# To interact or not to interact
# Fast waves and reduced MHD

Hervé Guillard

Université Côte d'Azur, Inria, CNRS, LJAD, France

# THE COMPRESSIBLE (ideal) MHD MODEL

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$$

$$\frac{\partial}{\partial t}\rho \boldsymbol{u} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \nabla p = \boldsymbol{J} \times \boldsymbol{B}$$

$$\frac{\partial}{\partial t}p + \boldsymbol{u} \cdot \nabla p + \gamma p \nabla \cdot \boldsymbol{u} = 0$$

$$\frac{\partial}{\partial t}\boldsymbol{B} + \nabla \times \boldsymbol{B} \times \boldsymbol{u} = 0$$

# MHD waves

Hyperbolic system with 3 different types of waves (+ entropy waves) If $\boldsymbol{n}$ is the direction of propagation of the wave

- Fast Magnetosonic waves : $\lambda_F = \boldsymbol{u}.\boldsymbol{n} \pm C_F$
  $C_F^2 = \frac{1}{2}(V_t^2 + v_A^2 + \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$
- Alfvén waves : $\lambda_F = \boldsymbol{u}.\boldsymbol{n} \pm C_A \;\; C_A^2 = (\boldsymbol{B}.\boldsymbol{n})^2/\rho$
- Slow Magnetosonic waves : $\lambda_S = \boldsymbol{u}.\boldsymbol{n} \pm C_S$
  $C_S^2 = \frac{1}{2}(V_t^2 + v_A^2 - \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$

$v_A^2 = |\boldsymbol{B}|^2/\rho$ $\quad v_A$ : Alfvén speed
$V_t^2 = \gamma p/\rho$ $\quad V_t$ : acoustic speed

# Transverse MHD waves

propagation speed depends on the direction w r to the magnetic field.

If $\boldsymbol{n} \cdot \boldsymbol{B} = 0$ (transverse waves) :

- Alfvén waves : $\lambda_F = 0$
- Slow Magnetosonic waves : $\lambda_S = 0$
- Fast Magnetosonic waves : $\lambda_F = \pm C_F$ with $C_F^2 = V_t^2 + v_A^2$

only the Fast Magnetosonic waves survive !

# Reduced MHD models

Prototypical example : two equation model of Strauss, Physics of Fluids, 19, p 134, 1976)
variables : $\phi$ : electric potential $\psi$ : magnetic flux

$$\frac{\partial U}{\partial t} + [\varphi, U] - [\psi, J] - \frac{\partial}{\partial z} J = 0$$

$$\frac{\partial \psi}{\partial t} + [\varphi, \psi] - \frac{\partial \varphi}{\partial z} = 0$$

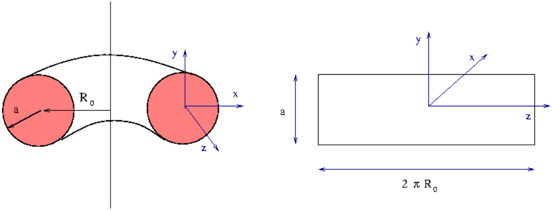
$$U = \nabla_\perp^2 \varphi \qquad J = \nabla_\perp^2 \psi$$

$[f, g] = \mathbf{z}.\nabla_\perp f \times \nabla_\perp g$

# Two equation RMHD model of Strauss

Assumptions :

1. Slab geometry




2. Large dominant magnetic field $\boldsymbol{B}_\perp / \boldsymbol{B}_z = \varepsilon << 1$
3. Low plasma $\beta$ : $\beta = p/B_0^2 \sim \varepsilon^2$
4. Constant density

# Reduced MHD models and wave filtering

Fast transverse waves are absent from RMHD system

"An important feature of RMHD systems is that they eliminate the fast compressional Alfvén wave" (Strauss, Reduced MHD in nearly potential magnetic fields, Journal of Plasma Physics 1997; 57(1):8387)

However the full MHD system is non linear and contains quadratic terms :

$$\mathcal{Q}(U,U) = (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} \quad \text{or} \quad (\boldsymbol{B}\cdot\nabla)\boldsymbol{B} \quad \text{or} \quad (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \cdots$$

But the fast wave do exist :

$$U_{MHD} = U_{RMHD} + U_{\text{Fast wave}}$$

thus

$$\begin{aligned}\mathcal{Q}(U,U) \quad &= \mathcal{Q}(U_{RMHD},U_{RMHD}) + \mathcal{Q}(U_{RMHD},U_{\text{Fast wave}}) + \mathcal{Q}(U_{\text{Fast wave}},U_{RMHD})\\ &+\mathcal{Q}(U_{\text{Fast wave}},U_{\text{Fast wave}})\end{aligned}$$

How do we know that the "Reynolds Stress" term : $\mathcal{Q}(U_{\text{Fast wave}},U_{\text{Fast wave}})$ is not important for the dynamics of the system ?

# Objective of this talk

1. prove that fast transverse magnetosonic waves do not affect the slow dynamics described by RMHD
2. prove that full MHD solutions converge to the solutions of RMHD

# Formulation of Reduced MHD in term of "primitive" variables

**"Almost 2D" Incompressible MHD**

$$\frac{\partial}{\partial\tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp\cdot\nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp\cdot\nabla_\perp)\mathcal{B}_\perp + \nabla_\perp\pi - \partial_z\mathcal{B}_\perp = 0 \quad (1.1)$$

$$\frac{\partial}{\partial\tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp\cdot\nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp\cdot\nabla_\perp)\boldsymbol{v}_\perp - \partial_z\boldsymbol{v}_\perp = 0 \quad (1.2)$$

$$\nabla_\perp\cdot\boldsymbol{v}_\perp = \nabla_\perp\cdot\mathcal{B}_\perp = 0 \quad (1.3)$$

Claim : Two equation RMHD model is equivalent to the system (1) :

$$\nabla_\perp\cdot\boldsymbol{v}_\perp = \nabla_\perp\cdot\mathcal{B}_\perp = 0 \Rightarrow \exists\ 2 \text{ scalar functions } \psi,\phi \text{ such that } :$$
$$\boldsymbol{v}_\perp = \mathbf{z}\times\nabla_\perp\phi \qquad \mathcal{B}_\perp = \mathbf{z}\times\nabla_\perp\psi$$

1. Plug these expression in eqs (1)
2. apply the operator $\mathbf{z}\cdot\nabla_\perp\times$ to (1.1) (vorticity form of the momentum equation)

**this gives the RMHD model**

# Reduced MHD - physical interpretation

In the presence of a large dominant magnetic field, the dynamic can be described by 2D incompressible MHD in the transverse direction and Alfvén waves propagating in the direction of the dominant magnetic field.

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp \pi \quad - \partial_z \mathcal{B}_\perp$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp \quad - \partial_z \boldsymbol{v}_\perp = 0$$

# Reduced MHD scaling

Start from the full compressible MHD system with constant density :

$$\frac{D}{Dt}\boldsymbol{u} + \nabla(p + \boldsymbol{B}^2/2) - (\boldsymbol{B}.\nabla)\boldsymbol{B} = 0 \quad (3.1)$$

$$\frac{D}{Dt}\boldsymbol{B} - (\boldsymbol{B}.\nabla)\boldsymbol{u} + \boldsymbol{B}\nabla.\boldsymbol{u} = 0 \quad (3.2)$$

$$\frac{1}{\gamma p}\frac{D}{Dt}p + \nabla.\boldsymbol{u} = 0 \quad (3.3)$$

# Scaling assumptions

| | | |
|---|---|---|
| Large Dominant Magnetic field | $\boldsymbol{B}_z = B_0(1 + \varepsilon \mathcal{B}_z)$ | (4.2) |
| Large Dominant Magnetic field | $\boldsymbol{B}_\perp = B_0 \varepsilon \mathcal{B}_\perp$ | (4.1) |
| Pressure : Low $\beta$ assumption | $p = \rho_0 v_A^2(\varepsilon^2 q)$ | (4.3) |
| Velocity : small transverse velocities | $\boldsymbol{u}_\perp = \varepsilon v_A \boldsymbol{v}_\perp$ | (4.4) |
| Velocity : very small parallel velocities | $\boldsymbol{u}_z = \varepsilon^2 v_A \boldsymbol{v}_z$ | (4.5) |
| Time : low frequencies | $t = \dfrac{a}{\varepsilon v_A} \tau$ | (4.6) |

# Scaled full MHD equations

$$\frac{\partial}{\partial \tau}\mathcal{B}_z + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z + \mathcal{B}_z \nabla_\perp \cdot \boldsymbol{v}_\perp + \boxed{\frac{1}{\varepsilon}\nabla_\perp \cdot \boldsymbol{v}_\perp} = \mathcal{O}(\varepsilon) \tag{5.1}$$

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp(\mathcal{B}^2/2 + q) - \partial_z \mathcal{B}_\perp + \boxed{\frac{1}{\varepsilon}\nabla_\perp \mathcal{B}_z} = \mathcal{O}(\varepsilon) \tag{5.2}$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \boldsymbol{v}_\perp - \partial_z \boldsymbol{v}_\perp = \mathcal{O}(\varepsilon) \tag{5.3}$$

# Some notations

$$\text{The variables} \quad : \quad \mathscr{V}^\varepsilon = (\mathcal{B}_z^\varepsilon, \mathbf{v}_\perp^\varepsilon, \mathcal{B}_\perp^\varepsilon)^t$$

$$\text{The equations} \quad : \quad \partial_t \mathscr{V}^\varepsilon + \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$\mathbb{H}(\mathscr{V}, \mathscr{V})$ is a non-linear operator (at most quadratic)

$$\mathbb{H}(\mathscr{V}, \mathscr{V}) = \begin{pmatrix} (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z + \mathcal{B}_z \nabla_\perp \cdot \mathbf{v}_\perp \\ (\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v} - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp(\mathcal{B}^2/2 + q) - \partial_z \mathcal{B}_\perp \\ (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \mathbf{v}_\perp - \partial_z \mathbf{v}_\perp \end{pmatrix}$$

$\mathbb{L}\mathscr{V}$ is the constant coefficient linear operator

$$\mathbb{L}\mathscr{V} = \begin{pmatrix} \nabla \cdot \mathbf{v}_\perp \\ \nabla \mathcal{B}_z \\ 0 \end{pmatrix}$$

# The proof strategy

S. Schochet, E. Grenier, P.L.Lions-N.Masmoudi, B. Desjardins...

1. Introduce a filtered variable $\tilde{\mathscr{V}}^\varepsilon = \mathcal{F}\mathscr{V}^\varepsilon$ to remove the oscillations
2. Prove that the filtered variable $\tilde{\mathscr{V}}^\varepsilon \to \tilde{\mathscr{V}}^0$ satisfying some equation $\partial_t \mathscr{V}^0 + \quad \mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$ with $\mathcal{H}$ time-independent.
3. Prove that the original variable $\mathscr{V}^\varepsilon \to \mathcal{F}^{-1}\tilde{\mathscr{V}}^0$
4. Since $\mathcal{F}^{-1}\tilde{\mathscr{V}}^0 \to P\tilde{\mathscr{V}}^0$ where $P$ is the $L^2$ projection on the kernel of $\mathbb{L}$

## Result

$\mathscr{V}^\varepsilon \to \overline{\mathscr{V}} = P\tilde{\mathscr{V}}^0$ and $\overline{\mathscr{V}}$ satisfies :

$$\partial_t \overline{\mathscr{V}} + \quad P\mathcal{H}(\mathscr{V}^0, \mathscr{V}^0) = 0$$

# The wave operator $\mathbb{L}$

$$\mathbb{L}\mathscr{V} = \left( \begin{array}{l} \nabla \cdot \boldsymbol{v}_\perp \\ \nabla \mathcal{B}_z \\ 0 \end{array} \right)$$

- $L^2(\Omega) \times (L^2(\Omega))^2 = \mathrm{Ker}\mathbb{L} \oplus \mathrm{Im}\mathbb{L}$
  $\mathrm{Ker}\mathbb{L} = \{(\mathcal{B}_z, \boldsymbol{v}); \mathcal{B}_z = cte, \nabla_\perp \cdot \boldsymbol{v} = 0\}$
  $\mathrm{Im}\mathbb{L} = \{(\mathcal{B}_z, \boldsymbol{v}); \int \mathcal{B}_z = 0, \exists \Phi \boldsymbol{v} = \nabla_\perp \Phi\}$
- Spectrum of $\mathbb{L}$ on $\mathrm{Im}\mathbb{L}$
  Let $\{\psi_k, k \geq 1\}$ the eigenvectors of the Laplace operator

$$-\Delta \psi_k = \lambda_k^2 \psi_k \qquad \lambda_k > 0$$

then the eigenvectors of $\mathbb{L}$ are :

$$\Phi_k^\pm = \left[ \begin{array}{c} \psi_k \\ \\ \pm \dfrac{\nabla \psi_k}{i\lambda_k} \end{array} \right] \qquad \text{with} \quad \mathbb{L}\Phi_k^\pm = \pm i \lambda_k \Phi_k^\pm$$

# The solution operator $\mathcal{L}$ of the wave equation

Let $\mathcal{L}(t)$ be the semi-group $(\mathcal{L}(t), t \in \boldsymbol{R})$ defined by

$$\mathcal{L}(t) = \exp(-\mathbb{L}t) \tag{6}$$

In other words

$$\mathscr{V}(t,x) = \mathcal{L}(t)\mathscr{V}_0(x) \qquad \text{means that} \quad \frac{\partial \mathscr{V}}{\partial t} + \mathbb{L}\mathscr{V} = 0 \qquad \text{with} \quad \mathscr{V}(t=0,x) = \mathscr{V}_0(x)$$

Using the expression of the spectrum of $\mathbb{L}$ we can have an explicit representation of the solution operator $\mathcal{L}(t)$ : Let $P$ be the $L^2$ projection on $\mathrm{Ker}\mathbb{L}$

## on the velocity component $\mathcal{L}_v(t)\mathscr{V}$

$$\text{if} \quad \mathscr{V} - P\mathscr{V} = \sum_{k,\pm} a_k^{\pm}\Phi_k^{\pm} \quad \text{then} \quad \mathcal{L}_v(t)\mathscr{V} = \pi \boldsymbol{v}_{\perp} + \sum_{k,\pm} \pm a_k^{\pm} e^{\pm i\lambda_k t}\frac{\nabla \psi_k}{i\lambda_k}$$

$a_k^- = (a_k^+)^*$ conjugate (real functions)

# Step 1 : Equation satisfied by the filtered variable $\tilde{\mathscr{V}}^\varepsilon$

$$\partial_t \mathscr{V}^\varepsilon + \mathbb{H}(\mathscr{V}^\varepsilon, \mathscr{V}^\varepsilon) + \frac{1}{\varepsilon} \mathbb{L}\mathscr{V}^\varepsilon = \mathcal{O}(\varepsilon)$$

$$\text{introduce the filtered variable } \tilde{\mathscr{V}}^\varepsilon = \mathcal{L}(-t/\varepsilon)\mathscr{V}^\varepsilon$$

with

$$\mathcal{L}(t) = \exp(-\mathbb{L}t)$$

From the definition of $\mathcal{L}$, we deduce that

$$\begin{aligned}
\frac{\partial \tilde{\mathscr{V}}^\varepsilon}{\partial t} &= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon + \mathcal{L}(-t/\varepsilon)\frac{\partial \mathscr{V}^\varepsilon}{\partial t} \\
&= \frac{\mathbb{L}}{\varepsilon}\tilde{\mathscr{V}}^\varepsilon - \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) - \mathcal{L}(-t/\varepsilon)\frac{\mathbb{L}}{\varepsilon}\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon + \mathcal{O}(\varepsilon) \\
&= -\mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) + \mathcal{O}(\varepsilon)
\end{aligned}$$

since $\mathcal{L}(t/\varepsilon)$ and $\mathbb{L}$ commute.

Initial data : $\tilde{\mathscr{V}}^\varepsilon(t=0) = \mathscr{V}^\varepsilon(t=0)$ since $\mathcal{L}(0)$ is the identity

# Limit Equation

## Step 2 : Limit Equation for the filtered variable $\tilde{\mathscr{V}}^\varepsilon$

$$\frac{\partial \tilde{\mathscr{V}}^\varepsilon}{\partial t} + \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^\varepsilon) = \mathcal{O}(\varepsilon)$$

$\tilde{\mathscr{V}}^0 = \lim_{\varepsilon\to 0} \tilde{\mathscr{V}}^\varepsilon$

$$\frac{\partial \tilde{\mathscr{V}}^0}{\partial t} + \mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = 0$$

where $\mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = \lim_{\varepsilon\to 0} \mathcal{L}(-t/\varepsilon)\mathbb{H}(\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0, \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0)$ is a time-independent operator whose expression can be computed explicitly (see next slides)

## Step 3 : Go back to the unfiltered variable $\mathscr{V}^\varepsilon$

$$\mathscr{V}^\varepsilon \to \mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0$$

# Limit for the original variable

But we have

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 \rightarrow P\tilde{\mathscr{V}}^0$$

since

$$\mathcal{L}(t/\varepsilon)\tilde{\mathscr{V}}^0 = P\tilde{\mathscr{V}}^0 + \sum_{k,\pm} \pm a_k^{\pm} e^{\pm i\lambda_k t/\varepsilon} \Phi_k^{\pm} \rightharpoonup P\tilde{\mathscr{V}}^0$$

Final result : weak limit of $\mathscr{V}^\varepsilon = P\tilde{\mathscr{V}}^0$ that satisfies

$$\frac{\partial P\tilde{\mathscr{V}}^0}{\partial t} + P\mathcal{H}(\tilde{\mathscr{V}}^0, \tilde{\mathscr{V}}^0) = 0$$

# Explicit form of the limit equation for $P\tilde{\mathscr{V}}^0$

example : computation of the quadratic term $(\boldsymbol{v}_\perp \cdot \nabla)\boldsymbol{v}_\perp = (\boldsymbol{v}_\perp)_j \partial_j \boldsymbol{v}_\perp$

$$
\begin{aligned}
&(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} = \\
&\{\sum_k (a_k^+ e^{i\lambda_k t/\varepsilon} - a_k^- e^{-i\lambda_k t/\varepsilon})\frac{\nabla \psi_k}{i\lambda_k}\}_j \partial_j \{\sum_l (a_l^+ e^{i\lambda_l t/\varepsilon} - a_l^- e^{-i\lambda_l t/\varepsilon})\frac{\nabla \psi_l}{i\lambda_l}\} = \\
&\sum_{k,l}[-a_k^+ a_l^+ e^{i(\lambda_k+\lambda_l)t/\varepsilon} - a_k^- a_l^- e^{-i(\lambda_k+\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k \lambda_l}(\nabla\psi_k)_j \partial_j (\nabla \psi_l) \\
&+ \sum_{k,l}[a_k^- a_l^+ e^{i(\lambda_l-\lambda_k)t/\varepsilon} + a_k^+ a_l^- e^{i(\lambda_k-\lambda_l)t/\varepsilon}]\frac{1}{\lambda_k \lambda_l}(\nabla\psi_k)_j \partial_j (\nabla \psi_l)
\end{aligned}
$$

$\lim_{\varepsilon \to 0}$ (distribution) of all the terms is 0 except when $k = l$ and we get :

$$
(\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \cdot \nabla)\mathcal{L}_v(t/\varepsilon)Q\mathscr{V} \to \sum_k [a_k^- a_k^+ + a_k^+ a_k^-]\frac{1}{\lambda_k^2}(\nabla\psi_k)_j \partial_j (\nabla\psi_k) = \sum_k \frac{|a_k^+|^2}{\lambda_k^2}\nabla(|\nabla\psi_k|^2/2)
$$

On the average (weak limit) fast k-waves interact with l-waves only if $k = l$ and the result is a gradient

no interaction between fast waves and slow dynamics

# Summary

Weak limit of the solutions of full MHD system :

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp(\mathcal{B}^2/2 + q) - \partial_z \mathcal{B}_\perp + \frac{1}{\varepsilon}\nabla_\perp \mathcal{B}_z = \mathcal{O}(\varepsilon)$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \boldsymbol{v}_\perp - \partial_z \boldsymbol{v}_\perp = \mathcal{O}(\varepsilon)$$

are the solutions of the "incompressible" reduced MHD system

$$\frac{\partial}{\partial \tau}\boldsymbol{v}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp + \nabla_\perp \pi - \partial_z \mathcal{B}_\perp = 0$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\boldsymbol{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\boldsymbol{v}_\perp - \partial_z \boldsymbol{v}_\perp = 0$$

$$\nabla_\perp \cdot \boldsymbol{v}_\perp = \nabla_\perp \cdot \mathcal{B}_\perp = 0$$

no interaction between fast waves and slow dynamics

# Perspectives

- Direct discretization of the "almost 2D incompressible" RMHD instead of the RMHD expressed in term of $\phi$ and $\psi$
- same proof for high $\beta$ ordering $\beta \sim \varepsilon$
- get rid of slab approximation