A Finite Volume Approximation for a Two-Temperature Plasma Fusion Model

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Outline

1. Framework: A Fortunate Meeting of Programs

1. **BORDEAUX:** C. Berthon[1], B. Dubroca[2], A. S.

   then E. Estibals, D. Arégba, J. Breil, S. Brull, ...

2. **NICE:** H. Guillard, B. Nkonga, A. S.

   then E. Estibals

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Assumptions:

- Unmagnetized quasineutral totally ionized plasma
- Particles undergoing the electric field **E** given by the Ohm's law:

$$c_i \mathrm{grad} p_e - c_e \mathrm{grad} p_i = n_e q_e \mathbf{E}$$

where: $c_e = \rho_e/\rho$ and $c_i = \rho_i/\rho$

The Model:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p_e + p_i)\mathbf{I}) = 0 \\ \partial_t(\rho_e \varepsilon_e + \frac{1}{2}\rho_e \mathbf{u}\cdot\mathbf{u}) + \mathrm{div}((\rho_e \varepsilon_e + \frac{1}{2}\rho_e \mathbf{u}\cdot\mathbf{u} + p_e)\mathbf{u}) \\ \qquad - (c_i \mathrm{grad} p_e - c_e \mathrm{grad} p_i)\cdot \mathbf{u} = \nu_{ei}^{\mathscr{E}}(T_i - T_e) \\ \partial_t(\rho_i \varepsilon_i + \frac{1}{2}\rho_i \mathbf{u}\cdot\mathbf{u}) + \mathrm{div}((\rho_i \varepsilon_i + \frac{1}{2}\rho_i \mathbf{u}\cdot\mathbf{u} + p_i)\mathbf{u}) \\ \qquad + (c_i \mathrm{grad} p_e - c_e \mathrm{grad} p_i)\cdot \mathbf{u} = -\nu_{ei}^{\mathscr{E}}(T_i - T_e) \end{cases}$$

The mathematical properties of the model

- The model is non-conservative

- The seminal work of *Coquel* and *Marmignon* [3] make it conservative under:

  assumption of **null jump of entropy across shocks**

- The same transformation was recently rederived in the internship of *Estibals*[4]

3

F. Coquel, C. Marmignon, Astro. Space Sc., **260, 1-2**, 15-27 (1998)

4

D. Arégba, J. Breil, S. Brull, B. Dubroca, E. Estibals, submitted for publication

The mathematical properties of the model

- The following conservative system is then obtained:

$$\left\{\begin{array}{lcc} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) & = & 0 \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p_e + p_i)\mathbf{I}) & = & 0 \\ \partial_t(\rho \mathscr{E}) + \mathrm{div}((\rho \mathscr{E} + p_i + p_e)\mathbf{u}) & = & 0 \\ \partial_t(\rho_e s_e) + \mathrm{div}(\rho_e s_e \mathbf{u}) & = & \nu_{ei}^{\mathscr{E}} \rho_e^{1-\gamma_e}(T_i - T_e) \end{array}\right.$$

where: $\rho \mathscr{E} = \rho_i \varepsilon_i + \frac{1}{2}\rho_i \mathbf{u} \cdot \mathbf{u} + \rho_e \varepsilon_e + \frac{1}{2}\rho_e \mathbf{u} \cdot \mathbf{u}$ is the total energy, $s_e = p_e \rho_e^{-\gamma_e}$ is the electron's entropy

The mathematical properties of the model

- The following conservative system is then obtained:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho \mathbf{u}) & = & 0 \\ \partial_t(\rho \mathbf{u}) + \mathrm{div}(\rho \mathbf{u} \otimes \mathbf{u} + (p_e + p_i)\mathbf{I}) & = & 0 \\ \partial_t(\rho \mathscr{E}) + \mathrm{div}((\rho \mathscr{E} + p_i + p_e)\mathbf{u}) & = & 0 \\ \partial_t(\rho s_e) + \mathrm{div}(\rho s_e \mathbf{u}) & = & \nu_{ei}^{\mathscr{E}} \; c_e^{-\gamma_e} \; \rho^{1-\gamma_e}(T_i - T_e) \end{cases}$$

- This system can be written in the following compact form:

$$\partial_t \mathscr{U} + \mathrm{div} \mathscr{F}(\mathscr{U}) = \mathscr{S}(\mathscr{U})$$

with:

$$\mathscr{U} = \begin{pmatrix} \rho \\ \rho \mathbf{u} \\ \rho \mathscr{E} \\ \rho s_e \end{pmatrix} \quad \mathscr{F}(\mathscr{U}) = \begin{pmatrix} \rho \mathbf{u} \\ \rho \mathbf{u} \otimes \mathbf{u} + (p_i + p_e)\mathbf{I} \\ (\rho \mathscr{E} + p_i + p_e)\mathbf{u} \\ \rho s_e \mathbf{u} \end{pmatrix} \quad \mathscr{S}(\mathscr{U}) = \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ \mathscr{S}_e \end{pmatrix}$$

$$\mathscr{S}_e = \nu_{ei}^{\mathscr{E}} \; c_e^{-\gamma_e} \; \rho^{1-\gamma_e}(T_i - T_e)$$

The mathematical properties of the model

- The solution $\mathscr{U}$ of the model belongs to the set of *physically admissible states* $\mathscr{O}$ defined by:

$$\mathscr{O} = \left\{ \mathscr{U} = (\rho, \rho\mathbf{u}, \rho\mathscr{E}, \rho s_e)^T \in \mathbb{R}^6, \quad \rho > 0, \quad \mathscr{E} - \frac{1}{2}\mathbf{u}\cdot\mathbf{u} > 0, \quad s_e > 0 \right\}$$

- A useful Lemma:

Lemma

Let $\mathscr{U} = (\rho, \rho\mathbf{u}, \rho\mathscr{E}, \rho s_e)^T$ be a solution of the model. Then the following systems are equivalent:

$$\left\{ \begin{array}{lcc} \partial_t \rho = \partial_t \mathbf{u} = \partial_t \mathscr{E} & = & 0 \\ \partial_t s_e & = & \nu_{ei}^{\mathscr{E}} \, c_e^{-\gamma_e} \, \rho^{-\gamma_e} (T_i - T_e) \end{array} \right.$$

$$\left\{ \begin{array}{lcc} \partial_t \rho = \partial_t \mathbf{u} & = & 0 \\ \partial_t T_e & = & \nu_{ei}^{\mathscr{E}} \, (T_i - T_e) \\ \partial_t T_i & = & - \nu_{ei}^{\mathscr{E}} \, (T_i - T_e) \end{array} \right.$$

The mathematical properties of the model

- The model is Galilean invariant
- Hyperbolicity

Theorem

The 1D version of the model without the source term $\mathscr{S}_e$ is hyperbolic. The eigenvalues are given by the set

$$\Xi = \{u - c_{ei},\, u,\, u,\, u,\, u + c_{ei}\}$$

where $c_{ei} = \sqrt{\dfrac{\partial(p_i + p_e)}{\partial \rho}}$.

The characteristic fields associated to the eigenvalues $u \pm c_{ei}$ are genuinely nonlinear while the characteristic fields associated to the eigenvalue u are linearly degenerated.

A numerical strategy to approximate the model based on our work reported in:

A. Bonnement, et al., *ESAIM Proceedings*, **32**, 163-176 (2011)

M. Bilanceri, et al., *ESAIM Proceedings*, **43**, 164-179 (2013)

- Toroidal geometry and cylindrical coordinates for a torus

- Tessellation:
  - ☐ unstructured mesh composed of triangles in polar planes
  - ☐ structured mesh in the toroidal direction

i.e. curved primastic elements in the toroidal direction

- Our finite volume approximation in curvilinear coordinates:
  - ☐ the divergence of the momentum equation is kept in local cylindrical coordinates

- Our finite volume approximation in curvilinear coordinates:
  - ☐ the divergence of the momentum equation is kept in local cylindrical coordinates
  - ☐ integration of this divergence form on control cells
  - ☐ definition of adequate discrete cylindrical base and projection of the result of the integration step on this base

This yields the following generic FV approximation:

$$\left|\Omega_\alpha^{3D}\right| \partial_t \begin{pmatrix} \rho_\alpha \\ \rho_\alpha \eta_\alpha \mathbf{u}_\alpha \\ \rho_\alpha \mathscr{E}_\alpha \\ \rho_\alpha s_{e\,\alpha} \end{pmatrix} + \sum_{S_{\alpha\beta} \in \mathscr{S}^{pol}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega$$

$$+ \sum_{S_{\alpha\beta} \in \mathscr{S}^{tor}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega = \int_{\Omega_\alpha^{3D}} \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ R\mathscr{S}_e \end{pmatrix} d\Omega$$

The details:

- Toroidal geometry and cylindrical coordinates

$$\begin{cases} x = R\cos\varphi \\ y = R\sin\varphi \\ z = Z \end{cases}$$

$(\mathbf{e}_R,\ \mathbf{e}_\varphi,\ \mathbf{e}_Z)$ is the cylindrical local base

- The conservative form of the model in cylindrical coordinates is given by:

$$\begin{cases} \partial_t(R\rho) + \partial_{\xi_k}(R\rho\mathbf{u}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho\mathbf{u}) + \partial_{\xi_k}(R\mathbf{T}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho\mathscr{E}) + \partial_{\xi_k}(R(\rho\mathscr{E} + p_i + p_e)\mathbf{u}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho s_e) + \partial_{\xi_k}(R\rho s_e\mathbf{u}\cdot\mathbf{e}^k) & = & R\,\mathscr{S}_e \end{cases}$$

where: $e_k \in \{\mathbf{e}_R,\ \mathbf{e}_\varphi,\ \mathbf{e}_Z\}$; $\mathbf{e}_k\cdot\mathbf{e}^l = \delta_k^l$;

$\mathbf{T} = \left(\rho u_k u_l + (p_i + p_e)\delta_k^l\right)\ \mathbf{e}_k\otimes\mathbf{e}_l$

$\mathscr{S}_e = \nu_{ei}^{\mathscr{E}}\ c_e^{-\gamma_e}\ \rho^{1-\gamma_e}(T_i - T_e)$

The details:

- Tessellation:

  □ triangles T_β in (R, Z)–coordinates to mesh polar planes

  □ interval of angles $(\varphi_k, \varphi_{k+1})$, where $k \in \{1, \cdots, N_{plan}\}$ to mesh the computational domain in the toroidal direction

This leads to curved primastic elements in the toroidal direction to partition the computational domain

□ INRIA (R, Z)–coordinates 2D control cells Ω_α leading to the $3D$ control cells Ω_α^{3D} associated to each node α of the mesh of the computational domain

□ The boundary of each control cell Ω_α^{3D} is composed of poloidal surfaces and toroidal surfaces

The details:

- Our finite volume approximation in curvilinear coordinates:
  - ☐ Two kind of equations:
    - ◇ scalar equations: continuity, energy, entropy
    - ◇ vectorial equation: momentum
  - ☐ Our procedure applied to scalar equations is same as the well-known FV scheme
  - ☐ For vectorial equations, our strategy proceeds as follows:
    - ◇ Integration of momentum equation over the control cell Ω_α^{3D}:

$$|\Omega_\alpha^{3D}|\, \partial_t \left(\frac{1}{|\Omega_\alpha^{3D}|} \int_{\Omega_\alpha^{3D}} R\rho\mathbf{u}\, d\Omega \right) + \int_{\Omega_\alpha^{3D}} \partial_{\xi_k} (R\mathbf{T} \cdot \mathbf{e}^k)\, d\Omega = 0$$

    - ◇ Crucial choice of components of the vector $\dfrac{1}{|\Omega_\alpha^{3D}|} \displaystyle\int_{\Omega_\alpha^{3D}} R\rho\mathbf{u}\, d\Omega$ to be stored in order to represent it

The details:

- Our finite volume approximation in curvilinear coordinates:
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    - ◇ Crucial choice of components of the vector $\frac{1}{|\Omega_\alpha^{3D}|}\int_{\Omega_\alpha^{3D}} R\rho\mathbf{u}\,d\Omega$ to be stored in order to represent it
    - ◇ We chose to store the components of the vector $\mathbf{u}_\alpha$ with respect to the local basis $(\mathbf{e}_R(\alpha),\ \mathbf{e}_Z(\alpha),\ \mathbf{e}_\varphi(\alpha))$ of the control cell Ω_α^{3D}
    - ◇ This automatically leads to:

$$\frac{1}{|\Omega_\alpha^{3D}|}\int_{\Omega_\alpha^{3D}} R\rho\mathbf{u}\,d\Omega = \rho_\alpha\left(\eta_\alpha u_{R,\alpha}\mathbf{e}_R(\alpha) + u_{Z,\alpha}\mathbf{e}_Z(\alpha) + \eta_\alpha u_{\varphi,\alpha}\mathbf{e}_\varphi(\alpha)\right)$$

with:

$$\eta_\alpha = \frac{\sin\left(\frac{\varphi_{\alpha+1/2}-\varphi_{\alpha-1/2}}{2}\right)}{\frac{\varphi_{\alpha+1/2}-\varphi_{\alpha-1/2}}{2}},\quad \mathbf{u}_\alpha = u_{R,\alpha}\mathbf{e}_R(\alpha)+u_{Z,\alpha}\mathbf{e}_Z(\alpha)+u_{\varphi,\alpha}\mathbf{e}_\varphi(\alpha)$$

The details:

- Our finite volume approximation in curvilinear coordinates:
  - For vectorial equations, our strategy proceeds as follows:
    - This yields the following equation:

$$|\Omega_\alpha^{3D}|\ \partial_t(\rho_\alpha\eta_\alpha\mathbf{u}_\alpha) + \int_{\Omega_\alpha^{3D}} \partial_{\xi_k}(R\mathbf{T}\cdot\mathbf{e}^k)\, d\Omega = 0$$

where: $\eta_\alpha\mathbf{u}_\alpha = \eta_\alpha u_{R,\alpha}\mathbf{e}_R(\alpha) + u_{Z,\alpha}\mathbf{e}_Z(\alpha) + \eta_\alpha u_{\varphi,\alpha}\mathbf{e}_\varphi(\alpha)$

- **This yields the following generic FV approximation:**

$$\left|\Omega_\alpha^{3D}\right|\ \partial_t \begin{pmatrix} \rho_\alpha \\ \rho_\alpha\eta_\alpha\mathbf{u}_\alpha \\ \rho_\alpha\mathscr{E}_\alpha \\ \rho_\alpha s_{e\alpha} \end{pmatrix} + \sum_{S_{\alpha\beta}\in\mathscr{S}^{pol}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right)\, d\partial\Omega$$

$$+ \sum_{S_{\alpha\beta}\in\mathscr{S}^{tor}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right)\, d\partial\Omega = \int_{\Omega_\alpha^{3D}} \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ R\mathscr{S}_e \end{pmatrix} d\Omega$$

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- **This yields the following generic FV approximation:**

$$\left|\Omega_\alpha^{3D}\right| \partial_t \begin{pmatrix} \rho_\alpha \\ \rho_\alpha \eta_\alpha \mathbf{u}_\alpha \\ \rho_\alpha \mathscr{E}_\alpha \\ \rho_\alpha s_{e\alpha} \end{pmatrix} + \sum_{S_{\alpha\beta} \in \mathscr{S}^{pol}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega$$

$$+ \sum_{S_{\alpha\beta} \in \mathscr{S}^{tor}} \int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega = \int_{\Omega_\alpha^{3D}} \begin{pmatrix} 0 \\ \mathbf{0} \\ 0 \\ R\mathscr{S}_e \end{pmatrix} d\Omega$$

where: $\mathscr{S}^{pol}$ are poloidal boundaries of Ω_α^{3D}
$\mathscr{S}^{tor}$ are toroidal ones

- **The fluxes** $\int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) d\partial\Omega$ **are computed with a new relaxation scheme**

The details:

- **The fluxes** $\displaystyle\int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega$ **are computed with a new relaxation scheme** based on

C. Berthon, B. Dubroca, A. S., *SINUM*, **50**, 468-491 (2012)

C. Berthon, B. Dubroca, A. S., *CMS*, **13**, 2119-2154 (2015)

and derived in

D. Arégba, J. Breil, S. Brull, B. Dubroca, E. Estibals, **submitted for publication**

☐ The 1D model without source term

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) & = & 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + (p_e + p_i)) & = & 0 \\ \partial_t(\rho v) + \partial_x(\rho v u) & = & 0 \\ \partial_t(\rho w) + \partial_x(\rho w u) & = & 0 \\ \partial_t(\rho \mathscr{E}) + \partial_x((\rho \mathscr{E} + p_i + p_e) u) & = & 0 \\ \partial_t(\rho s_e) + \partial_x(\rho s_e u) & = & 0 \end{cases}$$

The details:

- **The fluxes** $\displaystyle\int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right)\, d\partial\Omega$ **are computed with a new relaxation scheme**
  - ☐ A relaxation approximation of the 1D model without source term

$$\begin{cases}
\partial_t \rho + \partial_x(\rho u) & = & 0\\
\partial_t(\rho u) + \partial_x(\rho u^2 + (\pi_e + \pi_i)) & = & 0\\
\partial_t(\rho v) + \partial_x(\rho u v) & = & 0\\
\partial_t(\rho w) + \partial_x(\rho u w) & = & 0\\
\partial_t(\rho \mathscr{E}) + \partial_x((\rho \mathscr{E} + \pi_i + \pi_e) u) & = & 0\\
\partial_t(\rho s_e) + \partial_x(\rho s_e u) & = & 0\\
\partial_t(\rho \pi_e + c_e a^2) + \partial_x(\rho \pi_e u + c_e a^2 u) & = & \dfrac{1}{\tau}\rho(p_e - \pi_e)\\
\partial_t(\rho \pi_i + c_i a^2) + \partial_x(\rho \pi_i u + c_i a^2 u) & = & \dfrac{1}{\tau}\rho(p_i - \pi_i)\\
\partial_t(\rho a) + \partial_x(\rho a u) & = & 0
\end{cases}$$

3. A Finite Volume Approximation

The details:

- **The fluxes** $\int_{S_{\alpha\beta}} \mathbf{F}\left(\mathscr{U}_\alpha, \mathscr{U}_\beta, \mathbf{n}_{\alpha\beta}\right) \, d\partial\Omega$ **are computed with a new relaxation scheme**

  □ A relaxation approximation of the 1D model without source term

  ◇ $\tau \to 0 \Longrightarrow \pi_e \to p_e, \quad \pi_i \to p_i$

  ◇ The relaxation system for 1D model turns into:

$$\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = \frac{1}{\tau}\mathbb{T}(\mathbb{U})$$

Theorem

The 1D relaxation system $\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = 0$ is hyperbolic. The eigenvalues are given by: $\Lambda = \{u - a/\rho,\ u,\ u,\ u,\ u,\ u,\ u + a/\rho\}$.

All the associated characteristic fields are linearly degenerated.

◇ The Riemann problem can be solved exactely:

Lemma

Assume $\mathbb{U}_l$ *and* $\mathbb{U}_r$ *are constant states and consider*

$$\mathbb{U}_0(x) = \begin{cases} \mathbb{U}_l & \text{if} \quad x < 0 \\ \mathbb{U}_r & \text{if} \quad x > 0 \end{cases}$$

as the initial data for the system $\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = 0$. *Let* a_l *and* a_r *be positive real numbers* $a_l > 0$, $a_r > 0$, *satisfying:*

$$b_l = u_l - a_l/\rho_l < u^\star < u_r + a_r/\rho_r = b_r$$

where: $u^\star = \dfrac{a_l u_l + a_r u_r}{a_l + a_r} - \dfrac{(\pi_{i,r} + \pi_{e,r}) - (\pi_{i,l} + \pi_{e,l})}{a_l + a_r}$.

Then the weak solution of system $\partial_t \mathbb{U} + \partial_x \mathbb{F}(\mathbb{U}) = 0$ *with the initial data* $(\mathbb{U}_l, \mathbb{U}_r)$ *is given by*

$$\mathbb{U}_{\mathcal{R}}(x/t, \mathbb{U}_l, \mathbb{U}_r) = \begin{cases} \mathbb{U}_l, & \text{if} \quad b_l > x/t \\ \mathbb{U}_l^\star, & \text{if} \quad b_l \le x/t \le u^\star \\ \mathbb{U}_r^\star, & \text{if} \quad u^\star \le x/t \le b_r \\ \mathbb{U}_r, & \text{if} \quad b_r < x/t \end{cases}$$

◇ With $g = l$ or r, and

$$\begin{cases} u^\star = (a_l u_l + a_r u_r)/(a_l + a_r) - (\pi_{i,r} + \pi_{e,r} - \pi_{i,l} - \pi_{e,l})(a_l + a_r) \\ v^\star = (a_l v_l + a_r v_r)/(a_r + a_l) \\ w^\star = (a_l w_l + a_r w_r)/(a_r + a_l) \\ \pi^\star_{i,l} = \pi_{i,l} + c_i a_l(\pi_{i,r} + \pi_{e,r} - \pi_{i,l} - \pi_{e,l} - a_r(u_r - u_l))/(a_l + a_r) \\ \pi^\star_{e,l} = \pi_{e,l} + c_e a_l(\pi_{i,r} + \pi_{e,r} - \pi_{i,l} - \pi_{e,l} - a_r(u_r - u_l))/(a_l + a_r) \\ \pi^\star_{i,r} = \pi_{i,r} + c_i a_r(\pi_{i,l} + \pi_{e,l} - \pi_{i,r} - \pi_{e,r} - a_l(u_r - u_l))/(a_l + a_r) \\ \pi^\star_{e,r} = \pi_{e,r} + c_e a_r(\pi_{i,l} + \pi_{e,l} - \pi_{i,r} - \pi_{e,r} - a_l(u_r - u_l))/(a_l + a_r) \\ 1/\rho^\star_g = 1/\rho_g - (\pi^\star_{i,g} + \pi^\star_{e,g} - \pi_{i,g} - \pi_{e,g})/(a_g)^2 \\ \varepsilon^\star_{e,l} = \varepsilon_{e,l} + ((\pi^\star_{e,l} + \pi^\star_{i,l})^2 - (\pi_{e,l} + \pi_{i,l})^2)/(2(c_e a_l)^2) \\ \varepsilon^\star_{i,l} = \varepsilon_{i,l} + ((\pi^\star_{e,l} + \pi^\star_{i,l})^2 - (\pi_{e,l} + \pi_{i,l})^2)/(2(c_i a_l)^2) \\ \varepsilon^\star_{e,r} = \varepsilon_{e,r} + ((\pi^\star_{e,r} + \pi^\star_{i,r})^2 - (\pi_{e,r} + \pi_{i,r})^2)/(2(c_e a_r)^2) \\ \varepsilon^\star_{i,r} = \varepsilon_{i,r} + ((\pi^\star_{e,r} + \pi^\star_{i,r})^2 - (\pi_{e,r} + \pi_{i,r})^2)/(2(c_i a_r)^2) \\ s^\star_{e,g} = s_{e,g}, \quad a^\star_g = a_g \end{cases}$$

$\diamond$ The Riemann problem can be solved exactely:
Then the star intermediate states $\mathbb{U}_l^\star$ and $\mathbb{U}_r^\star$ are given by

$$\mathbb{U}_g^\star = \begin{pmatrix} \rho_g^\star \\ \rho_g^\star u^\star \\ \rho_g^\star v^\star \\ \rho_g^\star w^\star \\ \frac{1}{2}\rho_g^\star (u^\star)^2 + \rho_g^\star \varepsilon_{i,g}^\star + \rho_g^\star \varepsilon_{e,g}^\star \\ \rho_g^\star s_{e,g}^\star \\ \rho_g^\star \pi_{i,g}^\star + c_i (a_g^\star)^2 \\ \rho_g^\star \pi_{e,g}^\star + c_e (a_g^\star)^2 \\ \rho_g^\star a_g^\star \end{pmatrix}.$$

The details:

- **Practical implementation of our approximation**

At time t^n, we consider a piecewise constant approximation of the solution of the initial model given by,

$$\mathscr{U}^{\Delta}(R, Z, \varphi, t^n) = \mathscr{U}^n_\alpha, \quad (R, Z, \varphi) \in \Omega^{3D}_\alpha$$

where

$$\mathscr{U}^n_\alpha = \begin{pmatrix} \rho^n_\alpha \\ \rho^n_\alpha \mathbf{u}^n_\alpha \\ \frac{1}{2}\rho^n_\alpha \mathbf{u}^n_\alpha \cdot \mathbf{u}^n_\alpha + \rho^n_\alpha \varepsilon^n_{i,\alpha} + \rho^n_\alpha \varepsilon^n_{e,\alpha} \\ \rho^n_\alpha s^n_{e,\alpha} \end{pmatrix}$$

with $\rho^n_\alpha \varepsilon^n_{i,\alpha} = \dfrac{p^n_{i,\alpha}}{\gamma_i - 1}$, $\rho^n_\alpha \varepsilon^n_{e,\alpha} = \dfrac{p^n_{e,\alpha}}{\gamma_e - 1}$, $s^n_{e,\alpha} = p^n_{e,\alpha}(c_e\, \rho^n_\alpha)^{-\gamma_e}$

To evolve in time this approximation, we proceed in two steps:

The details:

To evolve in time this approximation, we proceed in two steps:

☐ *First step: Evolution step.* We set the relaxation state as:

$$\mathbb{U}_\alpha^n = \begin{pmatrix} \rho_\alpha^n \\ \rho_\alpha^n \mathbf{u}_\alpha^n \\ \frac{1}{2}\rho_\alpha^n \mathbf{u}_\alpha^n \cdot \mathbf{u}_\alpha^n + \rho_\alpha^n \varepsilon_{i,\alpha}^n + \rho_\alpha^n \varepsilon_{e,\alpha}^n \\ \rho_\alpha^n s_{e,\alpha}^n \\ \rho_\alpha^n p_{i,\alpha}^n + c_i (a_\alpha^n)^2 \\ \rho_\alpha^n p_{e,\alpha}^n + c_e (a_\alpha^n)^2 \\ \rho_\alpha^n a_\alpha^n \end{pmatrix}$$

where the positive real numbers a_α^n satisfy a subcharacteristic condition

The details:
To evolve in time this approximation, we proceed in two steps:

☐ *First step: Evolution step.*

- ◇ The relaxation scheme leads to the updated relaxation state $\widetilde{\mathbb{U}}_{\alpha}^{n}$ along the normal $\mathbf{n}_{\alpha\beta}$ of the interface $S_{\alpha\beta}$
- ◇ The fluxe $\mathbf{F}\left(\mathscr{U}_{\alpha}, \mathscr{U}_{\beta}, \mathbf{n}_{\alpha\beta}\right)$ is reconstructed as:

$$\mathbf{F}\left(\mathscr{U}_{\alpha}, \mathscr{U}_{\beta}, \mathbf{n}_{\alpha\beta}\right) = \mathscr{F}\left(\mathscr{N}\widetilde{\mathbb{U}}_{\alpha}^{n}\right)$$

where: $\mathscr{F}$ is the exact physical flux,
$\mathscr{N}$ is the projection operator of relaxation state space to real state space.

The details:
To evolve in time this approximation, we proceed in two steps:

- *First step: Evolution step.*
  - The discrete scheme

$$\begin{pmatrix} \widehat{\rho}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1}\eta_\alpha \widehat{\mathbf{u}}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1}\widehat{\mathscr{E}}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1}\widehat{s_e}_\alpha^{n+1} \end{pmatrix} = \begin{pmatrix} \rho_\alpha^{n} \\ \rho_\alpha^{n}\eta_\alpha \mathbf{u}_\alpha^{n} \\ \rho_\alpha^{n}\mathscr{E}_\alpha^{n} \\ \rho_\alpha^{n}{s_e}_\alpha^{n} \end{pmatrix} - \frac{\Delta t}{|\Omega_\alpha^{3D}|} \sum_{S_{\alpha\beta}\in\mathscr{S}^{pol}\cup\mathscr{S}^{tor}} |S_{\alpha\beta}|\,\mathscr{F}\left(\mathscr{N}\widetilde{\mathbb{U}}_\alpha^n\right)$$

solves, at time t^{n+1}, with the initial data $\mathscr{U}_\alpha^n$, the system:

$$\begin{cases} \partial_t(R\rho) + \partial_{\xi_k}(R\rho\mathbf{u}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho\mathbf{u}) + \partial_{\xi_k}(R\mathbf{T}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho\mathscr{E}) + \partial_{\xi_k}(R(\rho\mathscr{E} + p_i + p_e)\mathbf{u}\cdot\mathbf{e}^k) & = & 0 \\ \partial_t(R\rho s_e) + \partial_{\xi_k}(R\rho s_e\mathbf{u}\cdot\mathbf{e}^k) & = & 0 \end{cases}$$

The details:

To evolve in time this approximation, we proceed in two steps:

□ *Second step: Relaxation.* The following system is solved at time t^{n+1}:

$$\left\{ \begin{array}{lcc} \partial_t \rho & = & 0 \\ \partial_t \mathbf{u} & = & 0 \\ \partial_t \mathscr{E} & = & 0 \\ \partial_t s_e & = & \nu_{ei}^{\mathscr{E}} \; c_e^{-\gamma_e} \; \rho^{-\gamma_e} (T_i - T_e) \end{array} \right.$$

with the data $\widehat{\mathscr{U}}_\alpha^{n+1} = \begin{pmatrix} \widehat{\rho}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1} \eta_\alpha \widehat{\mathbf{u}}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1} \widehat{\mathscr{E}}_\alpha^{n+1} \\ \widehat{\rho}_\alpha^{n+1} \widehat{s_e}_\alpha^{n+1} \end{pmatrix}$ at time t^n.

The details:

To evolve in time this approximation, we proceed in two steps:

□ *Second step: Relaxation.* Thanks to the useful Lemma, it amounts to solve at time t^{n+1} the system:

$$\left\{\begin{array}{lcc} \partial_t \rho & = & 0, \\ \partial_t \mathbf{u} & = & 0, \\ \partial_t T_e & = & \nu_{ei}^{\mathscr{E}}\,(T_i - T_e), \\ \partial_t T_i & = & -\nu_{ei}^{\mathscr{E}}\,(T_i - T_e), \end{array}\right.$$

with the data $\widehat{\mathscr{U}}_\alpha^{n+1}$ at time t^n and temperatures $T_{i,\alpha}^n$ and $T_{e,\alpha}^n$. Solving this system leads to the temperatures $T_{i,\alpha}^{n+1}$ and $T_{e,\alpha}^{n+1}$, and then the energy $\mathscr{E}_\alpha^{n+1}$ and entropy state s_e^{n+1} are reconstructed. The state $\mathscr{U}_\alpha^{n+1}$ at time t^{n+1} is thus determined.

The details:

Time-step Δt issue:

Consider the 2D contol cell Ω_α that generates the 3D control cell Ω_α^{3D}. Let T_β be any generic triangle that enters in the construction of Ω_α. Let $h_{\alpha\beta}$ be the minimum of the heights of the triangle T_β. We set:

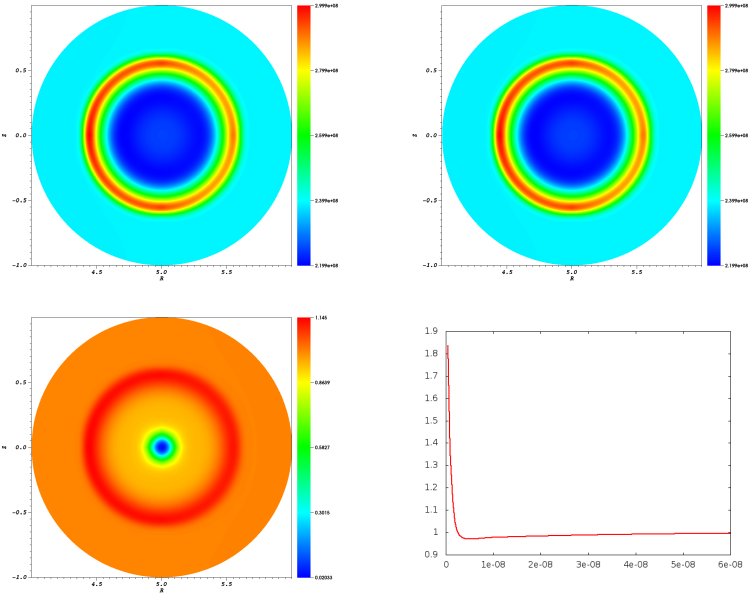
$$\begin{cases} \widehat{\lambda_\alpha} = \max\left\{ |u^n_{\varphi,\alpha} \pm c_{ei,\alpha}|,\ |u^n_{\varphi,\alpha} - \hat{c}_{ei,\alpha}|,\ |u^n_{\varphi,\alpha next} + \hat{c}_{ei,\alpha next}| \right\}, \\ \widehat{\lambda_{\alpha\beta}} = \max\left\{ |\mathbf{u}^n_\alpha \cdot \mathbf{n}_{\alpha\beta} \pm c_{ei,\alpha}|,\ |\mathbf{u}^n_\alpha \cdot \mathbf{n}_{\alpha\beta} - \hat{c}_{ei,\alpha}|,\ |\mathbf{u}^n_\beta \cdot \mathbf{n}_{\alpha\beta} + \hat{c}_{ei,\beta}| \right\}, \end{cases}$$

$$c_{ei,\alpha} = \sqrt{\frac{\partial(p_i + p_e)}{\partial \rho}}\Bigg|_{p_i = p^n_{i,\alpha}, p_e = p^n_{e,\alpha}, \rho = \rho^n_\alpha},\ \hat{c}_{ei,\alpha} = \max\left(\sqrt{\frac{\gamma_i p_{i,\alpha}}{c_i \rho_\alpha}}, \sqrt{\frac{\gamma_e p_{e,\alpha}}{c_e \rho_\alpha}}\right)$$

$$\text{Then:} \quad \Delta t = \min_\alpha \Delta t_\alpha, \quad \text{with} \quad \Delta t_\alpha = \min\left\{ \frac{\Delta \varphi_\alpha}{\widehat{\lambda_\alpha}}, \min_\beta \left(\frac{h_{\alpha\beta}}{\widehat{\lambda_{\alpha\beta}}} \right) \right\}$$

- **Sedov problem in 2D axisymmetric geometry**[5]
  - ☐ Uniform media: $T_i = T_e = 2.901 \times 10^4$ K
    Hot spot: $T_i = 5.802 \times 10^6$ K, and $T_e = 1.7606 \times 10^7$ K
    Other uniform parameters: $\rho = 1$ kg.m^{-3},
    $$u_R = u_Z = u_\phi = 0$$
  - ☐ A mesh made of 16384 triangles in (R, Z)-coordinates is used for the simulations

[5] S. Galéra, et al., *J. Comput. Phys.*, **229**, 5755-5787 (2010)

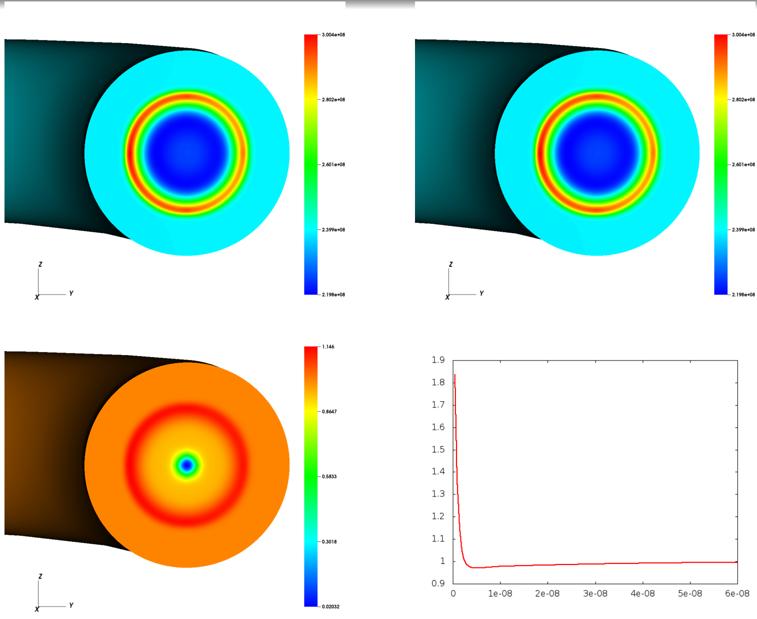


P_i, P_e, ρ, at $t = 9.7634 \times 10^{-6}$, and Te/Ti

- **Sedov problem in 3D axisymmetric geometry** [6]
  - ☐ Uniform media: $T_i = T_e = 2.901 \times 10^4$ K
    Hot spot: $T_i = 5.802 \times 10^6$ K, and $T_e = 1.7606 \times 10^7$ K
    Other uniform parameters: $\rho = 1$ kg.m^{-3},
    $u_R = u_Z = u_\phi = 0$
  - ☐ A mesh made of 16384 triangles in (R, Z)-coordinates is used for the simulations

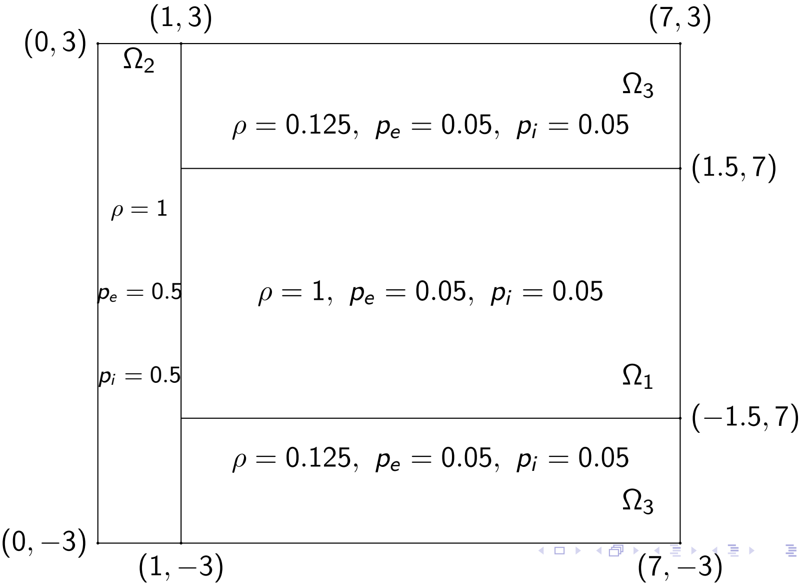
6

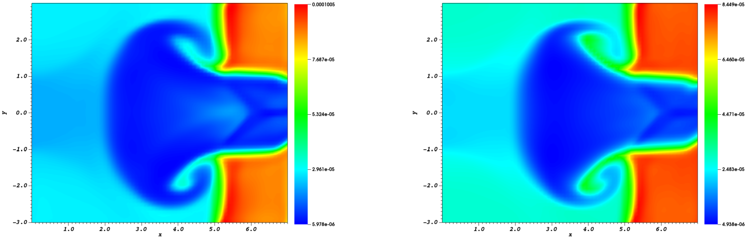
S. Galéra, et al., *J. Comput. Phys.*, **229**, 5755-5787 (2010)



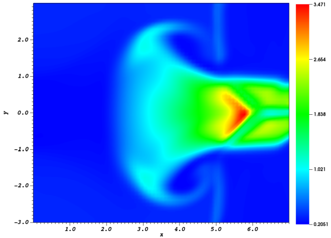
P_i, P_e, ρ, at $t = 9.7634 \times 10^{-6}$, and Te/Ti

- **Triple point problem in 2D Cartesian geometry**[7]





$T_i, \; T_e$ at $t = 3.5s$



The density ρ at $t = 3.5s$

- **Sedov problem in 2D axisymmetric geometry**

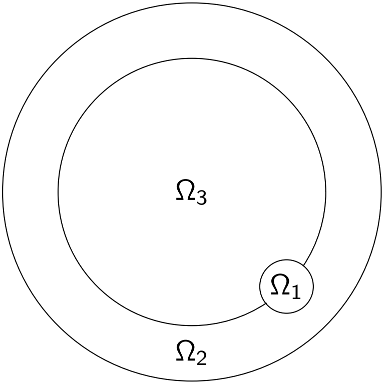
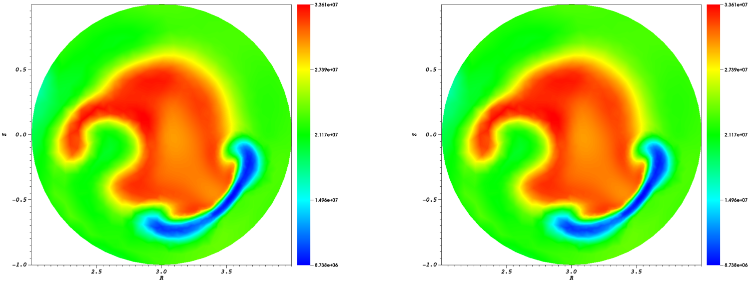
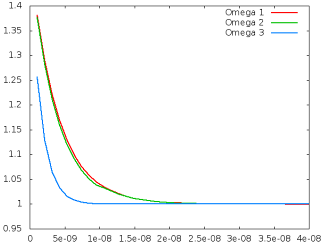


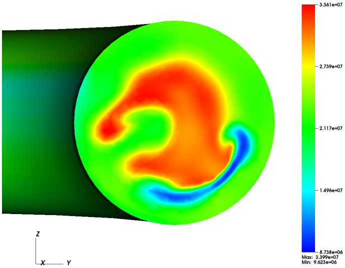
Figure : The three domain of the triple point problem in the (R, Z) plan.

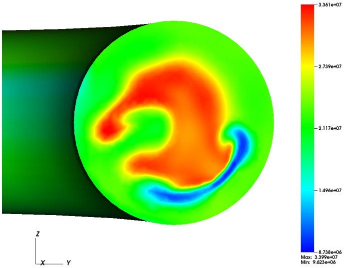


T_i, T_e at $t = 1.157410^{-5}s$

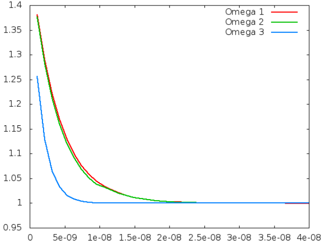


The ratio Te/Ti as function of time t





$T_i, \ T_e$ at $t = 1.157410^{-5}s$



The ratio Te/Ti as function of time t

- Presentation of a Two-Temperature model for fusion plasma
- Derivation of A Finite Volume approximation to compute the numerical solutions of this model implemented in *PlaTo*
- Numerical tests have shown the accuracy and robustness of the scheme
- Perspectives:
  - ☐ More numerical tests to simulate tokamak physics
  - ☐ Extension of the model to include magnetic field
  - ☐ Extension of the model to include heat flux and anisotropy

THANK YOU