# Robust Solvers for Elliptic Problems in Fluid Models Context

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# Outline

- **Aim** : We are interested in devising a robust and optimal solver for the Laplacian operator and the mass matrix.
- **Problem**: Hyperbolic models with small viscosities are in general strongly nonlinear and coupled and ill-conditioned.
- **Solutions**: Use algorithm (Physic based preconditioning, semi implicit scheme etc) which allows to split and reformulate the full system to some simpler systems.
- **Simple systems**: Laplacian, small advection or vector elliptic operators.
- **Key point**: Operator inversion under control (Convergence, computational cost, robustness, optimality).

# Context

- **Linearized Euler Equation**:

$$\left\{\begin{array}{ll} \partial_t \mathbf{u} + a\cdot\nabla\mathbf{u} + c\nabla p & = 0 \\ \partial_t p + a\cdot\nabla p + c\nabla\cdot\mathbf{u} & = 0 \end{array}\right. \longrightarrow \left\{\begin{array}{ll} \mathbf{u}^{n+1} + \Delta t a\cdot\nabla\mathbf{u}^{n+1} + c\Delta t\nabla p^{n+1} & = \mathbf{u}^n \\ p^{n+1} + \Delta t a\cdot\nabla p^{n+1} + c\Delta t\nabla\cdot\mathbf{u}^{n+1} & = p^n \end{array}\right.$$

- Using splitting scheme with reformulation, we obtain predictor-corrector scheme:

$$\left\{\begin{array}{l} \boldsymbol{u}^* + \Delta t a\cdot\nabla\boldsymbol{u}^* = \boldsymbol{u}^{\boldsymbol{n}} \\ p^{n+1} + \Delta t a\cdot\nabla p^{n+1} - c^2\Delta t^2\Delta p^{n+1} = p^n - c\Delta t\nabla\cdot\mathbf{u}^* \\ \mathbf{u}^{n+1} = \mathbf{u}^* - c\Delta t\nabla p^{n+1} \end{array}\right.$$

- **Conclusion**: when a is small we need to perform mass matrices inversion and stiffness matrix inversion. Aim: To come up with a Laplacian and mass solver.
- **Remark**: this approach is extendable to complex systems. Other approaches could give similar outcome.

# Outline

# Isogeometric Analysis

### Definition (IgA)

Is a recently developed computational approach that offers the possibility of integrating FEA into conventional CAD design tools.

- Finite Elements Analysis (FEA) models are created from CAD representations.
- Creating FEA models accounts more than 80 % of overall analysis time and is a major engineering bottelneck.
- The geometry is approximated in the FEA mesh
- Use B-Splines as basis functions in the Tokamak context:
  - Adapt geometry to the physics.
  - Using high order polynomial.

# B-Splines

## B-Splines

To create a family of *B-Splines*, we need a non-decreasing sequence of knots $T = (t_i)_{1 \leqslant i \leqslant N+k}$, also called **knot vector**, with $k = p + 1$. Each set of knots $T_j = \{t_j, \cdots, t_{j+p}\}$ will generate a *B-spline* $N_j$.

### Definition (B-Spline series)

The j-th B-Spline of order $k$ is defined by the recurrence relation:

$$N_j^k = w_j^k N_j^{k-1} + (1 - w_{j+1}^k) N_{j+1}^{k-1}$$

where,

$$w_j^k(x) = \frac{x - t_j}{t_{j+k-1} - t_j} \qquad N_j^0(x) = \chi_{[t_j, t_{j+1}[}(x)$$

for $k \geq 1$ and $1 \leq j \leq N$.

# Outline

# Multigrid

## Multigrid Idea

Project the system in a subspace, solve the resulting system in this subspace and interpolate the solution in order to improve the previous approximation.

## Multigrid Components

The Multigrid combines two iterative methods:

Smoother: a classic iterative method,

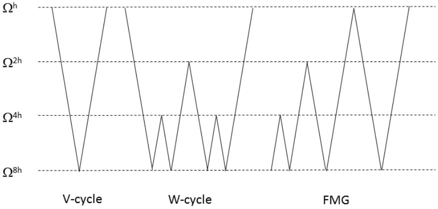
Coarse Grid Correction: projection of the error equation, solution of the restricted problem, interpolation.

# Multigrid Algorithm

## 2 Grid Cycle

1. **Iterate** on $A_f u = b_f$ to reach $u_f$ (in our case, 5 Gauss-Seidel steps).
2. **Restrict** the residual $r_f = b_f - A_f u_f$ to the coarse grid by $r_c = Rr_f$
3. **Solve** $A_c E_c = r_c$ (or come close to $E_c$ by 5 iterations from E $= 0$).
4. **Interpolate** $E_c$ back to $E_f = PE_c$. Add $E_f$ to $u_f$.
5. **Iterate** 5 more times on $A_f u = b_f$ starting from the improved $u_f + E_f$.

Types of cycles:



# Prolongation and Restriction Operators for B-Splines

These operators are based on the **knot insertion** algorithm. For the 1D case: one can insert a new knot t, where $t_j \leq t \leq t_{j+1}$.

For this purpose, we use the De Boor's algorithm.

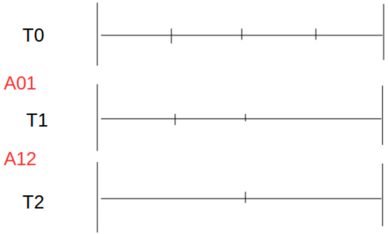
$$\tilde{T} = \{t_1, .., t_j, t, t_{j+1}, .., t_{N+k}\}$$

$$\alpha_i = \begin{cases} 1, & 1 \leq j \leq j-k+1 \\ \frac{t-t_i}{t_{i+k-1}-t_i}, & j-k+2 \leq i \leq j \\ 0, & j+1 \leq i \end{cases}$$

$$\mathbf{Q_i} = \alpha \mathbf{P_i} + (1-\alpha_i)\mathbf{P_{i-1}}$$

# Prolongation and Restriction Operators Cont'd

This can be written in the matrix form as: $\mathbf{Q} = A\mathbf{P}$. The basis transformation A is called the knot insertion matrix of degree k-1 from T to $\tilde{T}$.

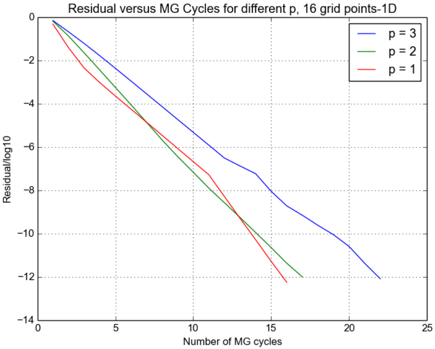



The insertion matrix from $T_0$ to $T_n$ is simply:

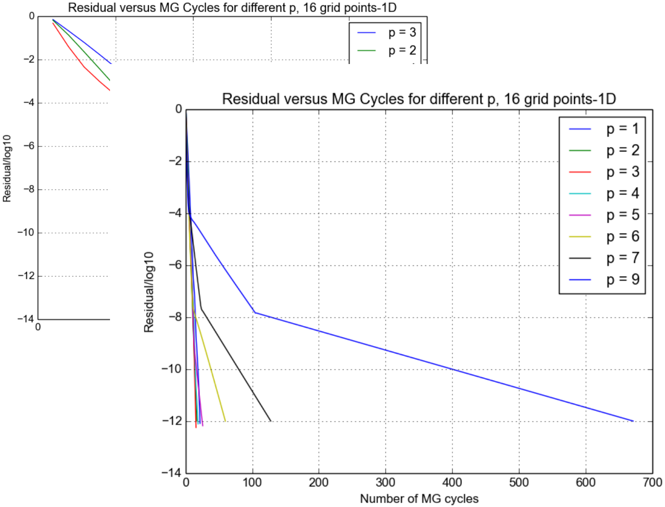
$$A := A_0^n = A_0^1 A_1^2 A_2^3 \ldots A_{n-1}^n$$

In 2D, the interpolation operator can be constructed using the Kronecker product.

# The Problem with Multigrid: Numerical Take-1D



# The Problem with Multigrid: Numerical Take-1D

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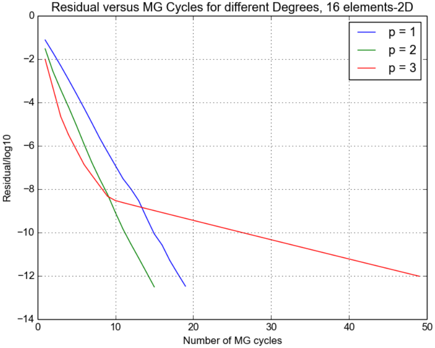
# The Problem with Multigrid: Numerical Take-2D



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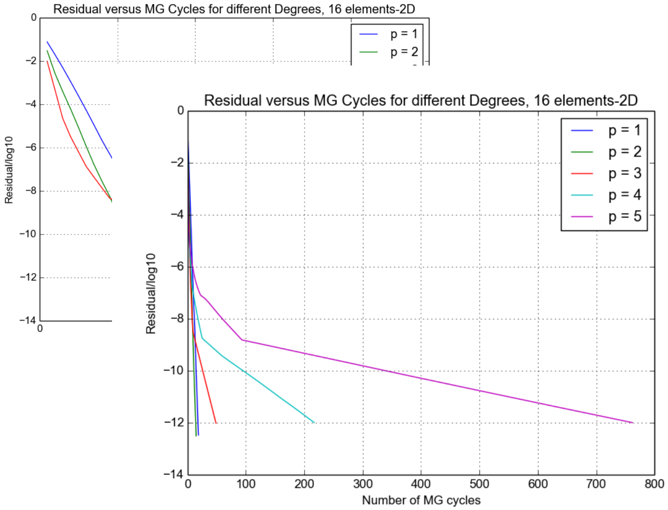

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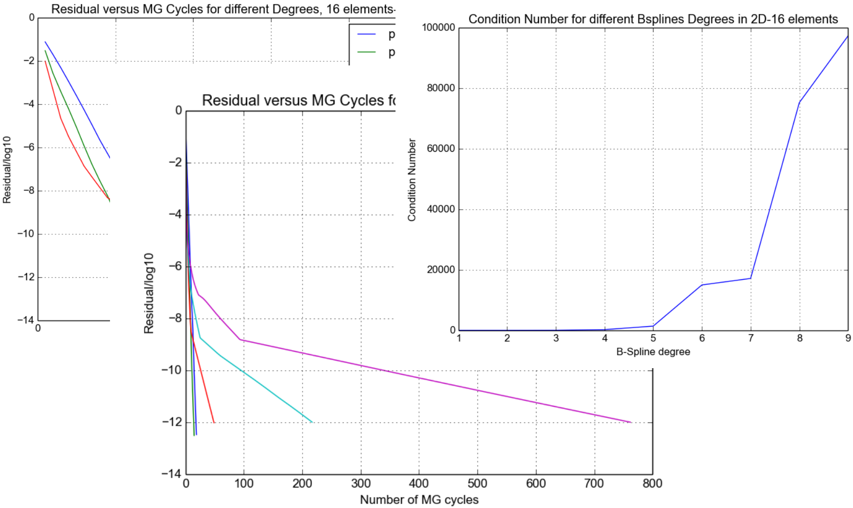


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# Outline

- **PDE:** $Lu = g$ after discretization gives $L_n u_n = g_n$ with $\{L_n\}_n$ is a sequence of discretizations.
- **GLT:** is a generalization of fourier analysis for the discretization of PDEs.
- Application to the Laplacian operator: using GLT in the fourier space:

$$L_n \approx M_p D_n$$

Where $M_p$ is homogenous to the mass matrix, and $D_n$ is the FD of the Laplacian operator in the fourier space.
- **Low frequency problem:** the eigenvalues of $D_n$ are $(2 - 2cos(\theta))$, solved via the MG.
- **High frequency problem:** $M_p$ tends to zero when p is large.
- **Idea**: preconditioning by $M^{-1}$ and use exploit kronecker product for the inversion.

# GLT: From Cardinal BSplines to the Toeplitz Matrix

## Theorem

The symbol $m_p$ is given by

$$\mathfrak{m}_p(x,\theta) := \mathfrak{m}_p(\theta) = \phi_{2p+1}(p+1) + 2\sum_{k=1}^{p} \phi_{2p+1}(p+1-k)\cos(k\theta)$$

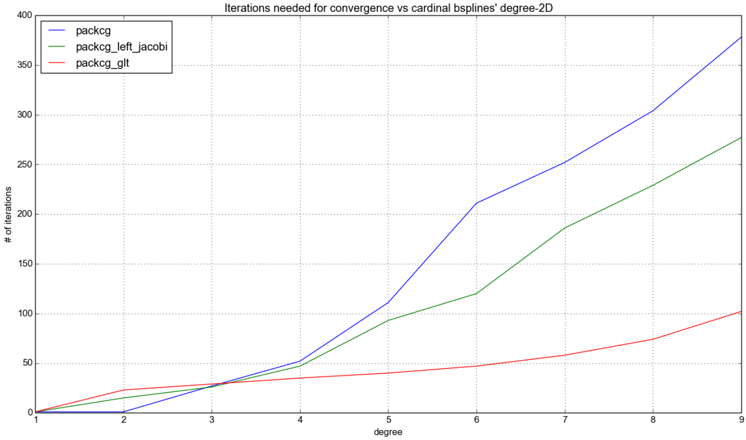
## Toeplitz matrix

is a real/complex valued $n \times n$ matrix $T_n = (t_{jk})_{j,k=0,\ldots,n-1}$, where $t_{jk} = t_{j-k}$, i.e.

$$T_n(f(\theta)) = \begin{pmatrix} t_0 & t_{-1} & t_{-2} & \ldots & t_{-(n-1)} \\ t_1 & t_0 & t_{-1} & \ldots & \\ t_2 & t_1 & t_0 & \ldots & \vdots \\ \vdots & & & \ddots & \\ t_{n-1} & \ldots & & \ldots & t_0 \end{pmatrix}$$

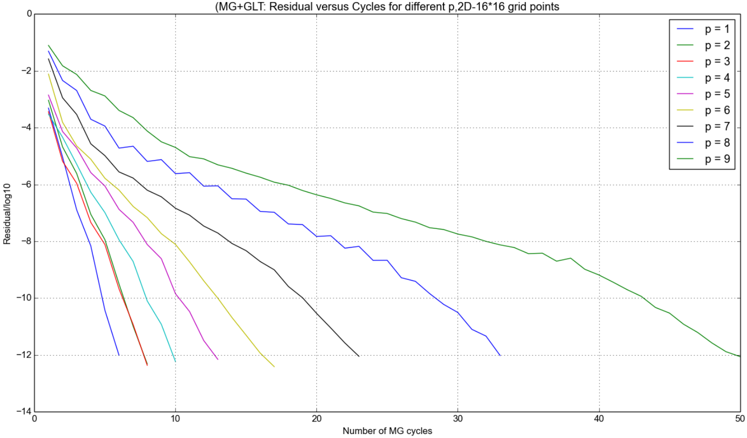
# Outline

# Results: GLT as a Smoother

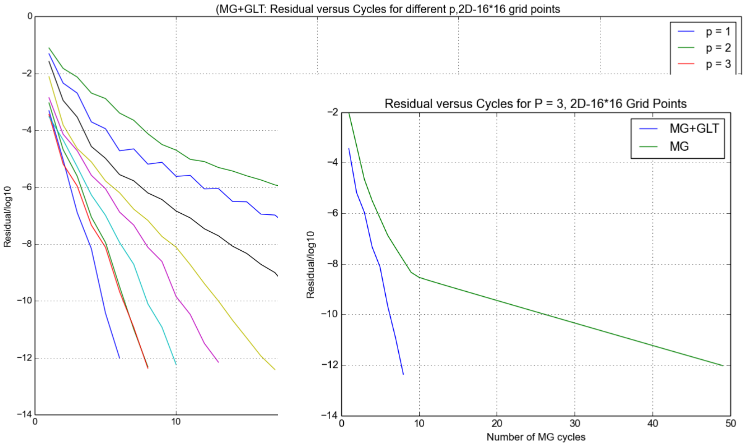


# Outline

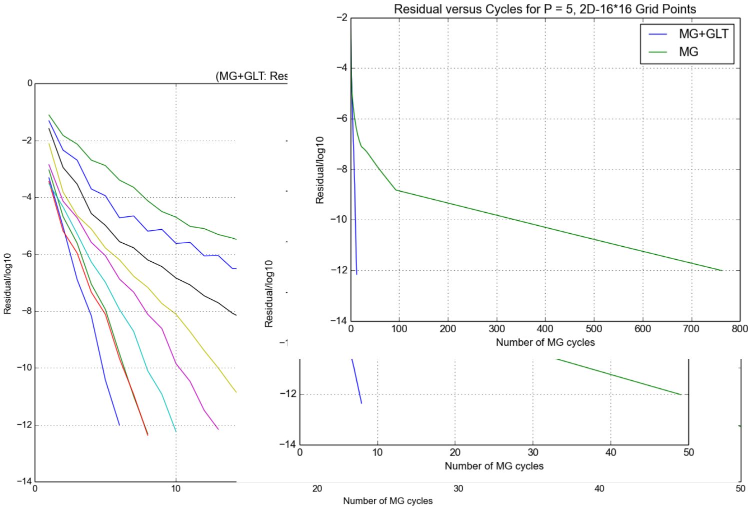
# Results: GLT + MG



# Results: GLT + MG



# Results: GLT + MG

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# Outline

# Mass on a circle and square

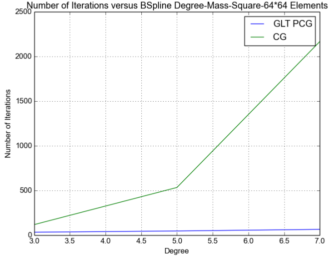



Table 1: Number of iterations-mass matrix on a square 32*32

| Degree | PCG | CG |
|---|---|---|
| 3 | 33 | 111 |
| 5 | 49 | 449 |
| 7 | 65 | 1777 |

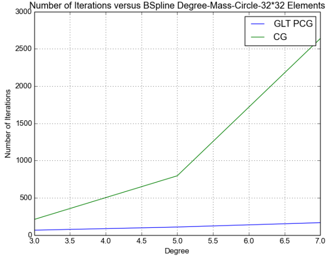
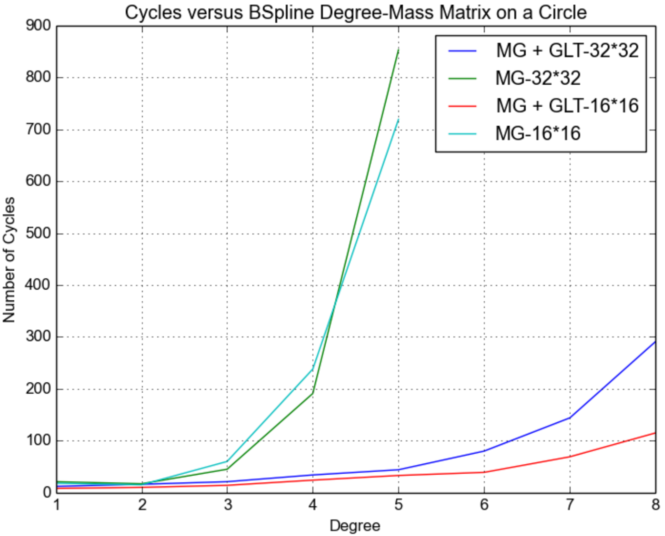



Table 2: Number of iterations-mass matrix on a circle 64*64

| Degree | PCG | CG |
|---|---|---|
| 3 | 98 | 340 |
| 5 | 160 | 1711 |
| 7 | 245 | >3000 |

# MG+GLT on a Circle



# Computational Costs

Table 4: Computational cost comparison for the Laplacian operator -2D 64*64 elements

| Degree/Scheme | MG + GLT | MG |
|---|---|---|
| 1 | 1.32 | 1.76 |
| 2 | 2.56 | 2.75 |
| 3 | 2.58 | 4.42 |
| 4 | 3.42 | 21.62 |
| 5 | 6.35 | 170.48 |
| 6 | 15.71 | 677.17* |
| 7 | 25.99 | 825.56* |
| 8 | 27.89 | 800.72* |
| 9 | 58.03 | 1098.94* |

This is not yet optimized!

# Outline

# Conclusions and Future Prospects

**Conclusions:**

- GLT treats the problem arising from the high degree component.
- GLT + MG gives a robust and optimal solver for the Laplacian operator.
- Can be used also for the mass matrix.

**Future work:**

- Short time:
  - Optimization
  - Apply the GLT theory to vector elliptic operators; grad div

$$\mu \boldsymbol{u} - \nabla \nabla \cdot \boldsymbol{u} = 0$$

  Ill-conditioned when $\mu \to 0$

- Long term: Applying this with time integration schemes for models like Euler and MHD.

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