New weak formulations for resonant Maxwell's equations and first numerical results

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Sponsors: ANR Chrome and Eurofusion/FRFCM/ENR (CEA-05)

Model of a cold plasma

Linearization of Maxwell-Newton for one species of electrons with charge $-e < 0$

$$\left\{ \begin{array}{lll} -\frac{1}{c^2}\partial_t \mathbf{E} & +\nabla \wedge \mathbf{B} & = \mu_0 \mathbf{J}, \\ \partial_t \mathbf{B} & +\nabla \wedge \mathbf{E} & = 0, \\ m_e \partial_t \mathbf{u}_e & & = -e\left(\mathbf{E} + \mathbf{u}_e \wedge \mathbf{B}_0(\mathbf{x})\right) - m_e \nu \mathbf{u}_e, . \end{array} \right. \qquad \mathbf{J} = -eN_e(\mathbf{x})\mathbf{u}_e,$$

The electronic density $N_e(\mathbf{x}) \geq 0$ is given. If the plasma if hot, then $N_e(\mathbf{x}) >> 1$.

The background (strong) magnetic field $\mathbf{B}_0(\mathbf{x}) \approx \mathbf{B}_0^{\mathrm{constant}}$ is given.

Last equation is the only kinetic one in the talk.

The collision frequency is $\nu > 0$. It corresponds to friction on a bath of static ions.

It is an extremely small quantity because the plasma has very low collisionality

$$\nu \approx 10^{-7} \text{ in a fusion plasma.}$$

Goal of this talk : pass to the limit $\nu = 0^+$.

Branbilla : Kinetic Theory of Plasma Waves-Homogeneous Plasmas, 1998.

Time harmonic domain : $\partial_t = i\omega$

Set $\omega_p(\mathbf{x})^2 = \frac{e^2 N_e(\mathbf{x})}{c^2\varepsilon_0}$, $\omega_c(\mathbf{x}) = \frac{e|\mathbf{B}_0(\mathbf{x})|}{m_e}$ and $\widetilde{\omega} = \omega + i\nu$,

$$\varepsilon^\nu = \begin{pmatrix} 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega\left(\widetilde{\omega}^2 - \omega_c^2\right)} & i\frac{\omega_c\omega_p^2}{\omega\left(\widetilde{\omega}^2 - \omega_c^2\right)} & 0 \\ -i\frac{\omega_c\omega_p^2}{\omega\left(\widetilde{\omega}^2 - \omega_c^2\right)} & 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega\left(\widetilde{\omega}^2 - \omega_c^2\right)} & 0 \\ 0 & 0 & 1 - \frac{\omega_p^2}{\omega\widetilde{\omega}} \end{pmatrix}.$$

- In the general case ε^ν is a normal matrix, and ε^0 is an hermitian complex matrix. In vacuum $N_e = 0$: then $\omega_p = 0$ and $\varepsilon^\nu = \mathbb{I}$.
- Special values of ω (for $\nu \approx 0$) are
  - $\omega = \omega_p(\mathbf{x})$ which is the cut-off (O-mode).
  - $\omega^2 = \omega_p(\mathbf{x})^2 + \omega_c(\mathbf{x})^2$ which is the hybrid resonance.
  - $\omega = \omega_c(\mathbf{x})$ which is the cyclotron resonance (not considered below).

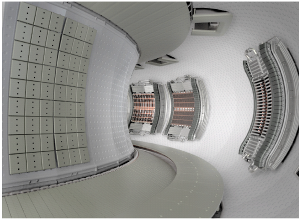
Maxwell's equations
with cold-plasma **anistropic dielectric tensor** $\varepsilon^\nu \in C^0\left(\overline{\Omega} : \mathcal{M}^{d\times d}\right)$, $(\varepsilon^0)^* = \varepsilon^0$,

$$\nabla \wedge \nabla \wedge \mathbf{E} - \frac{\omega^2}{c^2}\varepsilon^\nu \mathbf{E} = 0, \qquad \nabla \wedge \nabla \wedge = \text{curl curl} .$$

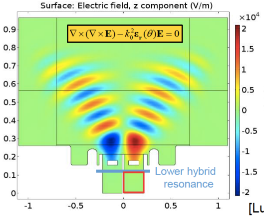
Labrunie, and al, "Electromagnetic wave propagation and absorption in magnetized plasmas . . ., 2015.

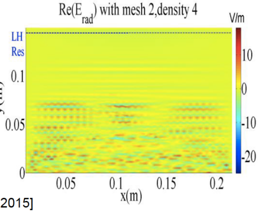
From (Ling-Feng) Lu PhD (Fusenet 2015)

Simulation results on RF wave propagation and variable density.



In this presentation, focus is on the mathematics of the hybrid resonance, motivated by the numerical difficulties reported by Lu-Colas.





[Lu 2015]

Toy problem of the talk : Budden problem 62'

• Restore an regularization (friction, damping, complex shift, ...) parameter $\nu > 0$

$$\left\{ \begin{array}{cccl} B_3^\nu & & -(E_2^\nu)' & = 0, \\ & -(\alpha + i\nu)E_1^\nu & -i\delta E_2^\nu & = 0, \\ -(B_3^\nu)' & +i\delta E_1^\nu & -(\alpha + i\nu)E_2^\nu & = 0, \end{array} \right.$$

$$\Longleftrightarrow \left\{ \begin{array}{ccl} B_3^\nu & -(E_2^\nu)' & = 0, \\ -(B_3^\nu)' & + \left(\frac{\delta^2}{\alpha + i\nu} - (\alpha + i\nu) \right) E_2^\nu & = 0, \end{array} \right.$$

$$\Longleftrightarrow -(E_2^\nu)'' + \left(\frac{\delta^2}{\alpha + i\nu} - (\alpha + i\nu) \right) E_2^\nu = 0.$$

The dielectric tensor is

$$\varepsilon = \begin{pmatrix} \alpha(x) & i\delta(x) \\ -i\delta(x) & \alpha(x) \end{pmatrix} = \begin{pmatrix} \varepsilon_\perp & i\varepsilon_\times \\ -i\varepsilon_\times & \varepsilon_\perp \end{pmatrix}$$

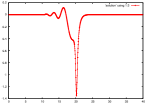
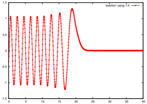
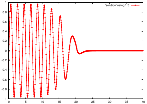
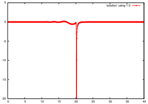
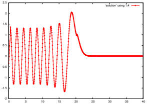
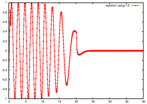
where the plasma-dependent parameters are α and δ.

• Important example is : $\alpha(x) = -x$ and $\delta(x) = 1$.
The hybrid resonance is at $x = 0$.

• Our goal is to pass to the limit $\nu \to 0^+$ with the limit absorption principle, in a form compatible with the general principles of FEM (Finite Element Methods).

-Budden, Radio wave in the ionosphere, 1961.

Numerical illustration (with FDTD)

	E_1^ν	E_2^ν	B_3^ν
$\nu = 0.1$			
$\nu = 0.01$			
$\nu = 0$	not stable		

Note the local Ansatz

$$E_1^\nu(x) = -\frac{i\delta(x)}{\alpha(x)+i\nu}E_2^\nu \approx -\frac{i\delta(0)}{rx+i\nu}E_2^\nu(0), \qquad r=\alpha'(0)\neq 0.$$

The resonant heating is generically positive

$$\mathcal{Q} = \lim_{\nu\to 0^+} \nu \int |\mathbf{E}^\nu|^2 = \lim_{\nu\to 0^+} \nu \int |E_1^\nu|^2 = \pi \frac{|\delta(0)|^2}{|r|} |E_2^{0^+}|^2 > 0.$$

Rigorous mathematical proofs in time-harmonic-domain can be found in :
- Hybrid resonance of Maxwell's equations in slab geometry, D.+Imbert-Gérard+Weder, JMPA 2014
- A new set of local techniques is with integral contour in the complex plane : D.+Imbert-Gérard-Lafitte (2016)

Section 1

New tools : manufactured solutions

Design of singular manufactured solutions

New tools : manufactured solutions

Numerical results

2D

Def : manufactured solution = quasi-solution with the main singularity and with 3 properties.
We will note $r = \alpha'(0)$

-Take analytical formulas inspired by the maths and the physics

P1 : $F_1^\nu = -\dfrac{1}{\alpha + i\nu}$, $F_2^\nu = \dfrac{i}{\delta}$, $C_3^\nu = i\dfrac{\delta(0)}{r}\left(\dfrac{1}{2}\log\left(r^2x^2+\nu^2\right) - i\arctan\left(\dfrac{rx}{\nu}\right)\right)$

- The quasi-ness is measured as follows

$$\text{P2}: \begin{cases} C_3^\nu - (F_2^\nu)' = q_3^\nu \in L^2_{\mathrm{unif}/\nu} \\ -(\alpha+i\nu)F_1^\nu + i\delta F_2^\nu = g_1^\nu = 0 \\ -(C_3^\nu)' - i\delta F_1^\nu - (\alpha+i\nu)F_2^\nu = g_2^\nu \in L^2_{\mathrm{unif}/\nu}. \end{cases}$$

The uniform boundedness is because

$$q_3^\nu = i\frac{\delta(0)}{r}\left(\frac{1}{2}\log\left(r^2x^2+\nu^2\right) - i\arctan\left(\frac{rx}{\nu}\right)\right) + i\frac{\delta'}{\delta^2}$$

and

$$g_2^\nu = i\frac{\delta(x)-\delta(0)}{rx+i\nu} + i\delta\left(\frac{1}{\alpha(x)+i\nu} - \frac{1}{rx+i\nu}\right) - i\frac{\alpha+i\nu}{\delta}.$$

The main trick

-Let $\varphi \in C_0^1(-1,1) = C_0^1(\Omega)$ be a smooth test function with compact support : $\mathrm{supp}(\varphi) = (-\epsilon, \epsilon)$.

- Check

$$\begin{aligned}
\int_\Omega (F_2^\nu B_3^\nu - E_2^\nu C_3^\nu)\,\varphi'\,dx &= -\int_\Omega (F_2^\nu B_3^\nu - E_2^\nu C_3^\nu)'\,\varphi dx \\
&= -\int_\Omega [(C_3^\nu - q_3^\nu)B_3^\nu + F_2^\nu(i\delta E_1^\nu - (\alpha + i\nu)E_2^\nu) \\
&\quad -B_3^\nu C_3^\nu - E_2^\nu(i\delta F_1^\nu - (\alpha + i\nu)F_2^\nu - g_2^\nu)]\,\varphi dx \\
&= \int_\Omega (q_3^\nu B_3^\nu - g_2^\nu E_2^\nu)\,\varphi dx + \int_\Omega i\delta\,(E_2^\nu F_1^\nu - E_1^\nu F_2^\nu)\,\varphi dx \\
&= \int_\Omega (q_3^\nu B_3^\nu - g_2^\nu E_2^\nu)\,\varphi dx.
\end{aligned}$$

- Pass to the limit $\nu \to 0^+$

$$\mathrm{P3}^{\mathrm{mod}} : \int_\Omega (F_2^+ B_3^+ - E_2^+ C_3^+)\,\varphi'\,dx = \int_\Omega (q_3^+ B_3^+ - g_2^+ E_2^+)\,\varphi dx, \quad \forall \varphi \in C_0^1(\Omega).$$

- not in any classical textbook on waves in plasmas.
- extremely well adapted to FE modeling since integrable.

Interpretation

- The identity

$$\mathrm{P3}^{\mathrm{mod}} : \int_\Omega \left(F_2^+ B_3^+ - E_2^+ C_3^+\right) \varphi' dx = \int_\Omega \left(q_3^+ B_3^+ - g_2^+ E_2^+\right) \varphi dx, \quad \forall \varphi \in C_0^1(\Omega)$$

makes sense because

$$E_2^+, B_3^+ \in L^2,$$

$$F_2^+ = \frac{i}{\delta} \in L^2, \qquad C_3^+ = i\frac{\delta(0)}{r}\left(\log(|rx|) - i\frac{\pi}{2}\mathrm{sign}(rx)\right) \in L^2,$$

$$g_2^+ = i\frac{\delta(x) - \delta(0)}{rx} + i\delta\frac{rx - \alpha(x)}{rx\alpha(x)} - i\frac{\alpha}{\delta} \in L^2$$

and

$$q_3^+ = i\frac{\delta(0)}{r}\left(\log(|rx|) - i\frac{\pi}{2}\mathrm{sign}(rx)\right) + i\frac{\delta'}{\delta^2} \in L^2.$$

- The causality is guaranteed by the term $\mathrm{sign}(rx) = \pm 1$.

- Loss of causality corresponds to $\mathrm{sign}(rx) = 0$.

- Locally $|\mathrm{sign}(rx)| << \log|x| << \frac{1}{|x|}$.

Well-posedness (after elimination of e_1)

Let $\Omega = (-1,1)$, $L^2 = L^2(\Omega)$, $H_0^1 = H_0^1(\Omega)$ and $H_{0,0}^1 = \{\psi \in H_0^1 \mid \psi(0) = 0\}$.

Find $(e_2, b_3) \in L^2 \times L^2$ with three conditions :

i) the weak formulations

$$\begin{cases} \int_\Omega (b_3\varphi_1 + e_2\varphi_1')dx & = 0, \quad \forall \varphi_1 \in H_0^1, \\ \int_\Omega \left(b_3\varphi_2' + \left(\frac{\delta^2}{\alpha} - \alpha\right) e_2\varphi_2\right) dx & = 0, \quad \forall \varphi_2 \in H_{0,0}^1, \end{cases}$$

ii) the boundary conditions

$$b_3(-1) + i\lambda e_2(-1) = f_- \quad \text{and} \quad b_3(1) - i\lambda e_2(1) = f_+,$$

in the sense of distributions

iii) one single integral relation with the singular manufactured solution, for one single test function $\varphi \in H_0^1 - H_{0,0}^1 : \varphi(0) \neq 0$.

Note that e_1 is eliminated and $\frac{\varphi_2}{\alpha} \in L^2$ thanks to Hardy's inequality.

Theor. (Campos-Pinto+D.) :
For all $(f_-, f_+) \in \mathbb{C}^2$, there exists a unique solution (e_2, b_3) and it coincides with the limit solution (E_2^+, B_3^+).

Dissipative inequalities

New tools : manufactured solutions

Numerical results

2D

This construction is closer to the initial idea of entropies, and more ambitious.

Change $F_1^\nu \leftarrow -F_1^\nu$ so that

$$\begin{cases} C_3^\nu - (F_2^\nu)' = q_3^\nu \\ -(\alpha + i\nu)F_1^\nu - i\delta F_2^\nu = 0 \\ -(C_3^\nu)' + i\delta F_1^\nu - (\alpha + i\nu)F_2^\nu = g_2^\nu \end{cases}$$

and for $k \in \mathbb{C}$

$$\begin{cases} (B_3^\nu - kC_3^\nu) - (E_2^\nu - kF_2^\nu)' = kq_3^\nu \\ -(\alpha + i\nu)(E_1^\nu - kF_1^\nu) - i\delta(E_2^\nu - kF_2^\nu) = 0 \\ -(B_3^\nu - C_3^\nu)' + i\delta(E_1^\nu - kF_1^\nu) - (\alpha + i\nu)(E_2^\nu - kF_2^\nu) = kg_2^\nu. \end{cases}$$

One obtains (after passing to the limit)

$$\text{P3}: \quad -\mathrm{Im} \int_\Omega (E_2^+ - kF_2^+)\overline{(B_3^+ - kC_3^+)}\varphi' dx$$

$$-\mathrm{Im} \int_\Omega \left(kq_3^+ \overline{(B_3^+ - kC_3^+)} - kg_2^+ \overline{(E_2^+ - kF_2^+)} \right) \varphi dx \geq 0, \quad \forall k \in \mathbb{C}, \ \forall \varphi \in C^1_{0,+}(\Omega).$$

- For $(e_2, b_3, k) \in L^2 \times L^2 \times \mathbb{C}$, define the quadratic functional

$$\mathcal{J}(e_2, b_3, k) := -\mathrm{Im} \int_\Omega (e_2 - kF_2^+)\overline{(b_3 - kC_3^+)}\varphi' dx$$

$$-\mathrm{Im} \int_\Omega \left(kq_3^+ \overline{(b_3 - kC_3^+)} - kg_2^+ \overline{(e_2 - kF_2^+)} \right) \varphi dx$$

The parallel with entropy techniques is evidenced with the notation $\mathcal{J}(e_2, b_3, k) = \mathcal{S}(U, k)$ with $U = (e_2, b_3)$.

- For $(e_2, b_3, k, \lambda_1, \lambda_2) \in L^2 \times L^2 \times \mathbb{C} \times H_0^1 \times H_{0,0}^1$, define the Lagrangian

$$\mathcal{L}(e_2, b_3, k; \lambda_1, \lambda_2) = \mathcal{J}(e_2, b_3, k)$$

$$+\mathrm{Im} \int_\Omega (\overline{b_3}\lambda_1 + \overline{e_2}\lambda_1')dx + \mathrm{Im} \int_\Omega \left(\overline{b_3}\lambda_2' + \left(\frac{\delta^2}{\alpha} - \alpha \right) \overline{e_2}\lambda_2 \right) dx \in \mathbb{R}.$$

Euler-Lagrange conditions

New tools : manufactured solutions

Numerical results

2D

The optimality conditions write

$$
\begin{cases}
\int_\Omega \left((b_3 - kC_3^+)\varphi' + kg_2^+\varphi + \lambda_1' + \left(\frac{\delta^2}{\alpha} - \alpha\right)\lambda_2 \right) u dx = 0, & \forall u \in L^2, \\
\int_\Omega \left(-(e_2 - kF_2^+)\varphi' - kq_3^+\varphi + \lambda_1 + \lambda_2' \right) v dx = 0, & \forall v \in L^2, \\
\int_\Omega \left(\left(\overline{C_3^+} e_2 - \overline{F_2^+} b_3 \right)\varphi' + \left(\overline{q_3^+} b_3 - \overline{g_2^+} e_2 \right)\varphi \right) dx + 2i\frac{\pi\varphi(0)}{|r|} k = 0, & \\
\int_\Omega (b_3\varphi_1 + e_2{\varphi_1}')dx = 0, & \forall \varphi_1 \in H_0^1, \\
\int_\Omega \left(b_3{\varphi_2}' + \left(\frac{\delta^2}{\alpha} - \alpha\right) e_2\varphi_2 \right) dx = 0, & \forall \varphi_2 \in H_{0,0}^1.
\end{cases}
$$

Theor. For all $(f_-, f_+) \in \mathbb{C}^2$, there exists a unique solution in the space $L^2 \times L^2 \times \mathbb{C} \times H_0^1 \times H_{0,0}^1$. It is
$(e_2, b_3, k, \lambda_1, \lambda_2) = \left(E_2^+, B_3^+, -i\delta(0)E_2^+(0), \left(-B_3^+ + kC_3^+\right)\varphi, \left(E_2^+ - kF_2^+\right)\varphi\right)$.

A by-product is the decomposition into regular part+singular part

$$
E_1^+ = (E_1^+ - kF_1^+) + kF_1^+ = \underbrace{\frac{i\delta}{\alpha}(E_2^+ - kF_2^+)}_{\in L^2_{\mathrm{loc}}} + kF_1^+.
$$

Numerical results

Firstly : design of an analytical solution : 1/3

- Take $\alpha = -x$ and $\delta = \sqrt{1 - x/4 + x^2}$. The Budden equation boils down to

$$-E_2'' + \left(\frac{1}{4} - \frac{1}{x}\right) E_2 = 0.$$

It is the Whittaker equation. Here it has (miraculously) 2 simple analytical solutions

$$u(x) = xe^{-x/2} \text{ and } v(x) = e^{x/2} - \left(\log|x| + \int_1^x \frac{e^y - 1}{y} dy\right) xe^{-x/2}$$

- Therefore

$$x < 0 : \quad E_2 = a_L u + b_L v, \quad a_L, b_L \in \mathbb{C}$$

and

$$x > 0 : \quad E_2 = a_R u + b_R v, \quad a_R, b_R \in \mathbb{C}$$

Design of an analytical solution : 2/3

We use the boundary conditions

$$b_3(-1) + i\lambda e_2(-1) = f_- \quad \text{and} \quad b_3(1) - i\lambda e_2(1) = f_+,$$

note that e_2 is continuous at the origin,
and verify the integral relation can be characterized under the form independent of φ

$$\pi b_G = -\frac{1}{2} i(a_R - a_L).$$

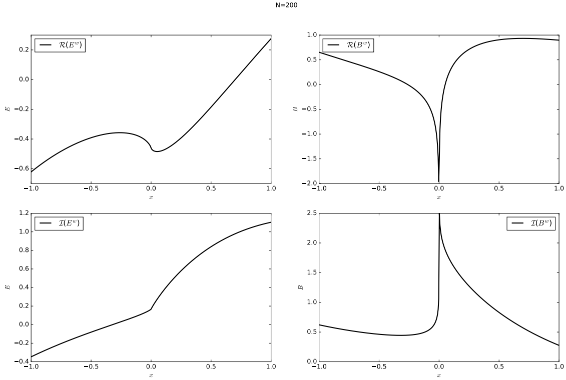
One gets the non singular linear system

$$\left(\begin{array}{cc|cc} u'(-1)+i\lambda u(-1) & v'(-1)+i\lambda v(-1) & 0 & 0 \\ 0 & 0 & u'(-1)-i\lambda u(1) & v'(1)-i\lambda v(1) \\ \hline 0 & 1 & 0 & -1 \\ i/2 & 0 & -i/2 & -\pi \end{array}\right) \begin{pmatrix} a_L \\ b_L \\ a_R \\ b_R \end{pmatrix} = \begin{pmatrix} f_- \\ f_+ \\ 0 \\ 0 \end{pmatrix}.$$

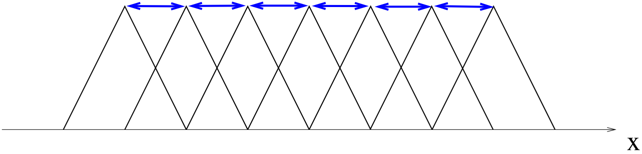
It gives $(a_L, b_L, a_R, b_R) \mapsto (E_2^+, B_3^+)$.

Design of an analytical solution : 3/3

Plot



Numerical solution with standard FE : 1/5



$$(b_3)_h \in P_0 \iff (b_3)_h = \sum_i \beta_{i+\frac{1}{2}} I_{x_i<x<x_{i+1}}$$

$$(e_2)_h \in P_1 \iff (e_2)_h = \sum_i \gamma_i T_{x_i<x<x_{i+2}}.$$

The first weak relation yields directly

$$\beta_{i+\frac{1}{2}} = \frac{\gamma_{i+1} - \gamma_i}{h}$$

The second weak relation yields

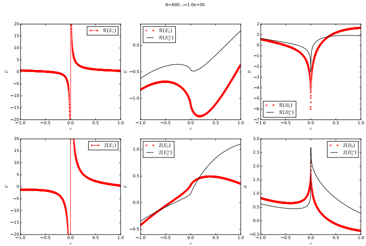
$$-\frac{\beta_{i+\frac{1}{2}} - \beta_{i-\frac{1}{2}}}{h} + \left(\frac{\delta^2}{\alpha + i\nu} - \alpha\right)_h \gamma_i = 0.$$

It is known that the quality of the solution is function of ν and h.

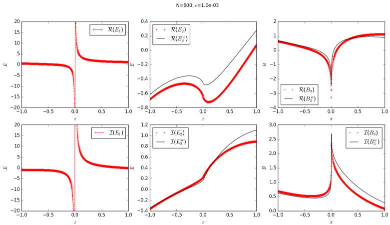
– More facts in : A numerical study of the solution of x-mode equations around the hybrid resonance, Caldini-Queiros, D. Imbert-Gérard, Kachanovska, 2016.

Numerical solution with standard FE : 2/5

The regularization is $\alpha \leftarrow \alpha + i\nu$ in the "dangerous" cell.

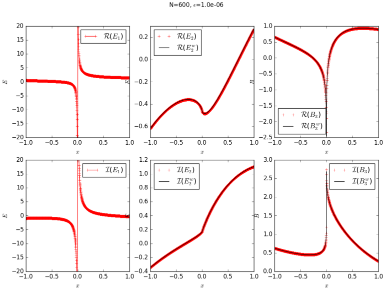


$\nu = 10^{-1}$

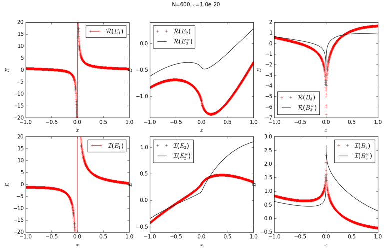


$\nu = 10^{-3}$

Numerical solution with standard FE : 3/5



$\nu = 10^{-6}$

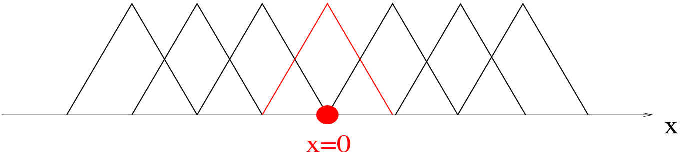


$\nu = 10^{-20}$

Principle of the new discretization

Standard F.E. discretization,
except at $x = 0$ which is replaced by one integral relation for one $\varphi \in C_0^1(\Omega)$

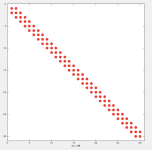
$$\begin{cases} \int_\Omega \left(b_3 \varphi_2' + \left(\frac{\delta^2}{\alpha} - \alpha \right) e_2 \varphi_2 \right) dx & = 0, \qquad \forall \varphi_2 \in H^1_{0,0}, \\ \int_\Omega \left(F_2^+ b_3 - e_2 C_3^+ \right) \varphi' dx & = \int_\Omega \left(q_3^+ b_3 - g_2^+ e_2 \right) \varphi dx. \end{cases}$$

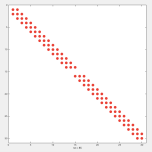
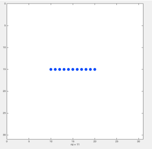
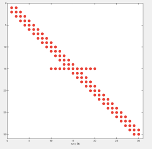


$$(e_2)_h \in P_h^1, \quad (b_3)_h \in P_h^0 \text{ and } (\varphi_2)_h \in P_h^1 - \psi_0$$

Principle : the same Finite Element, but one changes one line in the matrix.

Matrix structure

STANDARD : 

NEW :  $+$  $=$ 

The number of points in the lines is

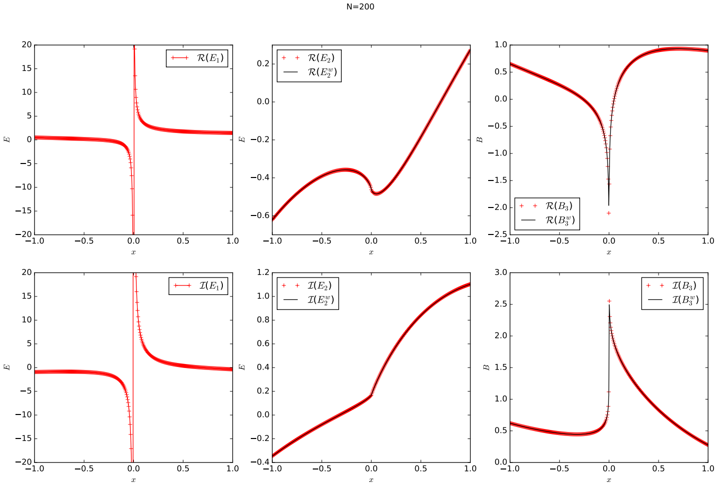
$$n_{\text{points}} = \frac{\text{length}\,(\text{sup}\varphi)}{\Delta x}$$

Comparison numerical solution/reference solution (A.N.)

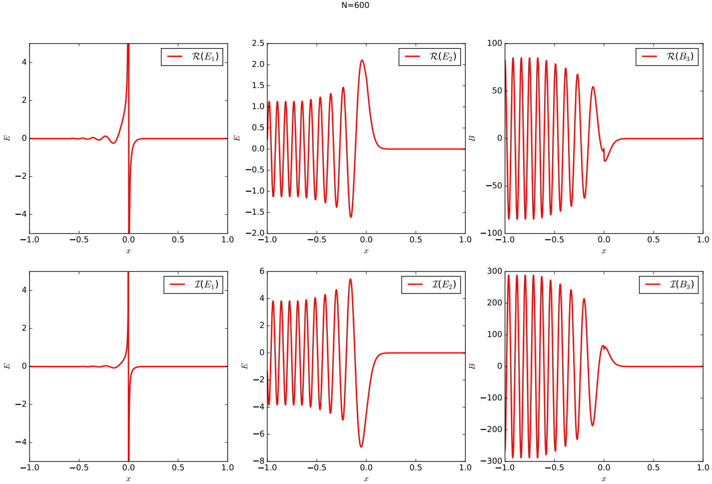
New tools : manufactured solutions

Numerical results

2D



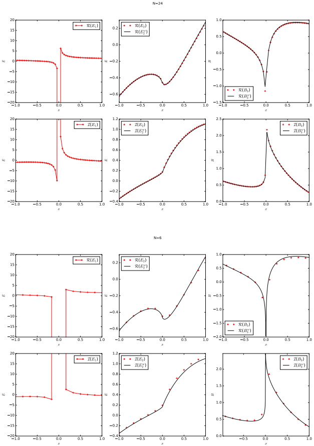
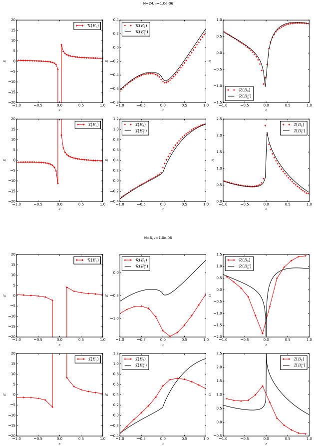
Comparison numerical solution/reference solution (A.N.)



Coarsening the mesh

For the same analytical solution, we compare the two techniques for a coarse mesh.

Number of cells is 48 then 12 : $\nu = 10^{-6}$ for the "standard" FEM method.

new	classic
	

Section 3

2D

2D extension

The method extends in multiD, where all previous techniques fail.

We consider the X-mode equations

$$\left\{\begin{array}{cccc} B_3^\nu & +\partial_y E_1^\nu & -\partial_x E_2^\nu & =0, \\ \partial_y B_3^\nu & -(\alpha+i\nu)E_1^\nu & -i\delta E_2^\nu & =0, \\ -\partial_x B_3^\nu & +i\delta E_1^\nu & -(\alpha+i\nu)E_2^\nu & =0. \end{array}\right.$$

The coefficients are $\alpha(x,y)$ and $\delta(x,y)$ in $C^2(\mathbb{R}^2)\cap L^\infty(\mathbb{R}^2)$.
The function α vanishes on the vertical line and only there, i.e.,

$$\left\{\begin{array}{lll} \alpha(x,y)<0, & x<0, & y\in\mathbb{R}, \\ \alpha(0,y)=0, & x=0, & y\in\mathbb{R} \\ \alpha(x,y)>0, & x>0 & y\in\mathbb{R}. \end{array}\right.$$

The function δ is uniformly positive : $0<\delta_-\leq\delta<\delta_+$.

Manufactured solutions

Set $r(y) = \partial_x \alpha(0, y) < 0$ and $\sigma(y) = \delta(0, y) > 0$.
The following limits hold in the same spaces as $\nu \to 0^+$

$$\begin{cases} F_1^{w,+}(x,y) = -\left(\dfrac{w(y)}{r(y)x}\right)\left(r(y)x + i0^+\right)^{-i\frac{\sigma'(y)}{r(y)}}, \\ F_2^{w,+}(x,y) = \dfrac{i}{\sigma(y)}\left(w(y)(r(y)x + i0^+)^{-i\frac{\sigma'(y)}{r(y)}} + \dots\right), \\ C_3^{w,+}(x,y) = \sigma(y)w(y)\left(\dfrac{1 - (r(y)x + i0^+)^{-i\frac{\sigma'(y)}{r(y)}}}{\sigma'(y)}\right) \end{cases}$$

where the complex powers are defined according to the principal value of the logarithm

$$(a+i0^+)^{i\lambda} := \lim_{\nu \to 0^+}(a+i\nu)^{i\lambda} = \begin{cases} e^{i\lambda \log|a|} & \text{if } a > 0, \\ e^{i\lambda(\log|a|+i\pi)} & \text{if } a < 0, \end{cases} \qquad \text{for } a \in \mathbb{R}^*, \ \lambda \in \mathbb{R}.$$

The new phase term highly depends on

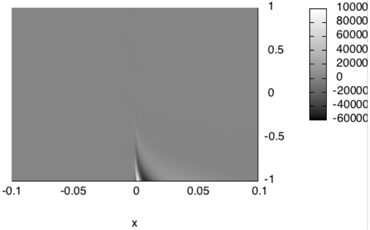
$$\sigma'(y) = \frac{\partial}{\partial s}\varepsilon_\times.$$

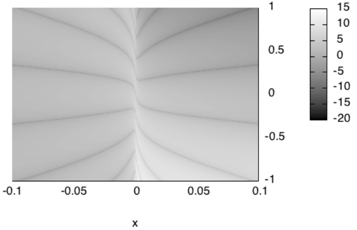
The singular manufactured solution satisfies the bound

$$\left\|F_1^{w,\nu}\right\|_{L^2_{x,\mathrm{loc}}(\Omega)} + \left\|F_2^{w,\nu}\right\|_{L^2_{\mathrm{loc}}(\Omega)} + \left\|C_3^{w,\nu}\right\|_{L^2_{\mathrm{loc}}(\Omega)} < \infty, \quad \mathrm{unif}/\nu.$$

Representation

The plasma-dependent parameters are $\alpha(x,y) = -x$ and $\delta(x,y) = y^2 + 1$.





Real part of the singular manufactured solution

$$\text{Real}\ \left(F_1^{1,+}\right) \approx \frac{1}{x}\cos(2y\log|x|) \times \begin{cases} 1, & x < 0 \\ \exp(-2\pi y), & x > 0 \end{cases}$$

on a 2D domain $\Omega = (-1,1)^2$, with dissipation $\nu = 0.001$ and weight $w \equiv 1$.

The right plot shows the same function in logscale, to improve the visibility of the oscillations along y outside the bottom-lower region where the negative sign of $\sigma'(y) := \partial_y \delta(y,0)$ creates an exponential growths in the $-y$ direction.

Not seen so far in any textbook in plasma physics.

Perspectives

- The mathematics of resonant Maxwell's equations is challenging. Simplification with $\nu = 0$ is wrong. New insights in the regime $\nu = 0^+$.
- The multiD theory is surprising : new divergence relation, new quasi-solutions with singularity $1/x$ combined with highly oscillatory phase.
- New approach fro FE solver where manufactured solutions (i.e. quasi-modes) are explicitly used for the discretization.

  Practical prescription : do not change the spaces, change a few lines in the matrix.

  The results are more accurate, even for coarse meshes. No tuning of the ν.
- An interesting problem (Lu-Colas, ...) is mode-coupling : in ITER, the angle is $\theta \approx 7$ deg

$$\underline{\underline{\varepsilon}} = \begin{pmatrix} \varepsilon_\perp \sin^2\theta + \varepsilon_\parallel \cos^2\theta & i\varepsilon_x \sin\theta & (\varepsilon_\parallel - \varepsilon_\perp)\sin\theta\cos\theta \\ -i\varepsilon_x \sin\theta & \varepsilon_\perp & i\varepsilon_x \cos\theta \\ (\varepsilon_\parallel - \varepsilon_\perp)\sin\theta\cos\theta & -i\varepsilon_x\cos\theta & \varepsilon_\perp\cos^2\theta + \varepsilon_\parallel \sin^2\theta \end{pmatrix}.$$

- Preprint (2016) : Martin Campos-Pinto -B.D., Constructive formulations of resonant Maxwell's equations.

- A. Nicolopoulos : M2 report, 2016.

Advertisement : official announcement very soon.

Summer School CEA-EDF-INRIA : "Waves and fusion plasmas" on modern modeling aspects and related numerical methods.

Scientific committee : Eric Sonnendrucker (Garching), B. Després (LJLL-UPMC, Paris), Martin Campos-Pinto (CNRS-LJLL, Paris), Lise-Marie Imbert-Gérard (Courant Institute, New York).

Date : first week July 2017. Where : Inria in Paris.

Tentative program.
Courses delivered by : Rémi Dumont (IRFM-CEA Cadarache), Omar Maj (IPP-Garching), Emanuele Poli (IPP-Garching), Lise-Marie Imbert-Gérard (Courant, NYU), Leslie Greengard (Courant, NYU), Laurent Colas (IRFM-CEA Cadarache).
Plus additional research talks.