)$
  - micro-part, $o(1)$
  - zero average

The two-scale equation decompose to

$$\begin{cases} \partial_t \underline{U} = \Pi\left(\mathcal{F}(t,\tau,\underline{U}+h)\right), \\ \partial_t h + \dfrac{1}{\varepsilon} L h = (I-\Pi)\left(\mathcal{F}(t,\tau,\underline{U}+h)\right). \end{cases}$$

$Lh := \partial_\tau h$. L has inverse: $L^{-1}h = (I-\Pi)\int_0^\tau h(t,\theta)d\theta$

Two-scale method for oscillatory equations

- $\underline{U}(t)$: independent of ε;
- look for asymptotic orders for $h(t,\tau)$

$$
\begin{aligned}
&h = \varepsilon A\left(\mathcal{F}(t,\tau,\underline{U}+h)\right) - \varepsilon L^{-1}(\partial_t h),\\
&\partial_t h = \varepsilon A\left[\partial_t \mathcal{F}(t,\tau,\underline{U}+h) + \partial_U \mathcal{F}(t,\tau,\underline{U}+h)(\partial_t \underline{U} + \partial_t h)\right]\\
&\qquad\quad - \varepsilon L^{-1}(\partial_t^2 h).
\end{aligned}
$$

$A := L^{-1}(I-\Pi)$.

- Assume $\partial_t h = O(1) \Rightarrow h = O(\varepsilon)$
- Assume $\partial_t^2 h = O(1) \Rightarrow \partial_t h = O(\varepsilon)$ & 1st order expansion:

$$
h(t,\tau) = h_1(t,\tau,\underline{U}) + O(\varepsilon^2), \text{ with } h_1(t,\tau,U) := \varepsilon A\left(\mathcal{F}(t,\tau,U)\right).
$$

- Further assume $\partial_t^3 h = O(1) \Rightarrow$ 2nd order expansion:

$$
\begin{aligned}
h(t,\tau) =& \varepsilon A\left(\mathcal{F}(t,\tau,\underline{U}+h_1)\right) - \varepsilon^2 A^2 \partial_t \mathcal{F}(t,\tau,\underline{U})\\
&- \varepsilon^2 A^2 \partial_U \mathcal{F}(t,\tau,\underline{U})\Pi\left(\mathcal{F}(t,\tau,\underline{U})\right) + O(\varepsilon^3)
\end{aligned}
$$

Two-scale method for oscillatory equations

- Valid at $t = 0$:

$$U(0,\tau) = \underline{U}(0) + h(0,\tau)$$

- $\underline{U}(0)$ known via:

$$u_0 = U(0,0) = \underline{U}(0) + h(0,0)$$

$\Rightarrow U(0,\tau) = u_0 + h(0,\tau) - h(0,0)$

  - 1st order expansion of $h \Rightarrow$ 1st order initial data
  - 2nd order expansion of $h \Rightarrow$ 2nd order initial data

Rigorous result (*Chartier, Crouseilles, Lemou and Méhats, 2015*): Under 2nd (or 1st) order initial data, $\partial_t^k U(t,\tau) = O(1)$ for $k = 0, 1, 2, 3$ (or $k = 0, 1, 2$) for all $0 < \varepsilon \leq 1$.

Two-scale method for oscillatory equations

Numerical methods for the two-scale equation:

$$\partial_t U(t,\tau) + \frac{1}{\varepsilon}\partial_\tau U(t,\tau) = F(t,\tau,U(t,\tau))$$

- In τ: Fourier spectral
- In t: finite difference (FD)
  - semi-implicit to avoid stability issue
  - need $\partial_t^{n+1} U = O(1)$ for n-th order accuracy.

A 2nd order scheme (*Chartier etc. 2015*):

$$\begin{cases} \dfrac{U^{n+1/2}(\tau) - U^n(\tau)}{\Delta t/2} + \dfrac{1}{\varepsilon}\partial_\tau U^{n+1/2}(\tau) = \mathcal{F}\left(t_n, \tau, U^n(\tau)\right), \\ \dfrac{U^{n+1}(\tau) - U^n(\tau)}{\Delta t} + \dfrac{1}{2\varepsilon}\partial_\tau \left(U^{n+1}(\tau) + U^n(\tau)\right) = \mathcal{F}\left(t_{n+1/2}, \tau, U^{n+1/2}(\tau)\right). \end{cases}$$

Two-scale method for oscillatory equations

- UA 2nd (or 1st) order FD method obtained under the 2nd (or 1st) order initial data (*Chartier etc. 2015*).

- In t (alternatively): exponential integrator (EI)

  - need $\partial_t^n U = O(1)$ for n-th order accuracy.

  - UA 2nd (or 1st) order EI method under the 1st (or 0th) order initial data (*details later*)

Two-scale method: application

2D beam model:

$$\partial_t f + \frac{v}{\varepsilon}\partial_r f + \left(E - \frac{r}{\varepsilon} + a\left(\frac{t}{\varepsilon}\right) r\right)\partial_v f = 0, \quad t > 0,\ r, v \in \mathbb{R},$$
$$\partial_r (rE) = r\int_{\mathbb{R}} f dv, \quad t > 0,\ r \in \mathbb{R},$$
$$f(0, r, v) = f_0(r, v), \quad r, v \in \mathbb{R}.$$

- $a(\cdot)$: given & 2π-periodic
- $E(t, r)$: self-consistent electrical field
- filter out main oscillation:

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} := e^{-Jt/\varepsilon}\begin{pmatrix} r \\ v \end{pmatrix} = \begin{pmatrix} \cos(t/\varepsilon) & -\sin(t/\varepsilon) \\ \sin(t/\varepsilon) & \cos(t/\varepsilon) \end{pmatrix}\begin{pmatrix} r \\ v \end{pmatrix},\ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$g(t, \xi) := f\left(t, \cos(t/\varepsilon)\xi_1 + \sin(t/\varepsilon)\xi_2, -\sin(t/\varepsilon)\xi_1 + \cos(t/\varepsilon)\xi_2\right)$$

Two-scale method: application

Beam model in rotating frame:

$$\begin{cases} \partial_t g(t,\xi) + \mathcal{E}(t,t/\varepsilon,\xi)\left[-\sin(t/\varepsilon)\,\partial_{\xi_1} g(t,\xi) + \cos(t/\varepsilon)\,\partial_{\xi_2} g(t,\xi)\right] = 0 \\ g(0,\xi) = f_0(\xi_1,\xi_2). \end{cases}$$

$$\begin{aligned}\mathcal{E}(t,\tau,\xi) := &\frac{1}{r(\tau,\xi)}\int_0^{r(\tau,\xi)}\int_{\mathbb{R}} s g\left(t, \cos(\tau)s - \sin(\tau)v, \sin(\tau)s + \cos(\tau)v\right) dvds \\ &+ a(\tau)[\cos(\tau)\xi_1 + \sin(\tau)\xi_2],\end{aligned}$$

$$r(\tau,\xi) := \cos(\tau)\xi_1 + \sin(\tau)\xi_2.$$

- Characteristics

$$\dot{\xi}_1(t) = -\sin(t/\varepsilon)\,\Xi(t,t/\varepsilon,\xi),$$
$$\dot{\xi}_2(t) = \cos(t/\varepsilon)\,\Xi(t,t/\varepsilon,\xi).$$

- Straightforward for the two-scale method: UA semi-Lagrangian! [1]

[1] *M. Lemou, F. Méhats, N. Crouseilles, X. Zhao, Uniformly accurate forward semi-Lagrangian methods for highly oscillatory Vlasov-Poisson equations, hal-01286947, preprint 2016.*

Long-time Vlasov-Poisson with strong magnetic field

Consider the Vlasov-Poisson equation:

$$\partial_t f + \frac{\mathbf{v}}{\varepsilon} \cdot \nabla_{\mathbf{x}} f + \frac{1}{\varepsilon}\left(\mathbf{E} + \frac{1}{\varepsilon}\mathbf{v}^{\perp}\right) \cdot \nabla_{\mathbf{v}} f = 0, \quad t > 0,\ \mathbf{x}, \mathbf{v} \in \mathbb{R}^2,$$

$$\mathbf{E}(t, \mathbf{x}) = -\nabla_{\mathbf{x}} \phi(t, \mathbf{x}), \quad -\Delta\phi(t, \mathbf{x}) = \int_{\mathbb{R}^2} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} - 1,$$

$$f(0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}), \quad \mathbf{x}, \mathbf{v} \in \mathbb{R}^2, t \in [0, T].$$

where $f = f(t, \mathbf{x}, \mathbf{v})$, $\mathbf{x} = (x_1, x_2)$, $\mathbf{v} = (v_1, v_2)$, $v^{\perp} = (v_2, -v_1)$.

- 4D drift-kinetic regime
- Strong magnetic field and long-time effect: $0 < \varepsilon \leq 1$.
- Main oscillation due to $\partial_t f + 1/\varepsilon^2 v^{\perp} \cdot \nabla_v f$
- f has wavelength in time $\mathcal{O}(\varepsilon^2)$

Long-time Vlasov-Poisson with strong magnetic field

- Theoretical wise: As $\varepsilon \to 0$, $f(t, \mathbf{x}, \mathbf{v})$ two-scale convergence to $g(t, \mathbf{x}, \mathbf{r}(\mathbf{v}, \tau))$ (*Frenód and Sonnerdrücker, 2000*)

$$\partial_t g(t, \mathbf{x}, \mathbf{v}) + \mathbf{E}^{\perp} \cdot \nabla_{\mathbf{x}} g - \frac{1}{2} (\nabla \cdot \mathbf{E}) \mathbf{v}^{\perp} \cdot \nabla_{\mathbf{v}} g = 0, \ \ t > 0, \ \ \mathbf{x} \in \mathbb{R}^2,$$
$$-\nabla \cdot \mathbf{E}(t, \mathbf{x}) = \int_{\mathbb{R}^2} g(t, \mathbf{x}, \mathbf{v}) d\mathbf{v} - 1,$$
$$g(0, \mathbf{x}, \mathbf{v}) = f_0(\mathbf{x}, \mathbf{v}), \quad \mathbf{r}(\mathbf{v}, \tau) = \mathrm{e}^{-\tau J} \mathbf{v}.$$

Further implies Guiding Center equation:

$$\partial_t \rho(t, \mathbf{x}) + \mathbf{E}^{\perp} \cdot \nabla_{\mathbf{x}} \rho = 0, \quad t > 0, \ \ \mathbf{x} \in \mathbb{R}^2,$$
$$-\nabla \cdot \mathbf{E}(t, \mathbf{x}) = \rho(t, \mathbf{x}) - 1,$$
$$\rho(0, \mathbf{x}) = \int_{\mathbb{R}^2} g(0, \mathbf{x}, \mathbf{v}) d\mathbf{v}.$$

- Numerical wise: multiscale numerical schemes
  - Exponential integrator (*Frénod, Hirstoaga, Lutz and Sonnendrücker, 2015*): a limit solver
  - AP semi-implicit Runge-Kutta (*Filbet and Rodrigues 2016*)

Long-time Vlasov-Poisson with strong magnetic field

Aim: construct a numerical scheme which

- is free from the constraint $\Delta t = \mathcal{O}(\varepsilon^p), p > 0$
- deals with all regimes: $\varepsilon = \mathcal{O}(1)$, $\varepsilon \ll 1$ and intermediate, with same cost and the same precision (uniform accuracy).

Long-time Vlasov-Poisson with strong magnetic field

Particle-in-Cell method: $f(t, \mathbf{x}, \mathbf{v})$ is sampled by

$$f_p(t, \mathbf{x}, \mathbf{v}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - \mathbf{x}_k(t)) \delta(\mathbf{v} - \mathbf{v}_k(t)), \quad t \geq 0, \ \mathbf{x}, \mathbf{v} \in \mathbb{R}^2,$$

where the position $\mathbf{x}_k(t)$ and velocities $\mathbf{v}_k(t)$ obeys

$$\begin{aligned} \dot{\mathbf{x}}_k(t) &= \frac{\mathbf{v}_k(t)}{\varepsilon}, \\ \dot{\mathbf{v}}_k(t) &= \frac{\mathbf{E}(t, \mathbf{x}_k(t))}{\varepsilon} + \frac{\mathbf{v}_k^{\perp}(t)}{\varepsilon^2}, \\ \mathbf{x}_k(0) &= \mathbf{x}_{k,0}, \quad \mathbf{v}_k(0) = \mathbf{v}_{k,0}. \end{aligned}$$

$\mathbf{E} = -\nabla_x \phi$ is determined from $(\mathbf{x}_k(t))_k$ at position $\mathbf{x}$ by

$$-\Delta \phi(t, \mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - \mathbf{x}_k(t)) - 1.$$

In practice, $\delta \leftrightarrow \varphi$ for example by the B-spline function

Long-time Vlasov-Poisson with strong magnetic field

- Self-consistent electrical field is oscillatory: $\mathbf{E}(t, \mathbf{x}) = \mathcal{E}(t, t/\varepsilon^2, \mathbf{x})$.
- Start to reformulate the characteristic to use the two-scale method.

First step: Filtering main oscillation

$$\mathbf{y}(t) = \mathrm{e}^{-tJ/\varepsilon^2}\mathbf{v}(t), \text{ with } J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \text{ and } \mathrm{e}^{sJ} = \begin{pmatrix} \cos s & \sin s \\ -\sin s & \cos s \end{pmatrix}.$$

Then the characteristic becomes

$$\dot{\mathbf{x}}(t) = \frac{1}{\varepsilon}\mathrm{e}^{tJ/\varepsilon^2}\mathbf{y}(t),$$
$$\dot{\mathbf{y}}(t) = \frac{1}{\varepsilon}\mathrm{e}^{-tJ/\varepsilon^2}\mathcal{E}(t, t/\varepsilon^2, \mathbf{x}(t)),$$
$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{y}(0) = \mathbf{v}_0.$$

with $\nabla_x \cdot \mathcal{E}(t, t/\varepsilon^2, \mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - \mathbf{x}_k(t)) - 1$.

Long-time Vlasov-Poisson with strong magnetic field

One possible way: the generalised two-scale method in diffusion scaling[2]: $X(t,\tau)$ and $Y(t,\tau)$ for $\tau \in [0, 2\pi]$ such that

$$X(t, \tau = t/\varepsilon^2) = \mathbf{x}_k(t), \quad Y(t, \tau = t/\varepsilon^2) = \mathbf{y}(t).$$

X and Y satisfies

$$\partial_t X(t,\tau) + \frac{1}{\varepsilon^2}\partial_\tau X(t,\tau) = \frac{1}{\varepsilon}\mathrm{e}^{\tau J} Y(t,\tau),$$
$$\partial_t Y(t,\tau) + \frac{1}{\varepsilon^2}\partial_\tau Y(t,\tau) = \frac{1}{\varepsilon}\mathrm{e}^{-\tau J} \mathcal{E}(t,\tau,X(t,\tau)),$$
$$X(0,0) = \mathbf{x}_0, \, Y(0,0) = \mathbf{y}_0,$$

where the electric field is given by

$$\nabla_x \cdot \mathcal{E}(t,\tau,\mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - X_k(t,\tau)) - 1.$$

[2] *M. Lemou, F. Méhats, X. Zhao, Uniformly accurate numerical schemes for the nonlinear Dirac equation in the nonrelativistic limit regime, arXiv:1605.02475[math.NA], preprint 2016.*

Long-time Vlasov-Poisson with strong magnetic field

Come back to the (diffusion scaling) two-scale problem

$$\partial_t X(t,\tau) + \frac{1}{\varepsilon^2}\partial_\tau X(t,\tau) = \frac{1}{\varepsilon}\mathrm{e}^{\tau J} Y(t,\tau),$$
$$\partial_t Y(t,\tau) + \frac{1}{\varepsilon^2}\partial_\tau Y(t,\tau) = \frac{1}{\varepsilon}\mathrm{e}^{-\tau J}\mathcal{E}(t,\tau,X(t,\tau)),$$
$$X(0,0) = \mathbf{x}_0,\ Y(0,0) = \mathbf{y}_0,$$
$$\nabla_x \cdot \mathcal{E}(t,\tau,\mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - X_k(t,\tau)).$$

- Could overcome temporal oscillation.
- However, $\mathcal{E}$ can not be obtained exactly!
- Spatial error and particle error are amplified by $1/\varepsilon$ which is even worse!!
- Seek for more transformation.

Long-time Vlasov-Poisson with strong magnetic field

Then, we introduce two new unknown functions $U_\pm(t,\tau)$:

$$U_+(t,\tau) := 2\left(X(t,\tau) + \varepsilon J \mathrm{e}^{\tau J} Y(t,\tau)\right), \quad U_-(t,\tau) := -2\varepsilon J Y(t,\tau),$$

which satisfy

$$\partial_t U_\pm + \frac{1}{\varepsilon^2}\partial_\tau U_\pm = \mathcal{F}_\pm(t,\tau,U_+,U_-),$$

with

$$\begin{aligned} \mathcal{F}_+(t,\tau,U_+,U_-) &= 2J\mathcal{E}\left(t,\tau,\tfrac{1}{2}\left(U_+ + \mathrm{e}^{\tau J}U_-\right)\right), \\ \mathcal{F}_-(t,\tau,U_+,U_-) &= -2J\mathrm{e}^{-\tau J}\mathcal{E}\left(t,\tau,\tfrac{1}{2}\left(U_+ + \mathrm{e}^{\tau J}U_-\right)\right), \end{aligned}$$

since

$$X = \frac{1}{2}\left(U_+ + \mathrm{e}^{\tau J}U_-\right), \qquad Y = \frac{1}{2\varepsilon}JU_-.$$

Long-time Vlasov-Poisson with strong magnetic field

Write the reformulated problem for (U_+, U_-) as

$$\begin{aligned} \partial_t U + \frac{1}{\varepsilon^2}\partial_\tau U - \mathcal{F}(t,\tau,U) &= 0, \\ U(t=0,\tau=0) &= \mathbf{u}_0 = (2(\mathbf{x}_0 + \varepsilon J\mathbf{y}_0), -2\varepsilon J\mathbf{y}_0). \end{aligned}$$

Then decompose $U(t,\tau) = \underline{U}(t) + \mathbf{h}(t,\tau)$, with $\underline{U}(t) = \Pi U(t,\cdot)$
Use first order the expansion derived before

$$U(t,\tau) = \underline{U}(t) + \varepsilon^2 L^{-1}(I-\Pi)\mathcal{F}(t,\tau,\underline{U}(t)) + \mathcal{O}(\varepsilon^4). \quad (*)$$

$\mathcal{F} \propto \mathcal{E} = \mathcal{E}[U](t,\tau,X)$ where $\mathcal{E}[U](t,\tau,X)$ is computed via

$$\nabla_x \cdot \mathcal{E}(t,\tau,\mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta\left(\mathbf{x} - \frac{1}{2}(U_+(t,\tau) + \mathrm{e}^{\tau J} U_-(t,\tau))_k\right) - 1.$$

Expansion $(*)$ is implicit. Need to approximate explicitly $\mathcal{E}$.

Long-time Vlasov-Poisson with strong magnetic field

To do so, from the relation between $\underline{U}_{\pm}$ and X, we have

$$X = X^{1st} + \mathcal{O}(\varepsilon^2), \quad \text{with } X^{1st}(t,\tau) := \frac{1}{2}\left[\underline{U}_{+}(t) + e^{\tau J}\underline{U}_{-}(t)\right].$$

Then, one can compute $\mathcal{E}^{1st} = \mathcal{E} + \mathcal{O}(\varepsilon^2)$ as

$$\nabla_x \cdot \mathcal{E}^{1st}(t,\tau,\mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - X_k^{1st}(t,\tau)) - 1,$$

Finally,

$$\mathbf{u}_0 =: U(0,\tau=0) = \underline{U}(0) + \varepsilon^2 h(\tau=0),$$

and we deduce

$$U(0,\tau=0) = \mathbf{u}_0 + \varepsilon^2(h(\tau) - h(0)), \quad (**)$$

where $h(\tau) = L^{-1}(I - \Pi)\mathcal{F}^{1st}(0,\tau,\mathbf{u}_0)$.

Long-time Vlasov-Poisson with strong magnetic field

Proposition (formal)

$$\partial_t U + \frac{1}{\varepsilon^2}\partial_\tau U - \mathcal{F}(t,\tau,U) = 0,$$
$$U(0,\tau) \text{ given by } (**)$$

Then we have (for a fixed $t \in [0, T]$)

$$\sup_{\varepsilon\in]0,1]} \|\partial_t^k U^\varepsilon(t,\cdot)\|_{L^\infty_\tau} \le C_1, \qquad k = 0,1,2.$$

$$\sup_{\varepsilon\in]0,1]} \|d_t^k \mathcal{F}(t,\cdot)\|_{L^\infty_\tau} \le C_2, \qquad k = 0,1,2.$$

Numerical scheme-exponential integrator

τ being periodic, we consider the Fourier transform in τ to get

$$\widehat{U}_\ell'(t) + \frac{i\ell}{\varepsilon^2}\widehat{U}_\ell(t) = \widehat{\mathcal{F}}_\ell(t).$$

Integrating between t_n and t_{n+1}, we have

$$\begin{aligned}
\widehat{U}_\ell(t_{n+1}) &= \mathrm{e}^{-\frac{i\ell\Delta t}{\varepsilon^2}}\widehat{U}_\ell(t_n) + \int_0^{\Delta t} \mathrm{e}^{-\frac{i\ell}{\varepsilon^2}(\Delta t - s)}\widehat{\mathcal{F}}_\ell(t_n+s)ds \\
&\approx \mathrm{e}^{-\frac{i\ell\Delta t}{\varepsilon^2}}\widehat{U}_\ell(t_n) + \int_0^{\Delta t} \mathrm{e}^{-\frac{i\ell}{\varepsilon^2}(\Delta t - s)}\left(\widehat{\mathcal{F}}_\ell(t_n) + s\frac{d}{dt}\widehat{\mathcal{F}}_\ell(t_n)\right)ds \\
&\approx \mathrm{e}^{-\frac{i\ell\Delta t}{\varepsilon^2}}\widehat{U}_\ell(t_n) + p_\ell\widehat{\mathcal{F}}_\ell(t_n) + q_\ell\frac{\widehat{\mathcal{F}}_\ell(t_n) - \widehat{\mathcal{F}}_\ell(t_{n-1})}{\Delta t},
\end{aligned}$$

with

$$p_\ell := \int_0^{\Delta t} \mathrm{e}^{-\frac{i\ell}{\varepsilon^2}(\Delta t - s)}ds, \quad q_\ell := \int_0^{\Delta t} \mathrm{e}^{-\frac{i\ell}{\varepsilon^2}(\Delta t - s)}s\,ds.$$

Proposition (formal)

Let consider the solution $U^n(\tau)$ of the following semi-discretized scheme

$$\widehat{U}_\ell^{n+1} = \mathrm{e}^{-\frac{i\ell\Delta t}{\varepsilon^2}}\widehat{U}_\ell^n + p_\ell \widehat{\mathcal{F}}_\ell^n + q_\ell \frac{1}{\Delta t}\left(\widehat{\mathcal{F}}_\ell^n - \widehat{\mathcal{F}}_\ell^{n-1}\right),$$

$\widehat{U}_\ell^0$ well-prepared

Then we have

$$\sup_{\varepsilon\in]0,1]} \|U^n - U(t^n)\|_{L^\infty_\tau} \leq C\Delta t^2.$$

$\forall n \leq N, N\Delta t = T.$

The simple choice $U_0(\tau) = \mathbf{u}_0$ will not work: indeed, one has $\partial_t^2 U^\varepsilon = \mathcal{O}(1/\varepsilon)$ which means that the numerical scheme behaves well for $\varepsilon = \mathcal{O}(1)$ and ε very small, but not for intermediate regimes.

Algorithm

- Initialization of $\mathbf{x}_k, \mathbf{v}_k$ from f_0.
- Compute the "well-prepared" initial data for $U_\pm(0,\tau)$

Time loop $n \to n+1$

- $X_k^n(\tau) = \frac{1}{2}(U_+^n(\tau) + e^{\tau J} U_-^n(\tau))_k$
- Solve the Poisson equation: $\nabla_x \cdot \mathcal{E}(t_n, \tau, \mathbf{x}) = \sum_{k=1}^{N_p} \omega_k \delta(\mathbf{x} - X_k^n(\tau))$
- Advance $(U_\pm^{n+1})_k$

From $(U_\pm^n)_k$, one can reconstruct the original solution

$$\mathbf{x}_k^n = X_k^n(\tau = t_n/\varepsilon^2) = \frac{1}{2}\left[U_+^n(\tau = t_n/\varepsilon^2) + e^{Jt_n/\varepsilon^2} U_-^n(\tau = t_n/\varepsilon^2)\right]_k,$$

$$\mathbf{v}_k^n = Y_k^n(\tau = t_n/\varepsilon^2) = \frac{J}{2\varepsilon} e^{Jt_n/\varepsilon^2} U_-^n(\tau = t_n/\varepsilon^2).$$

Numerical tests

Let consider the following initial condition (Kelvin-Helmholtz instability)

$$f_0(\mathbf{x}, \mathbf{v}) = \frac{1}{2\pi} \left(1 + \sin(x_2) + \eta \cos(k x_1)\right) \mathrm{e}^{-\frac{|\mathbf{v}|^2}{2}},$$

on a computational domain for $\mathbf{x}$ as $\Omega = [0, 2\pi/k] \times [0, 2\pi]$ for $k = 0.5$ and $\eta = 0.05$, $\mathbf{v} \in \mathbb{R}^2$.

Numerical parameters: 64 points in x_1-direction and 32 points in x_2-direction; the fifth order B-spline; $N_\tau = 32$ and $N_p = 204800$.

Uniform Accuracy

$$\rho(t, \mathbf{x}) = \int_{\mathbb{R}^2} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v}, \quad \mathbf{x} \in \Omega$$

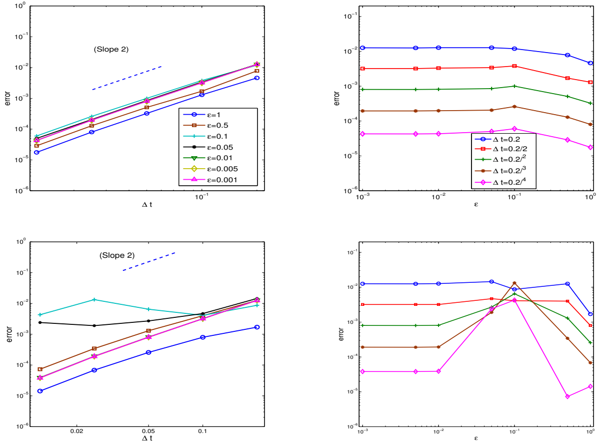

Figure: Temporal error at $t = 1$ in ρ with respect to Δt and ε: results of $(**)$ (1st row); results of $\mathbf{u}_0$ (2nd row).

Uniform Accuracy

$$\rho_{\mathbf{v},1}(t,\mathbf{x}) := \int_{\mathbb{R}^2} (|v_1| + |v_2|) f(t,\mathbf{x},\mathbf{v}) d\mathbf{v}, \ \ \rho_{\mathbf{v},2}(t,\mathbf{x}) := \int_{\mathbb{R}^2} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v}) d\mathbf{v}$$

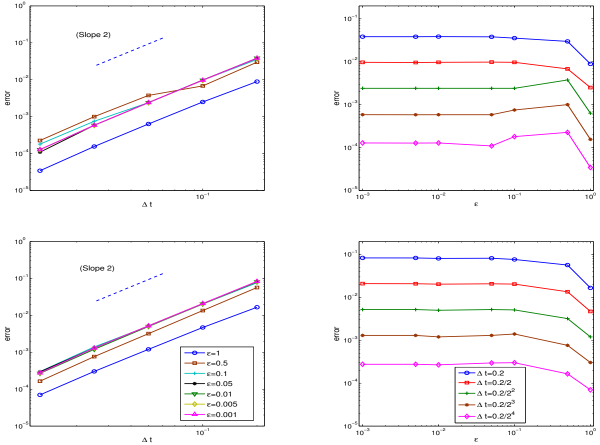

Figure: Temporal error under (∗∗): in $\rho_{\mathbf{v},1}$ (1st row); in $\rho_{\mathbf{v},2}$ (2nd row).

Uniform Accuracy

$$H(t) := \frac{1}{2}\int_{\mathbb{R}^2\times\mathbb{R}^2} |\mathbf{v}|^2 f(t,\mathbf{x},\mathbf{v})\,d\mathbf{x}\,d\mathbf{v} + \frac{1}{2}\int_{\mathbb{R}^2} |\mathbf{E}(t,\mathbf{x})|^2\,d\mathbf{x}$$

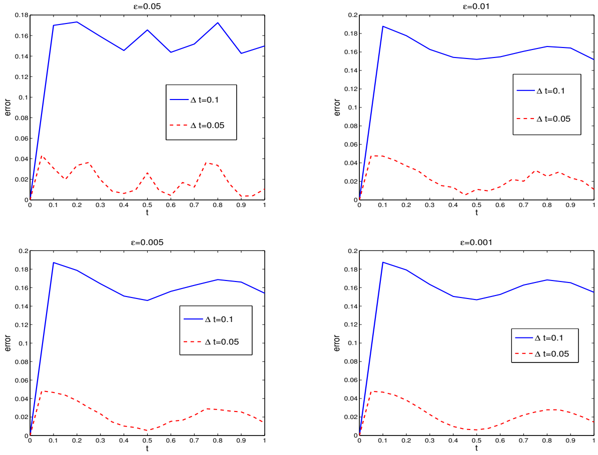

Figure: Energy error $|H(t_n) - H(0)|$.

Convergence to limit model

Convergence of the Vlasov-Poisson to the limit model as $\varepsilon \to 0$: in ρ and $\rho_{\mathbf{v},2}$

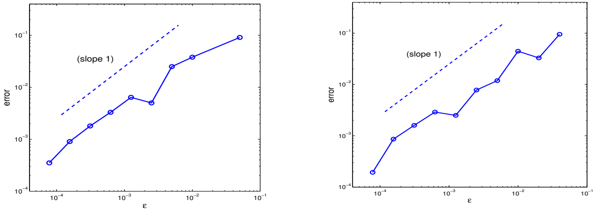

Figure: Maximum error in ρ (left) and in $\rho_{\mathbf{v},2}$ (right).

Dynamics

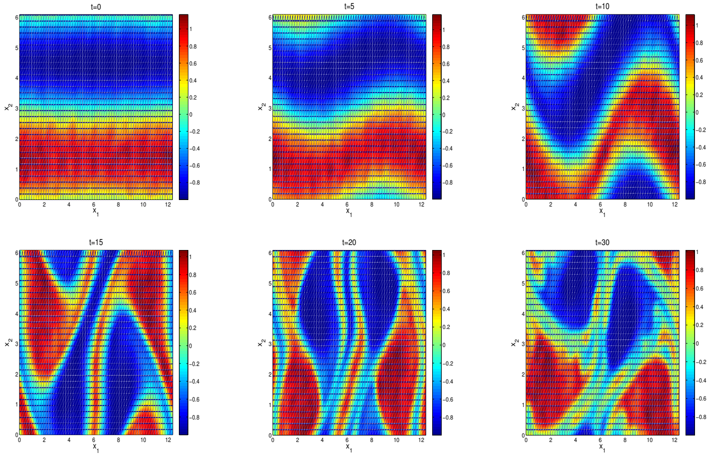

Figure: Contour plot of quantity $\rho(t, \mathbf{x}) - 1$ at different t with $\varepsilon = 0.005$.

Numerical results: non-isotropic initial condition

$$f_0(\mathbf{x},\mathbf{v}) = \frac{1}{4\pi}\left(1+\sin(x_2)+\eta\cos(kx_1)\right)\left(\mathrm{e}^{-\frac{(v_1+2)^2+v_2^2}{2}}+\mathrm{e}^{-\frac{(v_1-2)^2+v_2^2}{2}}\right).$$

with $\eta = 0.05, k = 0.5,\ N_p = 409600, N_\tau = 16.$

$$\chi(t,\mathbf{v}) := \int_{\mathbb{R}^2} f(t,\mathbf{x},\mathbf{v})d\mathbf{x}$$

Numerical results: non-isotropic initial condition

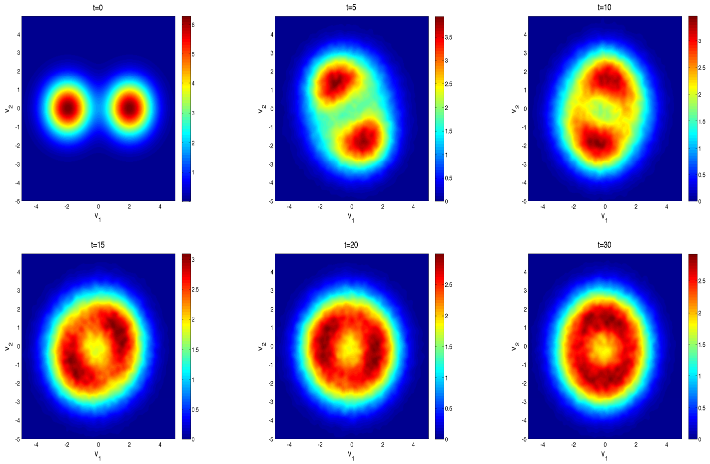

Figure: Contour plot of quantity $\chi(t, \mathbf{v})$ at different t with $\varepsilon = 0.005$.

Conclusion

- general method to construct uniformly accurate schemes for highly-oscillatory problems and application to strongly magnetized plasmas in a diffusion scaling
- not based on computations of averaged models

Perspectives

- Systematical comparison with other methods: limit solvers and AP schemes
- Extension to more general oscillations
- get rid of the variable τ