Asymptotic theory of Reduced MHD models for fusion plasmas.

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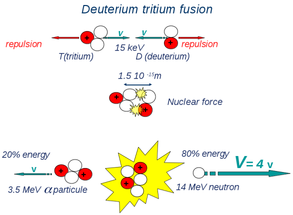
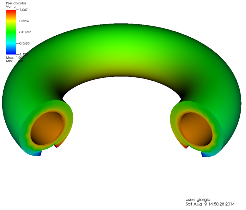
Physical Context

Goal : controlled nuclear fusion
"Lawson" criterion : $n\tau_E T > 5.10^{21} m^{-3}\ s\ keV$





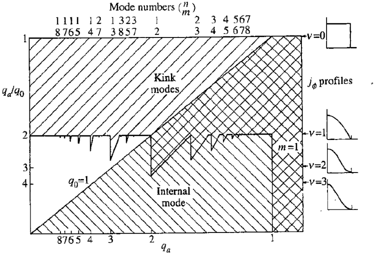
Tokamaks : Toroidal chamber where a very hot plasma ($150M\ ^oK$) is confined thanks to very large magnetic field (200 K x earth magnetic field)

Fusion plasma and instabilities

Very large number of possible instabilities → Numerical simulations

- to help identify possible instabilities
- to determine the stability domain constraining the operational range of the design parameters

stability domain of the safety factor $q = \frac{aB_T}{R_0 B_P}$



stability studies use the MHD model

THE (ideal) MHD MODEL

Hydrodynamics :

$$\frac{\partial}{\partial t}\rho + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\frac{\partial}{\partial t}\rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p = F_L$$

$$\frac{\partial}{\partial t}p + \mathbf{u} \cdot \nabla p + \gamma p \nabla \cdot \mathbf{u} = 0$$

+ Maxwell (pre-maxwell) equations :

$$\frac{\partial}{\partial t}\mathbf{B} + \nabla \times \mathbf{E} = 0$$

$$\cancel{\frac{1}{c^2}\frac{\partial}{\partial t}\mathbf{E}} + \nabla \times \mathbf{B} = \mathbf{J}$$

systems coupled by Ohm's law $E + \mathbf{u} \times \mathbf{B} = 0$
and the def of the Lorentz force $F_L = \mathbf{J} \times \mathbf{B}$

THE MHD MODEL

First-order Hyperbolic system intensively studied from a mathematical and numerical view point

1. Nice properties :
   - existence of a conservative form, existence of an entropy
   - symmetry form
   - hyperbolic
   - eigensystem with explicit analytic expression
2. Not so nice :
   - not strictly hyperbolic
   - some fields are neither gnl nor ld
   - existence of the involution $\nabla \cdot \mathbf{B} = 0$

MHD waves

Hyperbolic system with 3 different types of waves (+ entropy waves) If **n** is the direction of propagation of the wave

- Fast Magnetosonic waves : $\lambda_F = \mathbf{u.n} \pm C_F$
  $$C_F^2 = \frac{1}{2}(V_t^2 + v_A^2 + \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$$
- Alfvén waves : $\lambda_F = \mathbf{u.n} \pm C_A$ $C_A^2 = (\mathbf{B.n})^2/\rho$
- Slow Magnetosonic waves : $\lambda_S = \mathbf{u.n} \pm C_S$
  $$C_S^2 = \frac{1}{2}(V_t^2 + v_A^2 - \sqrt{(V_t^2 + v_A^2)^2 - 4V_t^2 C_A^2})$$

$v_A^2 = |\mathbf{B}|^2/\rho$ v_A : Alfvén speed
$V_t^2 = \gamma p/\rho$ V_t : acoustic speed

MHD waves are polarized

propagation speed depends on the direction w r to the magnetic field.
In particular if $\mathbf{n} \cdot \mathbf{B} = 0$:

- Alfvén waves : $\lambda_F = 0$
- Slow Magnetosonic waves : $\lambda_S = 0$
- Fast Magnetosonic waves : $\lambda_F = \pm C_F$ with $C_F^2 = V_t^2 + v_A^2$

only the Fast Magnetosonic waves survive !

MHD models in Tokamak simulation

8 variable $(\rho, \mathbf{u}, \mathbf{B}, p)$ MHD model is not used intensively for Tokamak simulations

Large majority of Tokamak simulations are done with
Reduced MHD models

Example (Strauss Model 76) :

$$\frac{\partial U}{\partial t} + [\varphi, U] - [\psi, J] - \frac{\partial}{\partial z} J = 0$$

$$\frac{\partial \psi}{\partial t} + [\varphi, \psi] - \frac{\partial \varphi}{\partial z} = 0$$

$$U = \nabla_\perp^2 \varphi \qquad J = \nabla_\perp^2 \psi$$

$[f, g] = \mathbf{z}.\nabla_\perp f \times \nabla_\perp g$

Reduced MHD models

- Models introduced in the 70 (Strauss, Physics of Fluids, 19, p 134, 1976)
- Understand the derivation of the reduced models
- Understand their properties
- What are the waves that are filtered out by these models ?
- Is it possible to obtain rigorous convergence results ?

Singular limit of hyperbolic PDEs

Let $\mathbf{W} \in \boldsymbol{R}^N$ solution of the hyperbolic system with a large operator

$$\begin{cases} A_0(\mathbf{W},\varepsilon)\partial_t \mathbf{W} + \sum_j A_j(\mathbf{W},\varepsilon)\partial_{x_j}\mathbf{W} + \dfrac{1}{\varepsilon}\sum_j C_j \partial_{x_j}\mathbf{W} = 0 \\ \mathbf{W}(0,\mathbf{x},\varepsilon) = \mathbf{W}_0(\mathbf{x},\varepsilon) \end{cases}$$

What is the behavior of the solutions when $\varepsilon \longrightarrow 0$?

Evolution equation : depends on the initial conditions ! $\Rightarrow 2 \neq$ cases :

slow case : $\sum_j C_j \partial_{x_j}\mathbf{W}(t=0) \sim \mathcal{O}(\varepsilon)$
$\mathbf{W}(t=0)$ close to the kernel of the large operator s.t $\sum_j C_j \partial_{x_j}\mathbf{W}/\varepsilon$ stays bounded.

fast case : $\sum_j C_j \partial_{x_j}\mathbf{W}(t=0) \sim \mathcal{O}(1)$

Singular limit of hyperbolic PDEs

Explicit linear example I

Consider the linear system

$$\frac{\partial r}{\partial t} \quad + \mathbf{a}.\nabla r \quad + \frac{1}{\varepsilon} div \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} \quad + \mathbf{a}.\nabla \mathbf{u} \quad + \frac{1}{\varepsilon} \nabla r = 0$$

Compact form :

$$\partial_t \mathbf{v} + \quad \mathbb{H}\mathbf{v} + \frac{1}{\varepsilon} \quad \mathbb{L}\mathbf{v} = 0$$

$\mathbb{H}\mathbf{v} = \mathbf{a}.\nabla \mathbf{v}$ is a constant convection operator

$$\mathbb{L} = \begin{pmatrix} 0 & \nabla\cdot \\ \nabla & 0 \end{pmatrix} \tag{3}$$

Singular limit of hyperbolic PDEs

Explicit linear example II

In Fourier space

$$\frac{\partial \hat{\mathbf{v}}(\mathbf{k})}{\partial t} + i[\hat{\mathbb{H}}(\mathbf{k}) + \frac{1}{\varepsilon}\hat{\mathbb{L}}\ (\mathbf{k})]\hat{\mathbf{v}}(\mathbf{k}) = 0 \quad \text{for} \quad \mathbf{k} \in Z^2 \tag{4}$$

where the matrix $\hat{\mathbb{H}}(\mathbf{k}) + 1/\varepsilon\hat{\mathbb{L}}(\mathbf{k})$ is equal to :

$$\begin{pmatrix} \mathbf{a}.\mathbf{k} & k_1/\varepsilon & k_2/\varepsilon \\ k_1/\varepsilon & \mathbf{a}.\mathbf{k} & 0 \\ k_2/\varepsilon & 0 & \mathbf{a}.\mathbf{k} \end{pmatrix} \tag{5}$$

This matrix is diagonalizable, its eigenvectors are :

$$s_1(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -k_1/\mid \mathbf{k} \mid \\ -k_2/\mid \mathbf{k} \mid \end{pmatrix}, \quad s_2(\mathbf{k}) = \frac{1}{\mid \mathbf{k} \mid} \begin{pmatrix} 0 \\ -k_2 \\ k_1 \end{pmatrix}$$
$$, \quad s_3(\mathbf{k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ k_1/\mid \mathbf{k} \mid \\ k_2/\mid \mathbf{k} \mid \end{pmatrix} \tag{6}$$

with associated eigenvalues $\lambda_1 = \mathbf{a}.\mathbf{k} - \frac{\mid \mathbf{k} \mid}{\varepsilon}$, $\lambda_2 = \mathbf{a}.\mathbf{k}$ and $\lambda_3 = \mathbf{a}.\mathbf{k} + \frac{\mid \mathbf{k} \mid}{\varepsilon}$.

Note : $\hat{\mathbb{L}} s_2(\mathbf{k}) = 0$;in physical space $s_2(\mathbf{k})$ corresponds to constant scalar fields ($\nabla r = 0$) and div free vectors ($\nabla \,.\, \mathbf{u} = 0$)

Singular limit of hyperbolic PDEs

Explicit linear example III

$$\hat{\mathbf{v}}(\mathbf{k},t) =$$

$$\begin{cases} \frac{1}{\sqrt{2}}(\hat{r}(\mathbf{k},0) - \frac{k_1}{|\mathbf{k}|}\hat{u}(\mathbf{k},0) - \frac{k_2}{|\mathbf{k}|}\hat{v}(\mathbf{k},0))e^{-i(\mathbf{a}.\mathbf{k}-|\mathbf{k}|/\varepsilon)t}s_1(\mathbf{k}) \\ +\frac{1}{|\mathbf{k}|}(-k_2\hat{u}(\mathbf{k},0) + k_1\hat{v}(\mathbf{k},0))e^{-i\mathbf{a}.\mathbf{k}t}s_2(\mathbf{k}) \\ +\frac{1}{\sqrt{2}}(\hat{r}(\mathbf{k},0) + \frac{k_1}{|\mathbf{k}|}\hat{u}(\mathbf{k},0) + \frac{k_2}{|\mathbf{k}|}\hat{v}(\mathbf{k},0))e^{-i(\mathbf{a}.\mathbf{k}+|\mathbf{k}|/\varepsilon)t}s_3(\mathbf{k}) \end{cases}$$

Singular limit of hyperbolic PDEs

Explicit linear example IV

Slow component belonging to the kernel of $\mathbb{L}$

$$\hat{\mathbf{v}}_s(\mathbf{k},\tau) = \frac{1}{\mid \mathbf{k} \mid}(-k_2\hat{u}(\mathbf{k},0) + k_1\hat{v}(\mathbf{k},0))e^{-i\mathbf{a}.\mathbf{k}t}s_2(\mathbf{k})$$

that satisfies the limit system

$$\begin{cases} \dfrac{\partial \mathbf{v}_s}{\partial t} + \mathbb{H}\mathbf{v}_s = 0 \\ \mathbb{L}\mathbf{v}_s = 0 \end{cases}$$

Singular limit of hyperbolic PDEs

Explicit linear example V

Fast oscillatory component $\hat{\mathbf{v}}_f(\mathbf{k}, t, t/\varepsilon)$

$$\frac{1}{\sqrt{2}}\left\{\begin{array}{l} (\hat{r}(\mathbf{k},0) - \frac{k_1}{\mid \mathbf{k} \mid}\hat{u}(\mathbf{k},0) - \frac{k_2}{\mid \mathbf{k} \mid}\hat{v}(\mathbf{k},0))e^{-i\mathbf{a}.\mathbf{k}t} s_1(\mathbf{k})\, e^{i|\mathbf{k}|t/\varepsilon} + \\ (\hat{r}(\mathbf{k},0) - \frac{k_1}{\mid \mathbf{k} \mid}\hat{u}(\mathbf{k},0) - \frac{k_2}{\mid \mathbf{k} \mid}\hat{v}(\mathbf{k},0))e^{-i\mathbf{a}.\mathbf{k}t} s_3(\mathbf{k})\, e^{-i|\mathbf{k}|t/\varepsilon} \end{array}\right.$$

that solves :

$$\left\{\begin{array}{l} \frac{\partial \mathbf{v}_f}{\partial \tau} + \mathbb{L}\mathbf{v}_s = 0 \\ \mathbf{v}_f(t,0) = \mathbf{W}(t) \end{array}\right.$$

Explicit linear example VI

Summary of the linear example

- $\mathbf{v}_\varepsilon(t,\mathbf{x}) = \mathbf{v}^s(t,\mathbf{x}) + \mathbf{v}^f(t/\varepsilon, t, \mathbf{x})$
- slow component : $\mathbf{v}^s(t,\mathbf{x})$ satisfies the limit system

$$\begin{cases} \dfrac{\partial \mathbf{v}_s}{\partial t} + \mathbb{H}\mathbf{v}_s = 0 \\ \mathbb{L}\mathbf{v}_s = 0 \end{cases}$$

- fast component : $\mathbf{v}^f(t/\varepsilon, \tau, \mathbf{x})$ satisfying

$$\begin{cases} \dfrac{\partial \mathbf{v}_f}{\partial \tau} + \mathbb{L}\mathbf{v}_s = 0 \\ \mathbf{v}_f(t,0) = \mathbf{W}(t) \end{cases}$$

Explicit linear example VII

Slow limit (Well-Prepared initial data)

- If the initial data $\in \mathrm{Ker}\mathbb{L}$, then the fast component does not exist
- If the initial data $\mathbf{v}(t=0) - P_{\mathrm{Ker}\mathbb{L}}\mathbf{v}(t=0) < \mathcal{O}(\varepsilon)$, then the fast component remains $< \mathcal{O}(\varepsilon)$

Slow Singular limit in the non-linear case :

Assume the initial data is close to the kernel of the large operator $\mathbb{L}$:

$$W_0(\mathbf{x}, \varepsilon) = W_0^0(\mathbf{x}) + \varepsilon W_0^1(\mathbf{x}, \varepsilon) \qquad \sum_j C_j \partial_j W_0^0 = 0$$

- Solutions exist on some time independent of ε ?
- Solutions converge to the solution of some limit system ?

Not always : counter-example (Schochet 1988) :
$a(u)\partial_t u + a(u)\partial_x u + \frac{1}{\varepsilon}\partial_y u = 0$
with $u_0(x, y, \varepsilon) = u_0^0(x) + \varepsilon u_0^1(y)$.

Slow limit

If

1. Initial data : $W_0(\mathbf{x},\varepsilon) = W_0^0(\mathbf{x}) + \varepsilon W_0^1(\mathbf{x},\varepsilon)$ where $\sum_j C_j \partial_j W_0^0 = 0$
2. Structure of the system
    1. A_0, A_j and C_j are symmetric and C^s continuous
    2. A_0 is positive definite and $A_0 = A_0(\varepsilon \mathbf{W})$
    3. The C_j are constant matrices

Then(Klainerman-Majda 1981-1982)
$W(t,\mathbf{x},\varepsilon)$ exist for a time T independent of ε and converge to the solution of the limit system :

1. $A_0(0)\partial_t \mathbf{W^0} + \sum_j A_j(\mathbf{W}^0, 0)\partial_{x_j}\mathbf{W^0} + \sum_j C_j \partial_{x_j}\mathbf{W^1} = 0$
2. $\sum_j C_j \partial_{x_j}\mathbf{W^0} = 0$
3. $\mathbf{W^0}(0,\mathbf{x}) = W_0^0(\mathbf{x})$

Slow limit : comments on the proof

- The C_j are constant matrices existence of solution : Iterative scheme (Lax) where each iterate is linear symmetric hyperbolic system where the energy identity of Friedrichs holds :

$$\partial_t E = (\nabla \cdot \vec{A} \;\; \mathbf{W}, \mathbf{W})$$

with $\nabla \cdot \vec{A} = \partial_t A_0 + \sum_j \partial_j A_j + \frac{1}{\varepsilon} \cancel{\partial_j C_j}$

- $A_0 = A_0(\varepsilon \mathbf{W})$

$$\begin{aligned} \partial_t A_0(\varepsilon \mathbf{W}) &= \frac{DA_0}{D\mathbf{W}} \, \varepsilon \partial_t \mathbf{W} \\ &= -\frac{DA_0}{D\mathbf{W}} \, \varepsilon A_0^{-1} [A_j \partial_j \mathbf{W} + \frac{1}{\varepsilon} \partial_j C_j \mathbf{W}] \end{aligned}$$

Slow limit : From the limit model to reduced model I.

$$A_0(0)\partial_t \mathbf{W^0} + A_j(\mathbf{W}^0, 0)\partial_{x_j}\mathbf{W^0} + C_j\partial_{x_j}\mathbf{W^1} = 0 \quad \text{and} \quad C_j\partial_{x_j}\mathbf{W^0} = 0$$

Assume that $\exists \mathcal{M}$ a parametrization of the kernel of $\mathbb{L} = \{C_j\partial_{x_j}\cdot\}$

$$\omega \in \boldsymbol{R}^n \rightarrow \quad \mathbf{W} = \mathcal{M}\omega \quad s.t \quad \mathbb{L}\mathcal{M}\omega = 0$$

Adjoint operator $\mathcal{M}^*$ $\boldsymbol{R}^N \rightarrow \boldsymbol{R}^n$ $(\mathcal{M}(\omega), \mathbf{W}) = (\omega, \mathcal{M}^*\mathbf{W})$

then $\mathcal{M}^*\mathbb{L} = 0$ (Since $\mathbb{L}$ is skew symmetric)

Limit model can be written as a reduced model for $\omega \in \boldsymbol{R}^n$

$$\left\{ \mathcal{M}^* A_0(0)\mathcal{M}\ \partial_t\omega + \mathcal{M}^* A_j(\mathcal{M}(\omega), 0)\mathcal{M}\ \partial_{x_j}\omega = 0 \right.$$

Slow limit : From the limit model to reduced model II.

In practice

$\mathbb{L} = \{C_j \partial_{x_j} \cdot\}$ is linear operator with constant coefficients

$$\mathbf{W} = \mathcal{M}\boldsymbol{\omega} = (\sum_{j=1}^{d} P_j \partial_{x_j} + P_0)\boldsymbol{\omega}$$

$\{P_j; j = 0, d\}$: $N \times n$ constant matrices.

$$\boldsymbol{\omega} = \mathcal{M}^*(\mathbf{W}) = -\sum_{j=1}^{d} P_j^t \partial_{x_j} \mathbf{W} + P_0^t \mathbf{W}$$

where $P_j^t; j = 0, \cdots, d$ are rectangular $n \times N$ matrices, transposes of the P_j.

Note that the reduced system is a third order differential equation !

$$\{ \ \mathcal{M}^* A_0(0)\mathcal{M} \ \partial_t \boldsymbol{\omega} + \sum_j \mathcal{M}^* A_j(\mathcal{M}(\boldsymbol{\omega}), 0)\mathcal{M} \ \partial_{x_j} \boldsymbol{\omega} = 0$$

Application to reduced MHD

Goal : Cast the MHD system in the previous general framework

$$\rho(p)\frac{D}{Dt}\mathbf{u} + \nabla(p + \mathbf{B}^2/2) - (\mathbf{B}.\nabla)\mathbf{B} = 0 \quad (7.1)$$

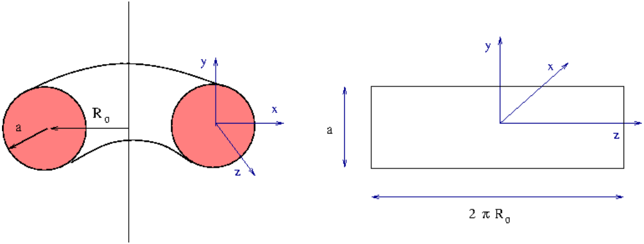
$$\frac{D}{Dt}\mathbf{B} - (\mathbf{B}.\nabla)\mathbf{u} + \mathbf{B}\nabla.\mathbf{u} = 0 \quad (7.2)$$

$$\frac{1}{\gamma p}\frac{D}{Dt}p + \nabla.\mathbf{u} = 0 \quad (7.3)$$

1. Identify the large operator
2. Make sure the assumptions on the structure of the system are verified
3. Apply the general result

Large aspect ratio theory : geometrical setting

1. Scaling of the space variables



two scale analysis $x = x/a, y = y/a$ but $z = z/R_0$ and we assume $\varepsilon = a/R_0$ is small

Definitions : poloidal (x, y) plan ; toroidal $z-$ direction

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_z \mathbf{z} \qquad \mathbf{v}_\perp = \mathbf{v}_x \mathbf{x} + \mathbf{v}_y \mathbf{y}$$

$$\nabla_\perp f = \frac{\partial f}{\partial x}\mathbf{x} + \frac{\partial f}{\partial y}\mathbf{y} \qquad \nabla_\perp \bullet \mathbf{v}_\perp = \frac{\partial \mathbf{v}_x}{\partial x} + \frac{\partial \mathbf{v}_y}{\partial y}$$

The large aspect ratio theory

2. Field scaling

Magnetic field : $$\mathbf{B} = \frac{F}{R}\mathbf{z} + \mathbf{B}_P = B_0(\mathbf{z} + \varepsilon\mathcal{B}) \qquad (8.1)$$

Pressure : $$p = P_0(\bar{p} + \varepsilon q) \qquad (8.2)$$

Velocity : $$\mathbf{u} = \varepsilon v_A \mathbf{v} \qquad (8.3)$$

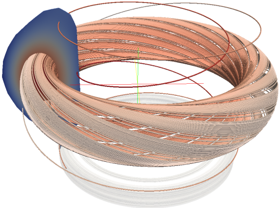
The large aspect ratio theory

3. Time scaling

Alfvén speed : $v_A^2 = B_0^2/\rho_0$

The relevant time scale in MHD is a/v_A

Not in Tokamaks
$\rightarrow$ The perturbations propagate along the field lines !



The relevant time scale in Tokamak is $R_0/v_A = a/(\varepsilon v_A)$

Scaled MHD equations

$$\rho(\bar{p} + \varepsilon q)\,[\frac{\partial}{\partial \tau}\mathbf{v} + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v}] + \partial_z(q + \mathcal{B}_z)\mathbf{z} + \nabla_\perp \mathcal{B}^2/2 - \partial_z \mathcal{B} - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B}$$
$$+\varepsilon(\rho v_z \partial_z \mathbf{v} + \partial_z(\mathcal{B}^2/2)\mathbf{z} - \mathcal{B}_z \partial_z \mathcal{B}) + \frac{1}{\varepsilon}\nabla_\perp(q + \mathcal{B}_z) = 0 \tag{9.2}$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_\perp + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \mathbf{v}_\perp - \partial_z \mathbf{v}_\perp$$
$$+\varepsilon(\mathbf{v}_z \partial_z \mathcal{B}_\perp - \mathcal{B}_z \partial_z \mathbf{v}_\perp + \partial_z \mathbf{v}_z \mathcal{B}_\perp) = 0 \tag{9.3}$$

$$\frac{\partial}{\partial \tau}\mathcal{B}_z + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathbf{v}_z + \mathcal{B}_z \nabla_\perp \cdot \mathbf{v}_\perp + \varepsilon \mathbf{v}_z \partial_z \mathcal{B}_z + \frac{1}{\varepsilon}\nabla_\perp \cdot \mathbf{v}_\perp = 0 \tag{9.4}$$

$$\frac{1}{\gamma(\bar{p} + \varepsilon q)}\,[\frac{\partial}{\partial \tau}q + (\mathbf{v}_\perp . \nabla_\perp)q + \varepsilon \mathbf{v}_z \partial_z q] + \partial_z \mathbf{v}_z + \frac{1}{\varepsilon}\nabla_\perp \cdot \mathbf{v}_\perp = 0 \tag{9.5}$$

Apply the results of the general theory :

The solution converge to the solution of the limit system :

$$\rho(\bar{p})[\frac{\partial}{\partial\tau}\mathbf{v} + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathbf{v}] + \partial_z(q + \mathcal{B}_z)\mathbf{z} + \nabla_\perp \mathcal{B}^2/2 - \partial_z \mathcal{B} - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathcal{B} + \boxed{\nabla_\perp(q^1 + \mathcal{B}_z^1)} = 0 \quad (10.2)$$

$$\frac{\partial}{\partial\tau}\mathcal{B}_\perp + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_\perp - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathbf{v}_\perp + \mathcal{B}_\perp \nabla_\perp \cdot \mathbf{v}_\perp - \partial_z \mathbf{v}_\perp = 0 \quad (10.3)$$

$$\frac{\partial}{\partial\tau}\mathcal{B}_z + (\mathbf{v}_\perp \cdot \nabla_\perp)\mathcal{B}_z - (\mathcal{B}_\perp \cdot \nabla_\perp)\mathbf{v}_z + \mathcal{B}_z \nabla_\perp \cdot \mathbf{v}_\perp + \boxed{\nabla_\perp \cdot \mathbf{v}_\perp^1} = 0 \quad (10.4)$$

$$\frac{1}{\gamma\bar{p}}[\frac{\partial}{\partial\tau}q + (\mathbf{v}_\perp . \nabla_\perp)q] + \partial_z \mathbf{v}_z + \boxed{\nabla_\perp \cdot \mathbf{v}_\perp^1} = 0 \quad (10.5)$$

with

$$\boxed{\nabla_\perp(q + \mathcal{B}_z) = 0}$$

$$\boxed{\nabla_\perp \cdot \mathbf{v}_\perp = 0}$$

Analysis of the limit system I

- $q + \mathcal{B}_z = f(z)$
- Combine toroidal Farady law and pressure equation to eliminate the corrective term, keep the pressure equation.
- In the perp momentum equation, combine the $\nabla_\perp \mathcal{B}^2/2$ and $\nabla_\perp(q^1 + \mathcal{B}_z^1)$ into a scalar “pressure” term $\nabla_\perp \lambda$

Analysis of the limit system II

Incompressible sub-system for the perpendicular dynamics :

$$\rho \frac{D^{\perp}}{Dt}\mathbf{v}_{\perp} - \nabla_{//}\mathcal{B}_{\perp} + \nabla_{\perp}\lambda = 0 \quad (11.1)$$

$$\frac{D^{\perp}}{Dt}\mathcal{B}_{\perp} - \nabla_{//}\mathbf{v}_{\perp} = 0 \quad (11.2)$$

$$\nabla_{\perp} \cdot \mathbf{v}_{\perp} = 0$$

Compressible 1D sub-system for the parallel dynamics :

$$\rho \frac{D^{\perp}}{Dt}\mathbf{v}_z + \nabla_{//} q = 0 \quad (11.3)$$

$$\left(\frac{1}{\gamma \bar{p}} + 1\right)\frac{D^{\perp}}{Dt} q + \nabla_{//}\mathbf{v}_z = 0 \quad (11.4)$$

$$\frac{D^{\perp}}{Dt}\cdot = \frac{\partial}{\partial \tau}\cdot + (\mathbf{v}_{\perp} \cdot \nabla_{\perp})\cdot \qquad \nabla_{//}\cdot = (\mathcal{B}_{\perp} \cdot \nabla_{\perp})\cdot + \partial_z \cdot$$

Analysis of the limit system III

- The incompressible subsystem does not depend upon the compressible one
- The compressible subsystem is a "slave" of the incompressible one
- Only the incompressible sub-system need to be solved
- $\nabla_\perp \cdot \mathcal{B}_\perp = 0$ is an involution.

$\Rightarrow$ concentrate on the incompressible sub-system

Reduced form of the limit model

$\mathcal{M}$ parametrization of the kernel of the large operator

$$K = \{(\mathbf{v}_\perp, \mathcal{B}_\perp); \nabla_\perp \cdot \mathbf{v}_\perp = \nabla_\perp \cdot \mathcal{B}_\perp = 0\}$$

Introduce 2 scalar functions ϕ, ψ such that

$$\mathbf{v}_\perp = \mathbf{z} \times \nabla\phi \qquad (12.1)$$
$$\mathcal{B}_\perp = \mathbf{z} \times \nabla\psi \qquad (12.2)$$

For any scalar function $F \in H^1$ if

$$\mathcal{M}(F) = \mathbf{z} \times \nabla F$$

The adjoint operator $\mathcal{M}^*$ is defined by :

$$\mathcal{M}^*(\mathbf{W}) = \mathbf{z} \cdot \nabla \times \mathbf{W}$$

(proof : Green formula)

Reduced model

$$\mathcal{M}^* A_0(0)\mathcal{M}\ \partial_t \omega + \sum_j \mathcal{M}^* A_j(\mathcal{M}(\omega),0)\mathcal{M}\ \partial_{x_j}\omega = 0$$

After some algebra :

$$\left[\begin{array}{c} \rho\,[\frac{\partial}{\partial \tau}\nabla_\perp^2\varphi + \mathbf{v}_\perp\cdot\nabla_\perp(\nabla_\perp^2\varphi)] - \mathcal{B}_\perp\cdot\nabla_\perp\nabla_\perp^2\psi - \frac{\partial}{\partial z}\nabla_\perp^2\psi = 0 \\ \\ \frac{\partial}{\partial \tau}\psi + \mathbf{v}_\perp\cdot\nabla_\perp\psi - \frac{\partial}{\partial z}\varphi = 0 \end{array}\right]$$

can be written (Strauss Model 76) :

$$\frac{\partial U}{\partial t} + [\varphi, U] - [\psi, J] - \frac{\partial}{\partial z} J = 0$$

$$\frac{\partial \psi}{\partial t} + [\varphi, \psi] - \frac{\partial \varphi}{\partial z} = 0$$

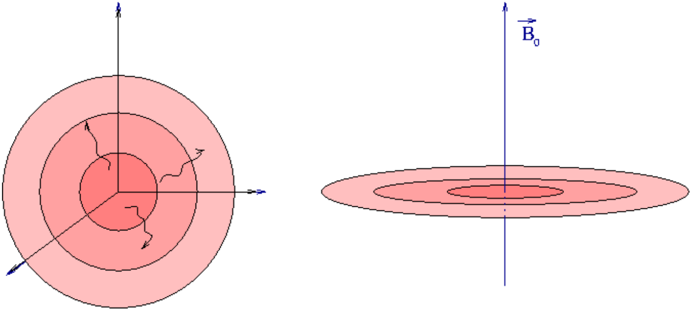
$$U = \nabla_\perp^2\varphi \qquad J = \nabla_\perp^2\psi \qquad [f,g] = \mathbf{z}.\nabla_\perp f \times \nabla_\perp g$$

Reduced model : waves filtering

solutions of the reduced model verify : $\mathbb{L}\mathbf{W} = 0$
On the fast (Alfvén) time scale a/v_A, the full model reduces to

$$\partial_t \mathbf{W} + \mathbb{L}\mathbf{W} = 0$$

whose solution are fast transverse magnetosonic waves traveling in the direction orthogonal to the magnetic field :



- Conclusions
  - Rigorous derivation of reduced MHD model
  - Identification of the waves filtered out by the reduced models
- Perspectives :
  - Study the fast limit
  - More complex models including curvature terms
    - Formulate the models in cylindrical coordinate
    - High order corrections : convergence of the asymptotic expansion of the solution
  - Relax the barotropic assumption
  - Relax the assumption of small aspect ratio a/R_0 and develop a theory based on the small parameter ρ^*