Back and forth from fluxes to residuals to construct conservative and steady state preserving multidimensional schemes

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Acknowledgements

Credit to them for the good stuff
blame me for the rest

Students and collaborators :

- Luca Arpaia, ISMAR Venice, Italy
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- Wassilij Barzukow, CNRS, Institut de Mathématiques de Bordeaux
- Davide Torlo, MathLab at SISSA, Italy
- Lorenzo Micalizzi, University of Zurich
- Rémi Abgrall, University of Zurich

Setting: balance laws

We want to solve numerically (hyperbolic) systems of balance laws

$$\partial_t U + \nabla \cdot F(U) = S(U; \varphi(x))$$

Typical examples

- Shallow water equations with topography/friction/Coriolis/etc
- Euler equations with gravity
- GPR model and hyperbolic reformulations of viscous/dispersive systems
- etc.

We consider both 1D and Multi-D problems

Setting: balance laws

We want to solve numerically (hyperbolic) systems of balance laws

$$\partial_t U + \nabla \cdot F(U) = S(U; \varphi(x))$$

Shopping list

- Arbitrary high order
- Possibly monotonicity/positivity preserving
- Steady state preserving

1. In one dimension we seek to enhance the resolution of

$$\partial_x F(U) = S(U; \varphi(x))$$

2. In multiD we seek to embed the contraint

$$\nabla \cdot F(U) = S(U; \varphi(x))$$

Not the same game...

Bibliography (incomplete) for 1D

1. Preserving $V(U, \varphi(x)) = V_0$ via WB differencing or generalized polynomial approximations
   - Roe Lect.Not.Math. 1986, Bermúdez,& Vázquez-Cendón CAF 1994, Greenberg & LeRoux SINUM 1996; Audusse et all SISC 2004, Parés & Castro M2AN 2004, Parés SINUM 2006, Castro et al M3AS 2007, Hernádez-Duenas & Karni JSC 2011, Ricchiuto JSC 2011, Xing JCP 2014, Cheng & Kurganov Comm.Math.Sci. 2016 etc.etc.

2. Reconstruction/evolution of fluctuations wrt a given equilibrium $U^*(x)$
   - Castro et al SINUM 2008, Gaburro et al MNRAS 2018, Klingenberg et al SISC 2019, Berberich et al CAF 2021, etc.etc.

3. Reconstruction/evolution of fluctuations wrt discrete equilibria (**approximate full well balanced**):
   - Castro & Parés J.Sci.Comp. 2020, Gómez-Bueno et al, Appl.Math.Comp. 2021, Gómez-Bueno et al Mathematics 2021, Guerrero Fernández et al Mathematics 2021, Gómez-Bueno et al, Appl.Num.Math. 2023, etc. etc.

4. Fully well balanced Riemann solver with 0-wave to enforce integral steady balance
   - Berthon & Chalons Math.Comp. 2016, Michel-Dansac et al JCP 2017, Castro et al SINUM 2018, Bulteau et al Calcolo 2021, etc.etc.

5. Well balanced via integration of the source term and global fluxes
   - Gascón & Corberán JCP 2001, Donat & Martinez-Gavara J.Sci.Comp. 2011, Chertock et al JCP 2018, Cheng et al J.Sci.Comp. 2019, Mantri&Noelle JCP 2021, Parés&Parés-Pulido JCP 2021, Carrillo et al JCP 2023, Ciallella et al. J.Sci.Comp. 2023, etc. etc.

Bibliography (incomplete) for 1D

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   - Roe Lect.Not.Math. 1986, Bermúdez,& Vázquez-Cendón CAF 1994, Greenberg & LeRoux SINUM 1996; Audusse et all SISC 2004, Parés & Castro M2AN 2004, Parés SINUM 2006, Castro et al M3AS 2007, Hernádez-Duenas & Karni JSC 2011, Ricchiuto JSC 2011, Xing JCP 2014, Cheng & Kurganov Comm.Math.Sci. 2016 etc.etc.
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Bibliography (incomplete) for multi-D

1. Compatible representations (staggered, mimetic, ad-hoc FE spaces, etc)
   - Nédélec Num.Math. 1980, Hyman & Shashkov CAMWA 1997, Balsara & Spicer JCP 1999, Hiptmair Acta Numerica 2002, Cockburn et al JCP 2004, Li & Shu J.Sci.Comp. 2005, Arnold et al Acta Numerica 2006, Li & Xu JCP 2012, Lipnikov et al JCP 2014, Balsara JCP 2009, Dumbser et al IJNMF 2018, Balsara et al Comm.Appl.Math.Comp. 2021, Boscheri et al JCP 2021, Chiocchetti & Dumbser J.Sci.Comp 2023 etc. etc.
2. Error cleaning
   - Powell NASA-CR 1994, Powell et al JCP 1999, Munz et al JCP 2000, Dedner et al JCP 2002, Dumbser et al JCP 2020, Chiocchetti et al JCP 2021, Boscheri et al JCP 2023 etc. etc.
3. Non-staggered multiD approximations
   - Torrilhon & Fey SINUM 2004, Torrilhon SISC 2005, Jelsch & Torrilhon BIT NM 2006, Mishra & Tadmor Comm.Comp.Phys. 2010, Mishra & Tadmor SINUM 2011, Balsara & Dumbser JCP 2015, Barsukow Math.Comp 2019, Barsukow et al J.Sci.Comp. 2019, Barsukow et al SISC 2021, etc. etc.

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TOC

The talk is split in three main chapters

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A motivational example to fix ideas

Shallow water eq.s

$$\partial_t \begin{pmatrix} h \\ hu \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \end{pmatrix} = -h \begin{pmatrix} 0 \\ gb'(x) + c_f u \end{pmatrix}$$

Incomplete taxonomy of steady states

1. Lake at rest
2. One dimensional frictionless flows with constant energy
3. One dimensional flows with friction
4. etc. etc.

Conservation, steady state preservation, and residuals

$$\partial_t U + \partial_x F(U) = \gamma \partial_x b(x)$$

Simple example: If γ is constant the state

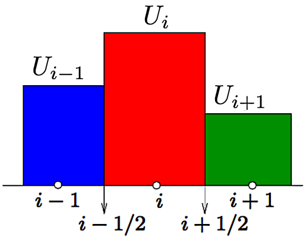
$$F(U(x)) - \gamma b(x) = \eta_0 = c^t$$

defines a steady equilibrium

Conservation, steady state preservation, and residuals

Conservative approximation ($\widehat{\cdot}$ denotes numerical fluxes/source)

$$\Delta x_i \frac{dU_i}{dt} + \widehat{F}_{i+1/2} - \widehat{F}_{i-1/2} - \gamma \widehat{\Delta b}_i = 0$$

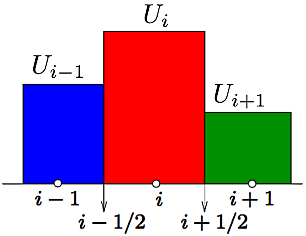


How to define $\widehat{\Delta b}_i$?

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

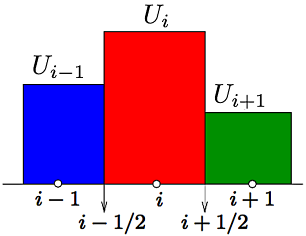
$$\Delta x_i \frac{dU_i}{dt} + \underbrace{\widehat{F}_{i+1/2} - F(U_i)}_{\psi_i^{i+1/2}} + \underbrace{F(U_i) - \widehat{F}_{i-1/2}}_{\psi_i^{i-1/2}} - \gamma\widehat{\Delta b}_i = 0$$



Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \underbrace{\widehat{F}_{i+1/2} - F(U_i)}_{\psi_i^{i+1/2}} + \underbrace{F(U_i) - \widehat{F}_{i-1/2}}_{\psi_i^{i-1/2}} - \gamma\widehat{\Delta b}_i = 0$$



Conservation

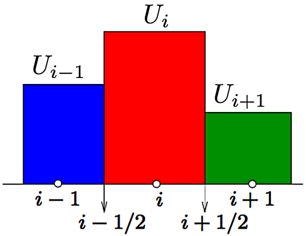
$$\Delta x_{i+1} \frac{dU_{i+1}}{dt} + \widehat{F}_{i+3/2} - F(U_{i+1})$$

$$+ \underbrace{F(U_{i+1}) - \widehat{F}_{i+1/2}}_{\psi_{i+1}^{i+1/2}} - \gamma\widehat{\Delta b}_{i+1} = 0$$

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \underbrace{\widehat{F}_{i+1/2} - F(U_i)}_{\psi_i^{i+1/2}} + \underbrace{F(U_i) - \widehat{F}_{i-1/2}}_{\psi_i^{i-1/2}} - \gamma\widehat{\Delta b}_i = 0$$



Conservation

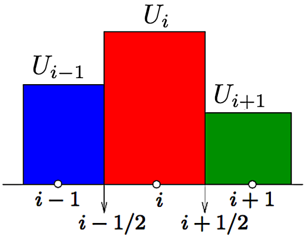
$$\psi^{i+1/2} := \psi_i^{i+1/2} + \psi_{i+1}^{i+1/2} = F(U_{i+1}) - F(U_i)$$

$$\psi^{i-1/2} := \psi_i^{i-1/2} + \psi_{i-1}^{i-1/2} = F(U_i) - F(U_{i-1})$$

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} - \gamma\widehat{\Delta b}_i = 0$$

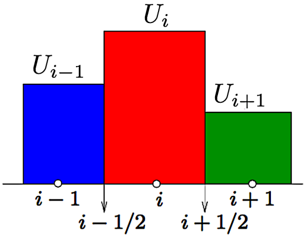


Well balanced

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} - \gamma\widehat{\Delta b}_i = 0$$



Well balanced

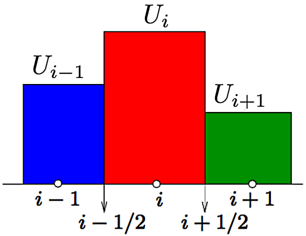
$$\gamma\widehat{\Delta b}_i = \gamma\widehat{\Delta b}_i^{i+1/2} + \gamma\widehat{\Delta b}_i^{i-1/2}$$

$$\gamma\widehat{\Delta b}_i^{i+1/2} + \gamma\widehat{\Delta b}_{i+1}^{i+1/2} = \gamma(b(x_{i+1}) - b(x_i)) = \int_{x_i}^{x_{i+1}} S$$

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \psi_i^{i+1/2} + \psi_i^{i-1/2} - \gamma\widehat{\Delta b}_i^{i+1/2} - \gamma\widehat{\Delta b}_i^{i-1/2} = 0$$



Well balanced

Design $\gamma\widehat{\Delta b}_i^{i\pm1/2}$ | if $F(U_i) - \gamma b(x_i) = \eta_0 \; \forall\, i$, then

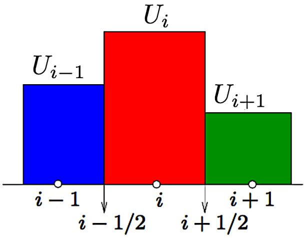
$$\phi_i^{i+1/2} = \psi_i^{i+1/2} - \gamma\widehat{\Delta b}_i^{i+1/2} = 0$$

$$\phi_i^{i-1/2} = \psi_i^{i-1/2} - \gamma\widehat{\Delta b}_i^{i-1/2} = 0$$

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$



Well balanced

Define

$$\phi^{i+1/2} := \int_{x_i}^{x_{i+1}} (\partial_x F - S)$$

$$= F(U_{i+1}) - F(U_i) - \gamma(b(x_{i+1}) - b(x_i))$$

If $\phi_i^{i+1/2} := \beta_i^{i+1/2}\phi^{i+1/2}$ the scheme is well balanced

Conservation, steady state preservation, and residuals

Conservative approximation in fluctuation form

$$\Delta x_i \frac{dU_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$

Examples:

$$\widehat{F}_{i+1/2} = \frac{F(U_i) + F(U_{i+1})}{2} \Rightarrow \phi_i^{i+1/2} = \frac{1}{2}\phi^{i+1/2}$$

$$\widehat{F}_{i+1/2} = \frac{F(U_i) + F(U_{i+1})}{2} - \frac{|\partial_U F|^{\mathsf{Roe}}_{i+1/2}}{2}(U_{i+1} - U_i) \Rightarrow \phi_i^{i+1/2} = \frac{1 - \mathsf{sign}(\partial_U F^{\mathsf{Roe}}_{i+1/2})}{2}\phi^{i+1/2}$$

Roe Lect.Not.Math. 1986,

Bermúdez,& Vázquez-Cendón CAF 1994

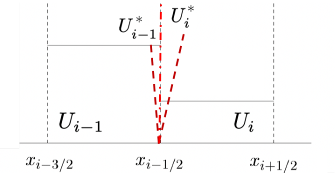
Conservation, steady state preservation, and residuals

Two intermediate states Riemann solv.

Berthon & Chalons Math.Comp. 2016,

Gallouet et al C&F 2003, LeVeque JCP 1998

Shallow water:
2 physical waves $u \pm \sqrt{gh}$ & a 0-wave.



$$\hat{F} = \frac{F_L + F_R}{2} - \frac{1}{2} \sum_k |\lambda_k| [\![U]\!]_k$$

Conservation, steady state preservation, and residuals

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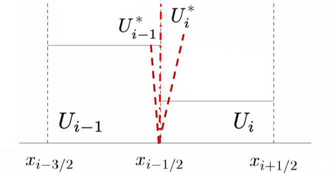


$$\hat{F} = \frac{F_L + F_R}{2} - \frac{1}{2} \sum_k |\lambda_k| [\![U]\!]_k$$

Consistency conditions

- Space time integration: 2 conditions :

$$[\![F]\!] - \sum_k \lambda_k [\![U]\!] = \int_{-\Delta x/2}^{\Delta x/2} S := \Delta x \hat{S}$$

- 2 conditions across 0-wave, e.g.

$$[\![hu]\!]^* = 0 \ , \ \ [\![u^2/2 + g(h+b)]\!]^* = -\int_{-\Delta x/2}^{\Delta x/2} c_f u^*$$

Conservative approximation in fluctuation form with

$$\phi_i^{i-1/2} = \frac{1}{2}\left(F_i - F_{i-1} - \sum_k |\lambda_k| [\![U]\!]_k - \Delta x \hat{S}_{i-1/2}\right)$$

Steady equilibrium:

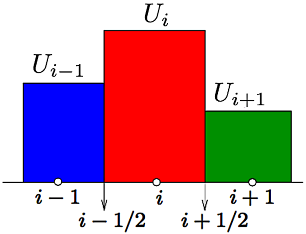
$$\phi_i^{i-1/2} = \frac{1}{2}\phi^{i-1/2} = \frac{1}{2}\int_{-\Delta x/2}^{\Delta x/2} (\partial_x F - S)$$

Conservation, steady state preservation, and residuals

From fluctuations back to (global) fluxes

$$\Delta x_i \frac{dU_i}{dt} + \underbrace{\widehat{F}_{i+1/2} - F(U_i)}_{\psi_i^{i+1/2}} + \underbrace{F(U_i) - \widehat{F}_{i-1/2}}_{\psi_i^{i-1/2}} - \gamma\widehat{\Delta b}_i = 0$$

Conservation



$$\Delta x_{i+1} \frac{dU_{i+1}}{dt} + \widehat{F}_{i+3/2} - F(U_{i+1})$$

$$+ \underbrace{F(U_{i+1}) - \widehat{F}_{i+1/2}}_{\psi_{i+1}^{i+1/2}} + \gamma\widehat{\Delta b}_{i+1} = 0$$

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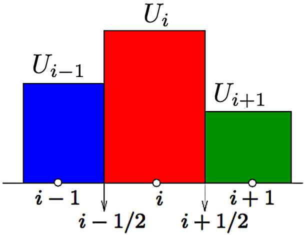
Conservation



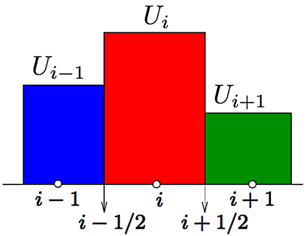
$$\widehat{F}_{i+1/2} = F(U_i) + \psi_i^{i+1/2} = F(U_{i+1}) - \psi_{i+1}^{i+1/2}$$

$$F(U)_{i+1} = F(U_i) + \psi_{i+1}^{i+1/2} + \psi_i^{i+1/2}$$

Conservation, steady state preservation, and residuals

From fluctuations back to (global) fluxes

$$\Delta x_i \frac{dU_i}{dt} + \phi_i^{i+1/2} + \phi_i^{i-1/2} = 0$$



Conservation and well balanced

$$\widehat{F}_{i+1/2} = F(U_i) + \psi_i^{i+1/2} = F(U_{i+1}) - \psi_{i+1}^{i+1/2}$$

$$F(U)_{i+1} = F(U_i) + \psi_{i+1}^{i+1/2} + \psi_i^{i+1/2}$$

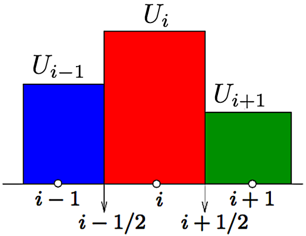
$$\widehat{G}_{i+1/2} = G_i + \phi_i^{i+1/2} = G_{i+1} - \phi_{i+1}^{i+1/2}$$

$$G_{i+1} = G_i + \phi^{i+1/2} = G_i + \int_{x_i}^{x_{i+1}} (\partial_x F - S)$$

Conservation, steady state preservation, and residuals

From fluctuations back to (global) fluxes

$$\Delta x_i \frac{dU_i}{dt} + \widehat{G}_{i+1/2} - \widehat{G}_{i-1/2} = 0$$



Conservation and well balanced

$$\widehat{G}_{i+1/2} = G_i + \phi_i^{i+1/2} = G_{i+1} - \phi_{i+1}^{i+1/2}$$

$$G_{i+1} = G_i + \phi^{i+1/2} = G_i + \int_{x_i}^{x_{i+1}} (\partial_x F - S)$$

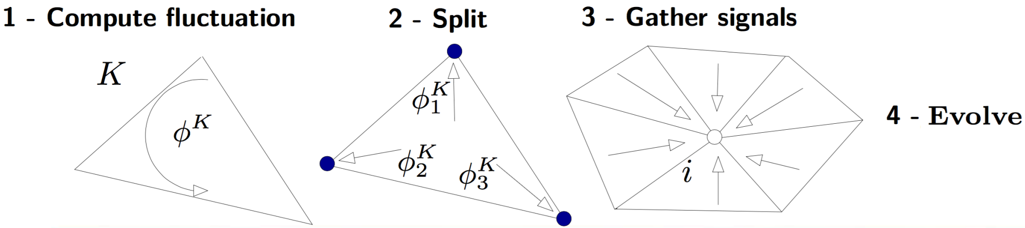
Steady state preserving:

$$G_{i+1} = G_i = G_0 \Rightarrow \widehat{G}_{i\pm1/2} = G_0$$

The RD approach

$$|C_i|\frac{dU_i}{dt} + \sum_{T} \Phi_i^T = 0$$

$$\sum_{j\in T} \Phi_j^T = \Phi^T := \oint_{\partial T} F(U_\mathrm{h})\cdot n - \int_T S(U_\mathrm{h},\varphi_\mathrm{h})$$



The RD approach

$$|C_i|\frac{dU_i}{dt} + \sum_T \Phi_i^T = 0$$

$$\sum_{j\in T}\Phi_j^T = \Phi^T := \oint_{\partial T} F(U_{\rm h})\cdot n - \int_T S(U_{\rm h},\varphi_{\rm h})$$

Well balanced RD (Brufau & Garcia-Navarro JCP 2003 $\longrightarrow$ Arpaia & MR JCP 2020):

$\Phi_i^T = \beta_i^T\Phi^T$, with β_i^T bounded (matrices). Steady state preserving encoded in Φ^T via

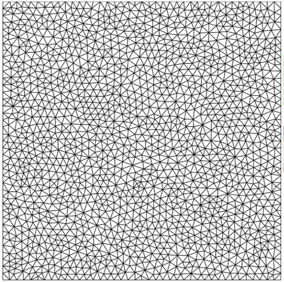
1. Data representation (bathymetry)
2. Polynomial approximation
3. Quadrature strategy

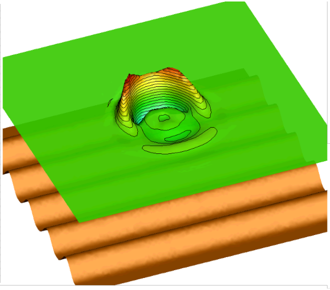
The RD approach: on fixed meshes

Proposition (Well balanced RD, MR, JCP 2015)
Consider a family of solutions characterized by the array of invariants $V(U,\varphi) = V_0$.
If the continuous polynomial approximation is written as of $V_{\mathrm{h}} = \sum \lambda_j V_j$ *any scheme of the form* $\Phi_i^T = \beta_i^T \Phi^T$ *with bounded (matrix) coefficients* β_i^T *verifies the following*

1. *Exact preservation of initial data* $V_j = V_0,\ \forall\, j$ *for exact quadrature*
2. *Consistency error* $\max(\mathrm{h}^q, \mathrm{h}^{k+1})$, *with* q *the quadrature order, and* k *the degree of the polynomial approximation (for regular enough data* φ*)*

The RD approach: on fixed meshes





Supercritical constant energy flow on a ribbed channel

	$Q_3 - u(v_h, b)$	$Q_5 - u(v_h, b)$	$Q_3 - u_h$
25/50	1.452714e-07	3.698282e-10	3.35738e-04
25/100	9.508237e-09	4.450410e-12	8.85116e-05
rate	3.947	6.399	1.930
25/200	6.584230e-10	4.688134e-14	2.36592e-05
rate	3.865	6.591	1.913

Proposition (Well balanced RD, MR, JCP 2015)
Consider a family of solutions characterized by the array of invariants $V(U,\varphi) = V_0$.
If the continuous polynomial approximation is written as of $V_\mathrm{h} = \sum \lambda_j V_j$ *any scheme of the form* $\Phi_i^T = \beta_i^T \Phi^T$ *with bounded (matrix) coefficients* β_i^T *verifies the following*

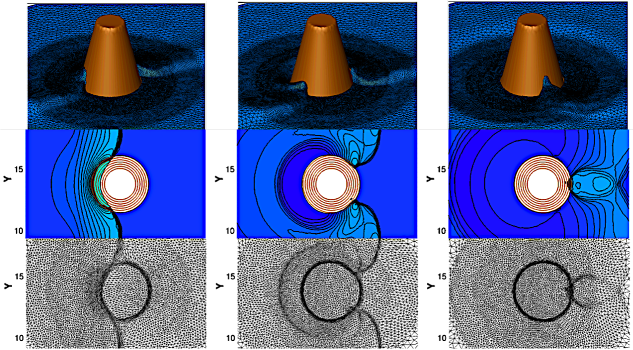
1. *Exact preservation of initial data* $V_j = V_0,\ \forall\, j$ *for exact quadrature*
2. *Consistency error* $\max(\mathrm{h}^q, \mathrm{h}^{k+1})$, *with* q *the quadrature order, and* k *the degree of the polynomial approximation (for regular enough data* φ*)*

The RD approach: on moving meshes

Proposition (Well balanced ALE RD, L. Arpaia & MR, C&F 2018)
Consider RD in ALE formulation of the form $\Phi_i^T = \beta_i^T \Phi^T$ *with bounded (matrix) coefficients* β_i^T*. The schemes conserves mass and lake at rest initial data* $\vec{u} = 0$ *and* $\eta = h + b = \eta_0$ *provided that*

1. *The continuous polynomial approximation is written as of* $\eta_{\rm h} = \sum \lambda_j \eta_j$
2. *The quadrature is exact in* $\eta_{\rm h}$
3. *The bathymetry remap is* *Lagrangian wrt the mesh velocity*

The RD approach: on moving meshes



Runup on conical island

- Adaptive ALE
- Mass conservative
- Well balanced (lake at rest)
- Positivity preserving

Proposition (Well balanced ALE RD, L. Arpaia & MR, C&F 2018)
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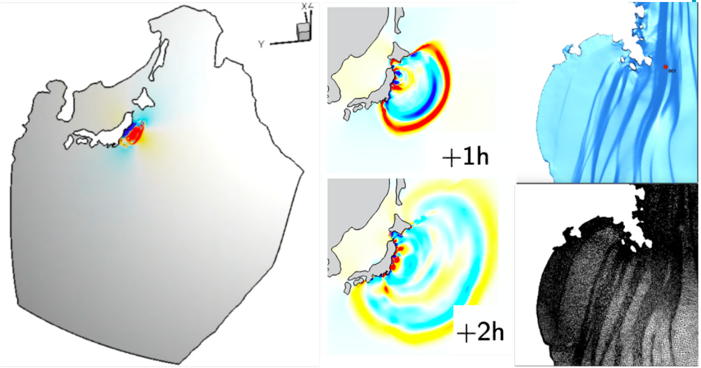
The RD approach: on moving meshes



2011 Tohoku tsunami

- Adaptive ALE
- Mass conservative
- Spherical coordinates
- Well balanced (lake at rest)
- Positivity preserving

Proposition (Well balanced ALE RD, L. Arpaia & MR, JCP 2020)

Consider RD in ALE curvilinear reference with $\Phi_i^T = \beta_i^T \Phi^T$ *and bounded (matrices)* β_i^T.
The schemes conserves mass and lake at rest initial data $\vec{u} = 0$ *and* $\eta = h + b = \eta_0$ *provided that*

1. *The continuous polynomial approximation is written as of* $\eta_h = \sum \lambda_j \eta_j$
2. *The quadrature is exact in* η_h
3. *The bathymetry remap is Lagrangian wrt the mesh velocity*

The RD approach

$$|C_i|\frac{dU_i}{dt} + \sum_T \Phi_i^T = 0$$

$$\sum_{j\in T}\Phi_j^T = \Phi^T := \oint_{\partial T} F(U_\mathrm{h})\cdot n - \int_T S(U_\mathrm{h},\varphi_\mathrm{h})$$

Well balanced RD (Brufau & Garcia-Navarro JCP 2003 $\longrightarrow$ Arpaia & MR JCP 2020):

$\Phi_i^T = \beta_i^T\Phi^T$, with β_i^T bounded (matrices). Steady state preserving encoded in Φ^T via

1. Data representation (bathymetry)
2. Polynomial approximation
3. Quadrature strategy

The RD approach

$$|C_i|\frac{dU_i}{dt} + \int_{\partial C_i} \widehat{G}_n = 0$$

$$\sum_{j\in T} \Phi_j^T = \Phi^T := \oint_{\partial T} F(U_\mathrm{h})\cdot n - \int_T S(U_\mathrm{h},\varphi_\mathrm{h})$$

Can be recast in a pseudo-conservative global flux form

- ✓ Explicit formulae for $\widehat{G}$ in R. Abgrall & MR, Numer.Fl.Dyn. Springer 2022
- ✘ Limited to solutions with some invariant set $V(U,\varphi) = V_0$ at steady state
- ✘ No characterization of the discrete steady state solution itself (only scheme consistency)
- ✘ What of truly multiD solutions ?

TOC

The talk is split in three main chapters

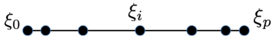
- **Ch.1** Finite volumes, residual distribution, and global fluxes
- **Ch.2** Global flux quadrature and finite elements: approximate full well balanced
- **Ch.3** Global flux quadrature in multiD: divergence preserving schemes

with Y. Mantri and P. Öffner

DG-SEM - Main notation

Discontinuous Galerkin Spectral Element Method

- Reference element $\xi \in [0,1]$
- $x(\xi)$ linear mapping $[0,1] \mapsto K$, for simplicity: $|K| = \mathrm{h}$
- $\{\xi_i\}_{i=0,p}$ the $p+1$ Gauss-Lobatto (GL) points
- $\{\phi_i(\xi)\}_{i=0,p}$ degree p Lagrange bases
- Set $U_\mathrm{h} = \sum_{i=0}^{p} \phi_i(x(\xi)) U_i$



DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

On an element K, start from discrete approximation arising from the variational form

$$\mathrm{h}\int_0^1 \varphi_i(\xi)\partial_t U_{\mathrm{h}} - \int_0^1 \partial_\xi \varphi_i(\xi) F_{\mathrm{h}} + (\varphi_i \hat{F}_{\mathrm{h}}(U_{\mathrm{h}}^-, U_{\mathrm{h}}^+))_{\xi=1} - (\varphi_i \hat{F}_{\mathrm{h}}(U_{\mathrm{h}}^-, U_{\mathrm{h}}^+))_{\xi=0} = \mathrm{h}\int_0^1 \varphi_i(\xi) S_{\mathrm{h}}$$

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM : quadrature based on the same GL nodes used for the polynomial expansion[1]

$$\mathcal{M}\dot{\mathbf{U}} - D_x^T \mathbf{F} + \mathcal{B}\widehat{\mathbf{F}} = \mathcal{M}\mathbf{S}$$

with the notation:

- $\mathcal{M} = \mathrm{diag}(\{w_i\}_{i=0,p})$ with $w_i = \mathrm{h}\int_0^1 \phi_i(\xi)d\xi$ the quadrature weights
- with $(D_x)_{ij} = w_i \partial_\xi \phi_j(\xi_i)$
- $\mathcal{B} = \mathrm{diag}(-1, \dots, 1)$ the matrix sampling boundary values
- $\mathbf{U}$, $\mathbf{F}$, $\widehat{\mathbf{F}}$, $\mathbf{S}$: elemental arrays of nodal solution/flux/num. flux/source values

[1] Kopriva & Gassner *J.Sci.Comp.* 44, 2010 ; Hesthaven & Warburton, Springer 2008

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

DG-SEM discrete equations in strong form (SBP property[2])

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F}) = \mathcal{M}\mathbf{S}$$

[2] Kopriva & Gassner *J.Sci.Comp.* 44, 2010; Gassner et al. *J.Comput.Phys.* 327, 2016

DG-SEM - Discrete variational form

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U,\varphi)$$

DG-SEM discrete equations in strong form

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\widehat{\mathbf{F}} - \mathbf{F}) = \mathcal{M}\mathbf{S}$$

Setting $\widehat{\mathbf{F}}(\mathbf{U}^-,\mathbf{U}^+) = \alpha\mathbf{F}^+ + (1-\alpha)\mathbf{F}^- - \mathcal{D}[\![\mathbf{U}]\!]$, we get the fully discrete method

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathcal{M}\mathbf{S}$$

This is our "reference" non well-balanced method

DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x G(U; \varphi(x)) = 0$$

with $G = F(U) + R(U; \varphi(x))$ and

$$R(U; \varphi(x)) = R_0 - \int_{x_0}^{x} S(U; \varphi(s))ds$$

DG-SEM - Discrete variational form using global fluxes

Consider the approximation of solutions of

$$\partial_t U + \partial_x F(U) = S(U, \varphi)$$

by locally recasting it in the pseudo-conservative form

$$\partial_t U + \partial_x F(U) = -\partial_x R(U; \varphi(x))$$

If we have the source primitive R at all GL nodes, we can readily write the DG-SEM scheme:

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = -D_x\mathbf{R} + \mathcal{B}(\alpha[\![\mathbf{R}]\!])$$

The key now is to define R at all GL nodes

DG-SEM - Discrete variational form using global fluxes

We compute nodal values of R:

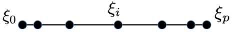
$$\partial_x R = -S$$

Choose

$$R_{\mathrm{h}} = \sum_{i=0,p} \varphi_i(x(\xi)) R_i \;, \quad S_{\mathrm{h}} = \sum_{i=0,p} \varphi_i(x(\xi)) S_i$$

and integrate on each element

1. For the (local) initial value we set $R_0 = R^-$
2. For all $i \in 1, p$ we compute $R_i = R_{i-1} - \mathrm{h} \sum_{l=0,p} \int_{\xi_{i-1}}^{\xi_i} \varphi_l(\xi) S_l ds$



DG-SEM - Discrete variational form using global fluxes

We compute nodal values of R:

$$\partial_x R = -S$$

Remark. *In compact notation we have*

$$\mathbf{R} = \mathbf{R}^- - \mathrm{h}\mathcal{I}\mathbf{S}$$

with $\mathcal{I}$ is the tableau of the $p+1$ stages RK-LobattoIIIA implicit collocation method[3]

[3] A. Prothero & A. Robinson, Math.Comp. 28, 1974

Source integration: initial value

The initial value R^- can be related to the last value on the neighbouring element:

$$R^- = [R(x_p)]^{K^-} + [\![R]\!] = [R(x_p)]^{K^-} + \lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} S$$

The critical term is for the shallow water example

$$\lim_{\epsilon \to 0} \int_{x_0-\epsilon}^{x_0+\epsilon} g h_{\mathrm{h}} \partial_x b_{\mathrm{h}}$$

For simplicity, let us rule out discontinuous bathymetry and set

$$R^- = [R(x_p)]^{K^-} \; , \quad [\![R]\!] = 0$$

DG-SEM - Discrete variational form using global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U;\varphi(x))$$

for all K evolve in time

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathrm{h}D_x\mathcal{I}\mathbf{S} \qquad \text{GF-DG}$$

DG-SEM - Discrete variational form using global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U; \varphi(x))$$

for all K evolve in time

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathrm{h}D_x\mathcal{I}\mathbf{S} \qquad \text{GF-DG}$$

Remark 1. Compared to standard DG-SEM a mass matrix change

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathcal{M}\mathbf{S} \qquad \text{DG}$$

DG-SEM - Discrete variational form using global fluxes

We seek solutions of the (hyperbolic) system of balance laws

$$\partial_t U + \partial_x F(U) = S(U;\varphi(x))$$

for all K evolve in time

$$\mathcal{M}\dot{\mathbf{U}} + D_x\mathbf{F} + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = \mathrm{h}D_x\mathcal{I}\mathbf{S} \qquad \text{GF-DG}$$

Remark 2. Residual formulation

$$\mathcal{M}\dot{\mathbf{U}} + D_x\Phi^K + \mathcal{B}(\alpha[\![\mathbf{F}]\!]) - \mathcal{B}(\mathcal{D}[\![\mathbf{U}]\!]) = 0$$

with array of integrated sub-cell residuals

$$(\Phi^K)_\ell = \int_{x_0}^{x_\ell} (\partial_x F_\mathrm{h} - S_\mathrm{h})$$

Main result

Proposition (Discrete steady state). *The DG-SEM with global flux quadrature preserves exactly* *continuous* *discrete steady states* $U_i^* = U(F_i)$ *with* F *obtained by integrating the ODE*

$$F' = S(U(F), \varphi(x))$$

using the implicit continuous collocation RK-LobattoIIIA method on spatial slabs of size h.

As long as $U(F)$ *is a one to one mapping* $U^*(x)$ *verifies the consistency estimates of the LobattoIIIA method*[4]*: Element endpoints are* $2p$*-order accurate, internal nodes have accuracy* h^{p+2}.

[4] See e.g. Theorem 7.10 in Hairer, Wanner and Norset, Solving Ordinary Differential Equations I., Springer 1993

Global flux quadrature

$$\int_K \phi_i f_{\rm h} \to {\rm h} D_x \mathcal{I} \mathbf{f}$$

- Approximate well balanced similar to Castro & Parés J.Sci.Comp. 2020 ; Gómez-Bueno, et al *Mathematics* 2021 : preservation of an enhanced discrete equilibirum
- Here: discrete equilibria solution of LobattoIIIA continuous collocation method

Main features

✓ No a-priori knowledge of equilibrium

✓ No need of compute the solution of the Cauchy problem .. (maybe for initialization)

✓ Considerable accuracy enhancements at steady state with minor change in code

× Order/type of collocation method not arbitrary

Numerical examples in 1d

Pseudo-1D rotating shallow water eq.s (Castro et al, SISC 31, 2008)

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} = -h \begin{pmatrix} 0 \\ \partial_x \varphi + c_f u - \phi v \\ c_f v + \phi u \end{pmatrix}$$

Notation.

h water depth

η free surface level

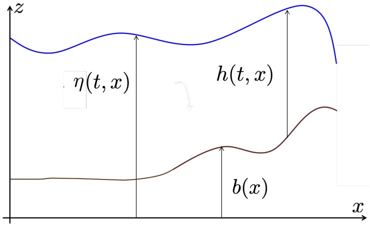
$\mathrm{v} = (u, v)$ horizontal velocity

$P = gh^2/2$ hydrostatic pressure (g gravity acceleration)

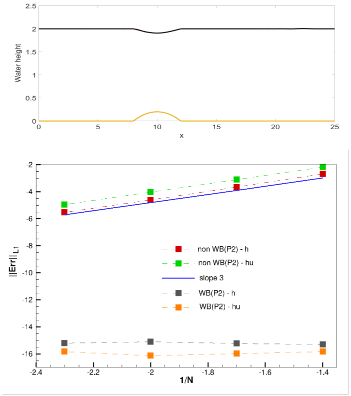
$\varphi = gb$ gravitational potential ($b(x, y)$ bottom topography)

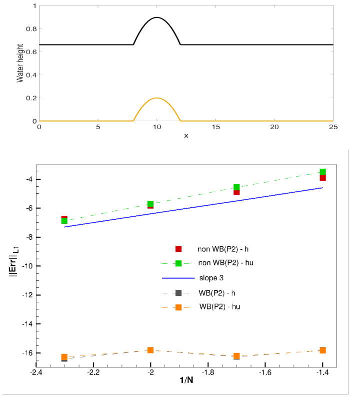
$c_f = c_f(h, \mathrm{v})$ friction coefficient

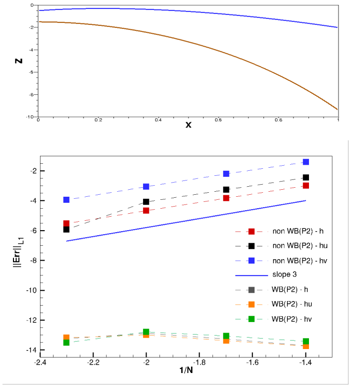
ϕ Coriolis coefficient



Numerical examples: error wrt ODE solver







Sub-critical with

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0$$

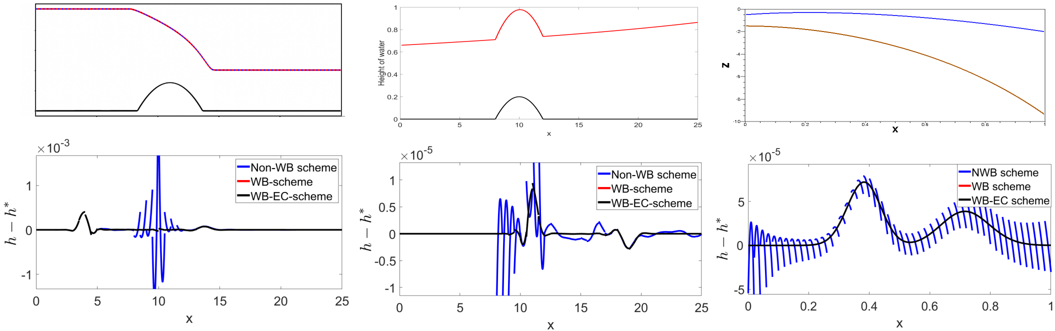
Super-critical with

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u$$

Sub-critical with $v = \phi x$ and

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi u v$$

Numerical examples: perturbations of moving equilibria



Trans-critical with

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0$$

Depth perturbation $\xi_h = 10^{-3}$

Super-critical with

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0 - \int_{x_0}^{x} c_f u$$

Depth perturbation $\xi_h = 10^{-5}$

Sub-critical with $v = \phi x$ and

$$hu = q_0$$
$$g\eta + u^2/2 = \mathcal{E}_0 + \int_{x_0}^{x} \phi u v$$

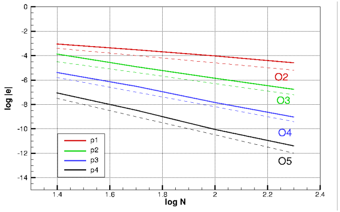
Depth perturbation $\xi_h = 10^{-5}$

Numerical examples: nodal super-convergence

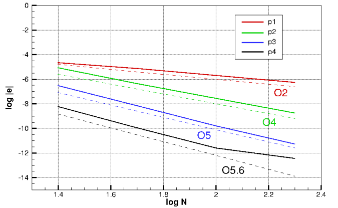
Theory for GF: internal points $= \mathcal{O}(\mathrm{h}^{p+2})$ - endpoints $= \mathcal{O}(\mathrm{h}^{2p})$

Super-critical channel with bump (no friction, no Coriolis)

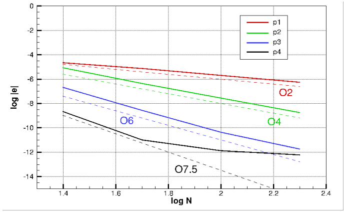
std DG-SEM



GF DG-SEM, all pts

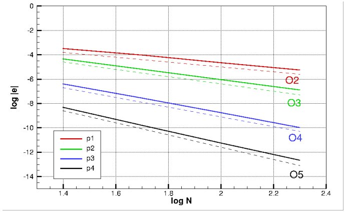


GF DG-SEM, end-pts

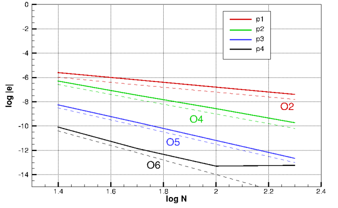


Sub-critical with Coriolis force

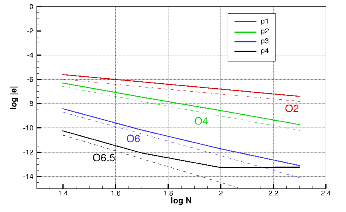
std DG-SEM



GF DG-SEM, all pts



GF DG-SEM, end-pts



Numerical examples: nodal super-convergence

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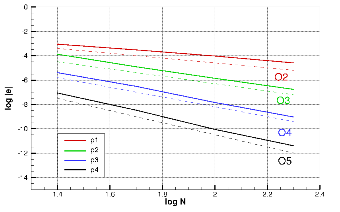
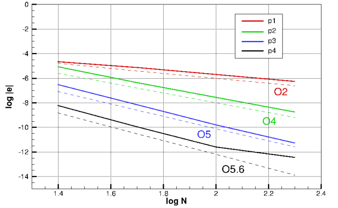
Super-critical channel with bump (no friction, no Coriolis)

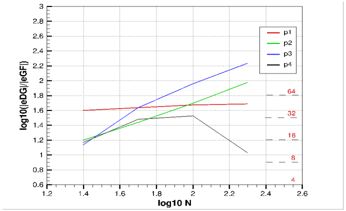
std DG-SEM



GF DG-SEM, all pts

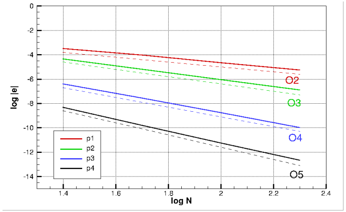


Error ratio $e_{\text{DG-SEM}}/e_{\text{GFq}}$, all-pts

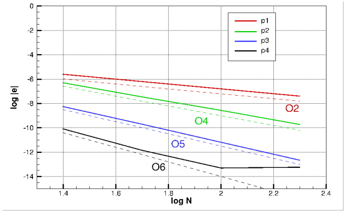


Sub-critical with Coriolis force

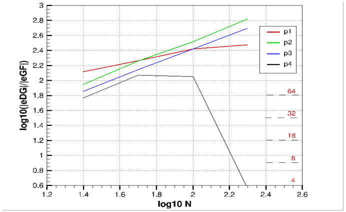
std DG-SEM



GF DG-SEM, all pts



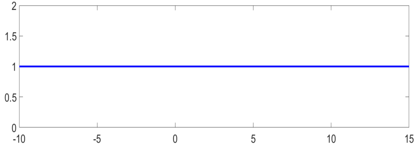
Error ratio $e_{\text{DG-SEM}}/e_{\text{GFq}}$, all-pts



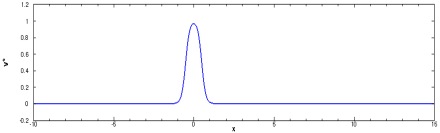
Numerical examples: Rossby Adjustment in an Open Domain[5]

$$\partial_t \begin{pmatrix} h \\ hu \\ hv \end{pmatrix} + \partial_x \begin{pmatrix} hu \\ hu^2 + P(h) \\ huv \end{pmatrix} = h \begin{pmatrix} 0 \\ \phi v \\ -\phi u \end{pmatrix}$$

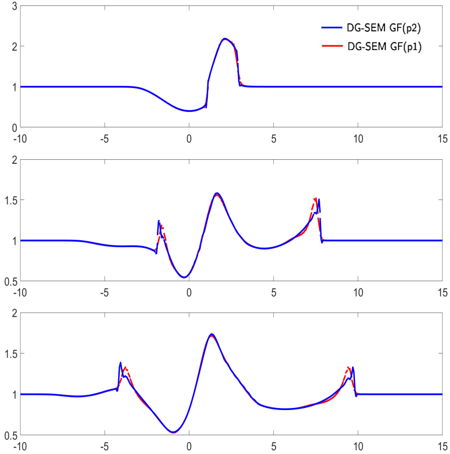
Initially flat free surface and quiescent water



Initial transverse velocity (transverse jet)



Free surface evolution



[5] Bouchut et al.J.Fluid Mech. 514, 2004 ; Castro et al.SISC 31, 2008

TOC

The talk is split in three main chapters

Ch.1 Finite volumes, residual distribution, and global fluxes

Ch.2 Global flux quadrature and finite elements: approximate full well balanced

Ch.3 Global flux quadrature in multiD: divergence preserving schemes

with W. Barsukow and D. Torlo

Well balanced in multi-D: structure/constraint preserving

$$\nabla \cdot F(U) = S(U; \varphi(x))$$

The form of the tensor F determines the type of differential constraints to preserve.

Well balanced in multi-D: structure/constraint preserving

We are going to work with the following example (with $\mathsf{v} = (u, v)$)

$$\begin{aligned}\partial_t P + \nabla \cdot \mathsf{v} &= s(x,y)\\ \partial_t u + \partial_x P &= \phi v - c_f u + \tau_x\\ \partial_t v + \partial_y P &= -\phi u - c_f v + \tau_y\end{aligned}$$

Linear waves with Coriolis, friction, mass source

P pressure

$\mathsf{v} = (u, v)$ velocity

$s(x, y)$ mass source

c_f friction coefficient

ϕ Coriolis coefficient

$\tau = (\tau_x, \tau_y)$ momentum forcing (e.g. wind for free suface waves)

Well balanced in multi-D: structure/constraint preserving

We are going to construct schemes preserving (simultaneously) the constrains

$$\nabla \cdot \mathsf{v} = s(x, y)$$

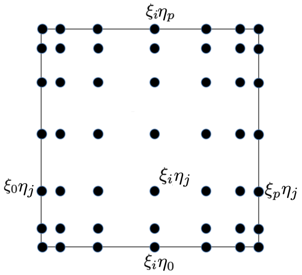
$$\nabla P = \mathsf{q}$$

- one multi-D constraint : $\nabla \cdot \mathsf{v} = s(x, y)$ (hard)
- two (pseudo-)1D constraints : $\nabla P = \mathsf{q}$ (easyer)

Discrete Framework: SEM

Tensor product spectral finite element method (SEM)

- $\mathrm{x}(\xi) = (x(\xi), y(\eta))$ mapping $[0,1]^2 \mapsto K$, for simplicity: $|K| = \mathrm{h}^2$
- $\{\xi_i\}_{i=0,p}$ and $\{\eta_j\}_{j=0,p}$ the $p+1$ Gauss-Lobatto (GL) points
- $\{\phi_i(x)\}_{i=0,p}$ and $\{\psi_j(y)\}_{j=0,p}$ 1d degree p Lagrange bases
- For node ij: $\lambda_{ij}(x(\xi), y(\eta)) = \phi_i(x)\psi_j(y)$
- Set $U_\mathrm{h} = \sum_{i,j} \lambda_{ij}(x(\xi), y(\eta)) U_{ij}$



Discrete Framework: SEM

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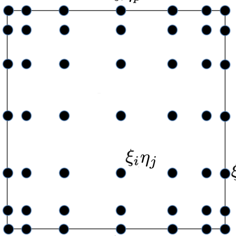

Notation: tensor product matrices

Mass matrix entries

$$\int_K \lambda_{ij}\lambda_{lm} = \int_{x_0}^{x_p} (\phi_i(x)\phi_l(x)) \times \int_{y_0}^{y_p} (\psi_j(y)\psi_m(y)) \Rightarrow M = M_x \otimes M_y = M_y \otimes M_x$$

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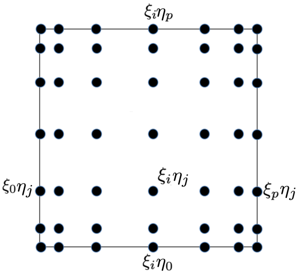

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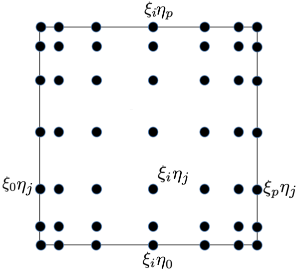

Notation: tensor product matrices

Derivative matrix entries

$$\int_K \lambda_{ij}\partial_x\lambda_{lm} = \int_{x_0}^{x_p} (\phi_i(x)\partial_x\phi_l(x)) \times \int_{y_0}^{y_p} (\psi_j(y)\psi_m(y)) \Rightarrow \quad D_x M_y = M_y D_x$$

Discrete Framework: SEM

We are going to work with the following example (with $\mathsf{v} = (u, v)$)

$$\begin{aligned}\partial_t P + \nabla \cdot \mathsf{v} &= s(x, y)\\ \partial_t u + \partial_x P &= q_x\\ \partial_t v + \partial_y P &= q_y\end{aligned}$$

Standard continuous SEM approximation (no stabilization) in strong form

$$\begin{aligned}M\dot{\mathbf{P}} + M_y D_x \mathbf{U} + M_x D_y \mathbf{V} &= M\mathbf{S}\\ M\dot{\mathbf{U}} + M_y D_x \mathbf{P} &= M\mathbf{Q_x}\\ M\dot{\mathbf{V}} + M_x D_y \mathbf{P} &= M\mathbf{Q_y}\end{aligned}$$

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Standard continuous SEM approximation (no stabilization) in strong form

$$M\dot{\mathbf{P}} + M_y D_x \mathbf{U} + M_x D_y \mathbf{V} = M\mathbf{S} = \sum_K M^K \mathbf{S}^K$$
$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = M\mathbf{Q_x}$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = M\mathbf{Q_y}$$

Abuse of notation:
for continuous SEM the matrix formulation involve local elemental assembly.
We omit it for brevity, and explicitly mention only when necessary

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Stabilization using standard methods in continuous FEM:
SUPG, or orthogonal subgrid scales term[6].
More later (if time)

[6] see e.g. (Michel et al, J.Sci.Comp. 94, 2023) for a review

The div constraint. Homogeneous case

Consider now the multiD (steady) constraint on the div.

$$\partial_x u + \partial_y v = 0$$

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1. We look at it as two 1D relations in one
2. We apply to each the GF quadrature as if we where working on 2 balance laws
3. We combine the two to get a discrete divergence operator

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Remarks (batch 1)

- Note that $\mathcal{I}_x = \mathcal{I}_y$, both corresponding to the LobattoIIIA tableau.
- We keep the subscripts $_x$ and $_y$ for better understanding
- Recall that the standard SEM divergence operator is

$$\text{div}_{\text{SEM}} := M_y D_x\mathbf{U} + M_x D_y\mathbf{V}$$

2 Global Flux Quadrature in MultiD

The div constraint. Homogeneous case

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Remark: from nodal to face unknowns

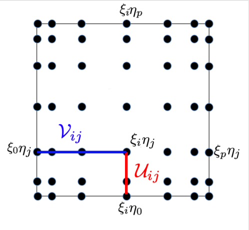
The arrays

$$\mathcal{U} := \mathcal{I}_y\mathbf{U}\ , \quad \mathcal{V} := \mathcal{I}_x\mathbf{V}$$

are sub-cell face-averages of normal velocity components.

These are the usual basis of mimetic div preserving methods.

Some links to e.g. Torrilhon&Fey SINUM 2004, Mishra&Tadmor Comm.Comp.Phys. 2010, etc.



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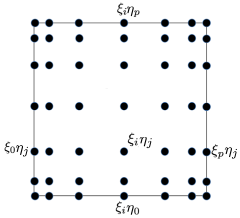
Proposition (The div residual) *The local assembly of the divergence operator can be written as*

$$\text{div} = \sum_K D_x^K D_y^K \Phi^K$$

where on each element Φ^K *is* the array of integrated divergences

$$(\Phi^K)_{lm} := \int_{y_0}^{y_m} (u_{\mathrm{h}}(x_l, s) - u_{\mathrm{h}}(x_0, s)) + \int_{x_0}^{x_l} (v_{\mathrm{h}}(s, y_m) - v_{\mathrm{h}}(s, y_0))$$



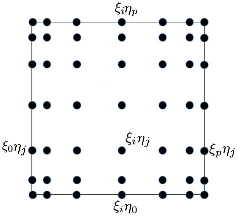
(non-stabilized) SEM-GF for the acoustics system

The SEM-GF semi-discrete approximation of the acoustics system (with abuse of notation) reads

$$M\dot{\mathbf{P}} + D_x D_y \Phi = 0$$
$$M\dot{\mathbf{U}} + M_y D_x \mathbf{P} = 0$$
$$M\dot{\mathbf{V}} + M_x D_y \mathbf{P} = 0$$

Proposition (Steady states) *The SEM-GF scheme preserves exactly initial states for which* $P_i = P_0 \,\forall\, i$*, and which verify within each element* K *and for every pair* $l, m \geq 1$

$$(\Phi^K)_{lm} = \int_{y_0}^{y_m} (u_{\mathrm{h}}(x_l, s) - u_{\mathrm{h}}(x_0, s)) + \int_{x_0}^{x_l} (v_{\mathrm{h}}(s, y_m) - v_{\mathrm{h}}(s, y_0)) = 0$$



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\end{aligned}
$$

Definition (LobattoIIIA line-by-line/row-by-row projection) *Let* $\mathrm{v}_e = (u_e(x,y), v_e(x,y))$ *be a given* div *free velocity. Consider the initialization obtained by means of the LobattoIIIA method*

$$[\mathcal{I}_y \mathbf{U}(x_l)]_m = \int_{y_0}^{y_m} u_e(x_l, y)\ , \quad [\mathcal{I}_x \mathbf{V}(y_m)]_l = \int_{x_0}^{x_l} v_e(x, y_m)$$

with local ICs given by the last value of the previous elements, and ICs on the lowest/left boundaries $u_{\mathrm{h}}(x_i, y=0) = u_e(x_i, y=0)$ *and* $v_{\mathrm{h}}(x=0, y_j) = v_e(x=0, y_j)$.

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Proposition (LobattoIIIA projection preservation) *Let* $\mathrm{v}_e = (u_e(x,y), v_e(x,y))$ *be a* div *free velocity, and consider the initial state consisting of the LobattoIIIA line-by-line/row-by-row projection of* v_e*, and* $P_i = P_0\ \forall i$*. The SEM-GF scheme*

1. *Preserves the initial condition within the accuracy of the evaluation of the integrals of the components of* v_e*. The IC is preserved exactly if the quadrature is exact.*
2. *It the nodal consistency order* $\mathcal{O}(\mathrm{h}^{p+2})$ *associated to the LobattoIIIA method*

(non-stabilized) SEM-GF for the full system: main ingredients

(non-stabilized) SEM-GF for the full system: main ingredients

Pressure equation: sources included using the GF recipe in x and y. Achieved setting

$$\Phi_{lm} = \int_{x_0}^{x_l} \int_{y_0}^{y_m} (\partial_x u_{\mathrm{h}} + \partial_y v_{\mathrm{h}} - s_{\mathrm{h}})$$

On each element

$$D_x D_y \Phi = \mathrm{h} D_x D_y \mathcal{I}_x \mathbf{U} + \mathrm{h} D_y D_x \mathcal{I}_x \mathbf{V} - \mathrm{h}^2 D_x \mathcal{I}_x D_y \mathcal{I}_y \mathbf{S}$$

(non-stabilized) SEM-GF for the full system: main ingredients

Velocity equations: we treat the 2 pseudo-1D constraints independently

$$\begin{array}{l} \partial_x P = q_x \\ \partial_y P = q_y \end{array} \quad \Rightarrow \quad \begin{array}{l} D_x \mathbf{P} = \mathrm{h} D_x \mathcal{I}_x \mathbf{Q_x} \\ D_y \mathbf{P} = \mathrm{h} D_y \mathcal{I}_y \mathbf{Q_y} \end{array}$$

More compactly, we can write the above as

$$\begin{array}{l} \partial_x P = q_x \\ \partial_y P = q_y \end{array} \quad \Rightarrow \quad \begin{array}{l} D_x(\mathbf{P} - \mathrm{h}\mathcal{I}_x \mathbf{Q_x}) = D_x \Phi_u \\ D_y(\mathbf{P} - \mathrm{h}\mathcal{I}_y \mathbf{Q_y}) = D_x \Phi_v \end{array}$$

having introduced the residuals

$$(\Phi_u)_{lm} = \int_{x_0}^{x_l} (\partial_x P_\mathrm{h}(x, y_m) - q_{x\mathrm{h}}) \text{ and } (\Phi_v)_{lm} = \int_{y_0}^{y_m} (\partial_y P_\mathrm{h}(x_l, y) - q_{y\mathrm{h}})$$

(non-stabilized) SEM-GF for the full system: main ingredients

We obtain the full non-stabilized SEM-GF discretization

$$
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All steady state preserving properties are enconded in the residual arrays $(\Phi,\, \Phi_u, \Phi_v)$

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All steady state preserving properties are enconded in the residual arrays $(\Phi,\ \Phi_u, \Phi_v)$

Stabilization

Use systematically GF quadrature to express all operators in terms of Φ, Φ_u and Φ_v

- We carry on the steady state preserving properties
- This is not the case for classical stabilization operators !

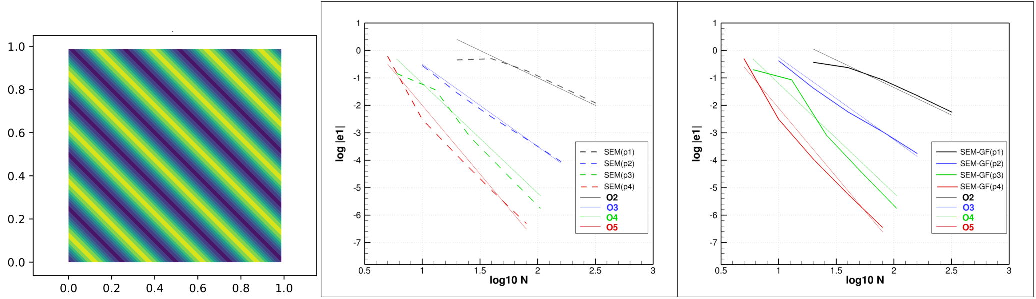
The initial solution

At the moment we can follow two approaches

1. Use a given IC (analytical or tabulated)
2. Given a div free velocity field v_0 (analytical or tabulated)
   - project v_0 on the space of discrete equilibria to obtain v_{h0}
   - Integrate the pressure in the each direction with LobattoIIIA, using v_{h0} to evaluate the RHS
   - Use div and $curl$ conditions to combine the two and obtain a single admissible pressure initial state

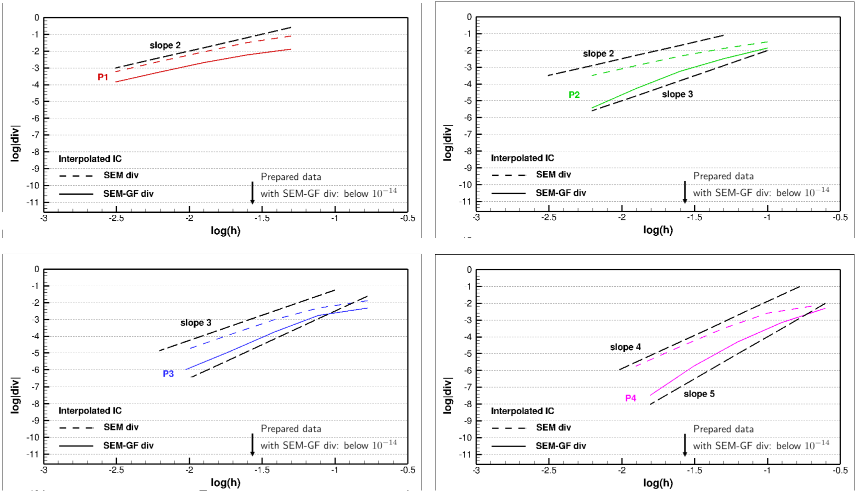
Numerical examples in 2d

Travelling waves (acoustics)



Numerical examples in 2d

How good is the div-GF



Numerical examples in 2d

Linear waves with Coriolis, friction, mass source

$$\partial_t \begin{pmatrix} P \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ P \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ P \end{pmatrix} = \begin{pmatrix} s(x,y) \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}$$

Notation.

P pressure

v $= (u, v)$ velocity

$s(x, y)$ mass source

c_f friction coefficient

ϕ Coriolis coefficient

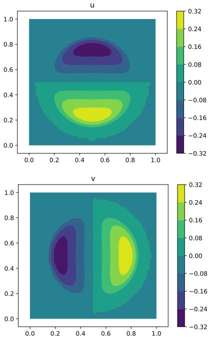
$\tau = (\tau_x, \tau_y)$ momentum forcing (e.g. wind for free suface waves)

Numerical examples in 2d

Vortex solutions

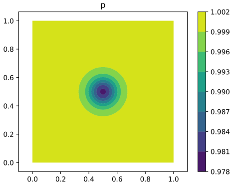
div-free exponential

$$r = \|\mathsf{x} - \mathsf{x}_0\|$$
$$\mathsf{v} = (\mathsf{x} - \mathsf{x}_0)^{\perp} f(r)$$
$$P = 1$$



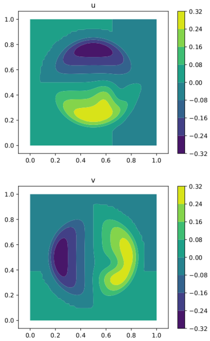
div-free exponential + Coriolis

$$r = \|\mathsf{x} - \mathsf{x}_0\|$$
$$\mathsf{v} = \mathsf{v}_{div-\mathsf{free}} + \nabla \varphi_2$$
$$P = 1 - \phi g(r)$$



div-free exponential + mass source

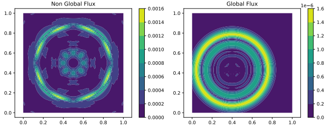
$$\mathsf{v} = \mathsf{v}_{div-\mathsf{free}} + \nabla \varphi_2$$
$$P = 1$$
$$s = \Delta \varphi_2$$



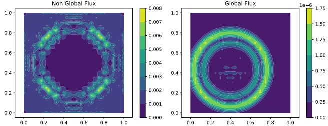
Numerical examples in 2d

Vortex solutions perturbations : $p = p_{\text{steady}} + 10^{-5}\delta_p$

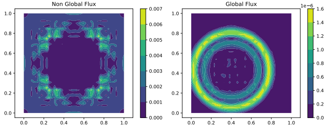
div-free exponential



$p1$ on 80×80 grid

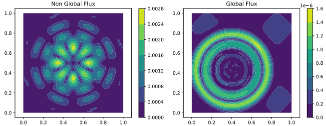


$p2$ on 20×20 grid



$p3$ on 13×13 grid

div-free exponential + Coriolis



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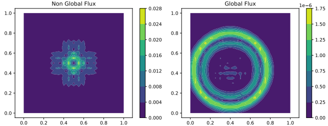
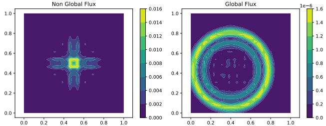
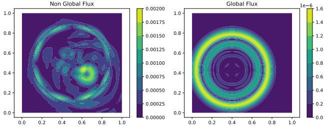


$p2$ on 20×20 grid



$p3$ on 13×13 grid

div-free exponential + mass source



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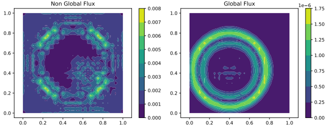
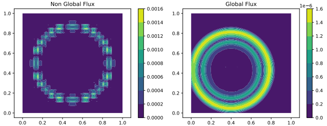


$p2$ on 20×20 grid



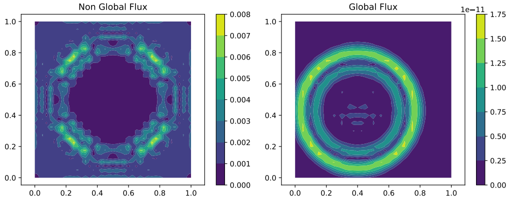
$p3$ on 13×13 grid

Mario Ricchiuto

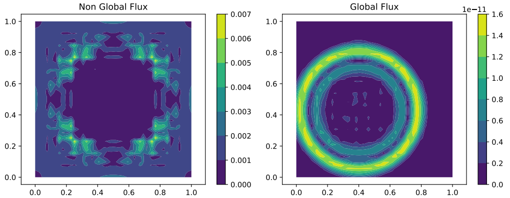
Numerical examples in 2d

Vortex solutions perturbations : $p = p_{\text{steady}} + 10^{-10}\delta_p$

div-free exponential

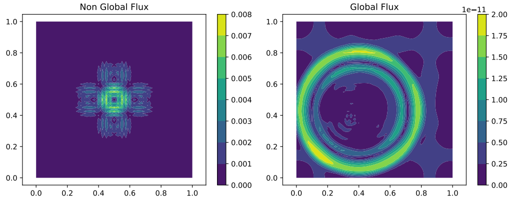


$p2$ on 20×20 grid

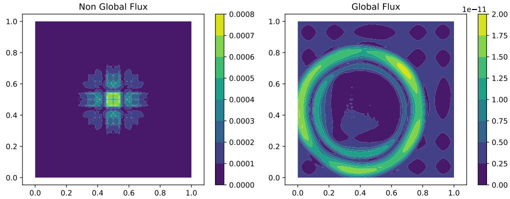


$p3$ on 13×13 grid

div-free exponential + Coriolis



$p2$ on 20×20 grid

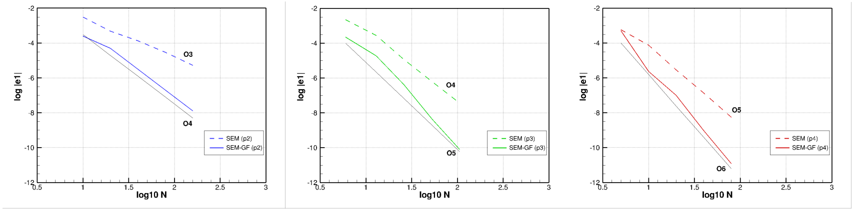


$p3$ on 13×13 grid

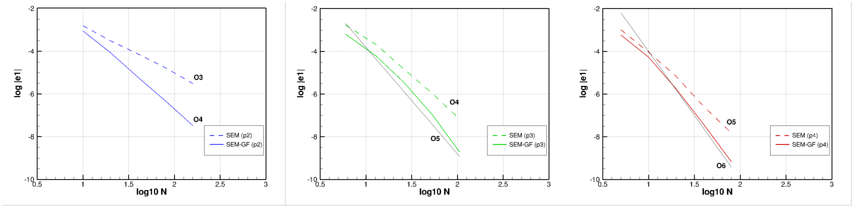
Numerical examples in 2d

Vortex solutions perturbations : super-convergence - theory GF = $\mathcal{O}(h^{p+2})$

div-free exponential + Coriolis



div-free exponential + mass source



Numerical examples in 2d

The Stommel Gyre

H. Stommel, The westwards intensification of wind-driven ocean currents, Trans.Amer.Geophys.Union 29(2), 1948

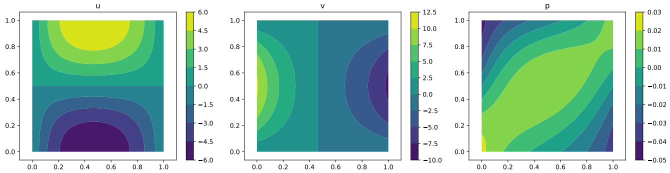
$$\partial_t \begin{pmatrix} p \\ u \\ v \end{pmatrix} + \partial_x \begin{pmatrix} u \\ p \\ 0 \end{pmatrix} + \partial_y \begin{pmatrix} v \\ 0 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ \phi v - c_f u + \tau_x \\ -\phi u - c_f v + \tau_y \end{pmatrix}$$

Notation.

Friction: c_f constant

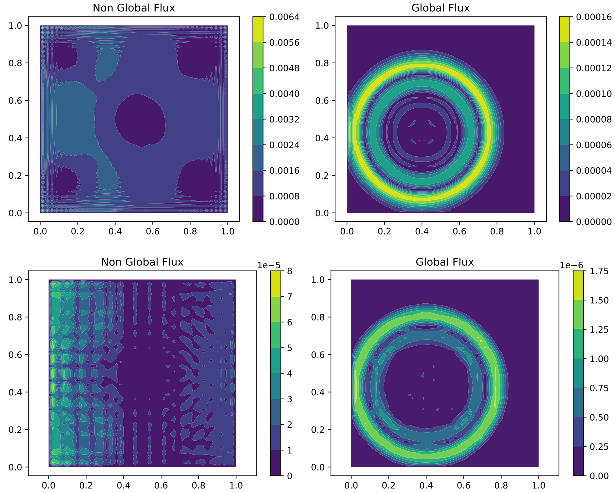
Coriolis: $\phi = \phi_0 + f_0 y$

Forcing: $\tau = \tau_0(0, \cos(\beta y))$



Numerical examples in 2d

The Stommel Gyre

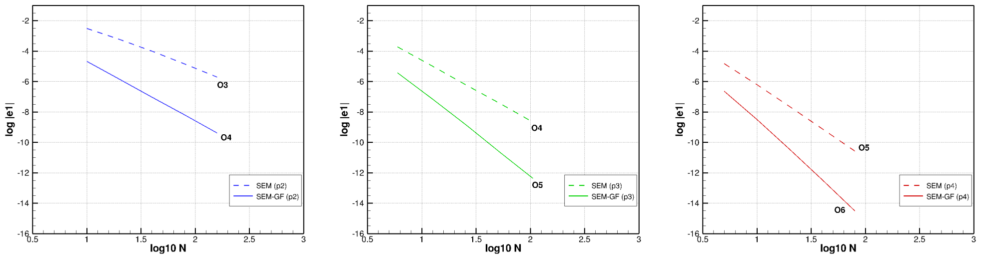


Pressure perturbation $10^{-3}\delta_p$: $p1$ on 80×80mesh

Pressure perturbation $10^{-5}\delta_p$: $p3$ on 13×13mesh

Numerical examples in 2d

The Stommel Gyre



Theory for GF: internal points = $\mathcal{O}(\mathrm{h}^{p+2})$

This is the last slide

Summary

- Global flux quadrature approach: considerable accuracy enhancements at steady state
- In one dimension discrete equilibria can be generated a-priori if necessary
- In two dimensions fully div preserving schemes can be designed (projection to the exact div free space available)

Outlook

- FD and FV variant using ODE operators independent of the underlying method
- Curl preserving version
- Explicit expressions for associated involutions (stabilized case)
- DG-SEM with GFq for multiD scalar conservation laws and nonlinear variants

спасибо