Well Balanced Discontinuous Galerkin scheme for the shallow water equations in spherical geometry for flooding applications

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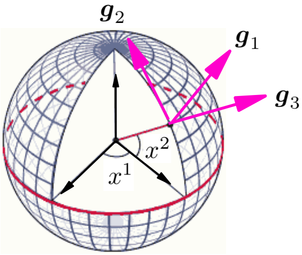
Shallow Water Equations in spherical geometry

Shallow Water equations:

$$\begin{aligned} \frac{\partial h}{\partial t} + \nabla \cdot h\boldsymbol{u} &= 0 \\ \frac{\partial h\boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{T} &= \boldsymbol{S} \end{aligned}$$

Momentum and flux can be expressed in 3D Cartesian basis or 2D covariant basis:

$$\begin{aligned} h\boldsymbol{u} &= hu^x \boldsymbol{e}_x + hu^y \boldsymbol{e}_y + hu^z \boldsymbol{e}_z \\ &= hu^1 \boldsymbol{g}_1^* + hu^2 \boldsymbol{g}_2^* \\ \boldsymbol{T} &= T^{ij} \boldsymbol{e}_i \boldsymbol{e}_j \\ &= T^{\alpha\beta} \boldsymbol{g}_\alpha^* \boldsymbol{g}_\beta^* \end{aligned}$$



Cartesian and covariant tangent reference systems

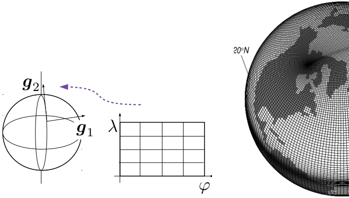
2D approach in ocean models

Divergence operator:

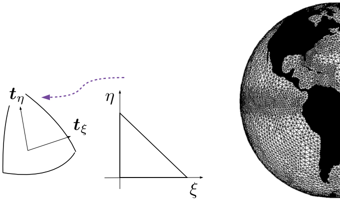
$$\nabla \cdot h\boldsymbol{u} = \frac{1}{J}\left(\frac{\partial}{\partial x^1}\left(Jhu^1\right) + \frac{\partial}{\partial x^2}\left(Jhu^2\right)\right)$$

with $J = \det \boldsymbol{J}$ and Jacobian $\boldsymbol{J} = \frac{\partial x^i}{\partial x^\alpha}$.

> latitude-longitude parametrization is typically used for structured grids and high order finite differences:
>
> $$J = \sqrt{G}, \quad x^1 = \varphi, \quad x^2 = \lambda.$$



> a local finite element map is typically used for unstructured grids and high order finite elements:
>
> $$J = J_A, \quad x^1 = \xi, \quad x^2 = \eta.$$



3D approach in ocean models

Divergence operator:

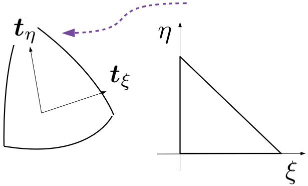
$$\nabla \cdot h\boldsymbol{u} = \frac{\partial h u^x}{\partial x} + \frac{\partial h u^y}{\partial y} + \frac{\partial h u^z}{\partial z}$$

$+$ Lagrange multiplier to constrain the currents to remain tangent to the sphere [Cote et al., 1988; Bonev et al. 2018]:

$$\frac{\partial h\boldsymbol{u}}{\partial t} + \nabla \cdot \boldsymbol{T} = \boldsymbol{S} + \mu \boldsymbol{x}$$

Mixed 2D/3D formulation (with finite element map)

[Bernard et al., 2008] local finite element map for tangent space : reference $\mapsto$ spherical triangle



$$hu = hu^\xi t_\xi^* + hu^\eta t_\eta^*$$

Differently from the 2D approach, momentum is treated in a hybrid manner:

1) Projection on the tangent plane ($t_\xi^* \cdot t_\xi^* = 1$, $t_\xi^* \cdot t_\eta^* \neq 0$):

$$\frac{\partial}{\partial t}(h\boldsymbol{u} \cdot \boldsymbol{t}_\alpha^*) + (\nabla \cdot \boldsymbol{T}) \cdot \boldsymbol{t}_\alpha^* \quad = \quad \boldsymbol{S} \cdot \boldsymbol{t}_\alpha^* \,, \qquad \alpha = \xi, \eta$$

Mixed 2D/3D formulation (with finite element map)

2) Time derivative is expressed in 2D while right-hand side is expressed in 3D.

e.g. for component $\alpha = \xi$:

$$
\begin{aligned}
\frac{\partial}{\partial t}(hu^\xi \underbrace{\boldsymbol{t}^*_\xi \cdot \boldsymbol{t}^*_\xi}_{=1} + hu^\eta \underbrace{\boldsymbol{t}^*_\eta \cdot \boldsymbol{t}^*_\xi}_{\neq 0}) \quad & + \quad \left(\frac{\partial T^{xx}}{\partial x} + \frac{\partial T^{xy}}{\partial y} + \frac{\partial T^{xz}}{\partial z}\right) t^{*x}_\xi \\
& + \quad \left(\frac{\partial T^{yx}}{\partial x} + \frac{\partial T^{yy}}{\partial y} + \frac{\partial T^{yz}}{\partial z}\right) t^{*y}_\xi \\
& + \quad \left(\frac{\partial T^{zx}}{\partial x} + \frac{\partial T^{zy}}{\partial y} + \frac{\partial T^{zz}}{\partial z}\right) t^{*z}_\xi = S^j t^{*j}_\xi
\end{aligned}
$$

Mixed 2D/3D formulation (with finite element map)

Combines some of the the advantages of 2D and 3D methods:

- number of unknowns kept at a minimum
- differential oerators are kept in Cartesian form:
  - no need of complex manifold differential transformations
  - 3D line integrals are invariant under coordinate transformation, thus independent on the parametrization
  - Riemann solvers are formulated easily in 3D Cartesian framework.

NEW: Mixed 2D/3D formulation with exact map

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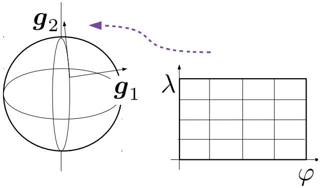
Covariant basis for tangent space:



$$hu = hu^1 g_1^* + hu^2 g_2^*$$

1) Projection on the tangent plane ($g_1^* \cdot g_1^* = 1$, $g_1^* \cdot g_2^* = 0$):

$$\frac{\partial}{\partial t}(h\boldsymbol{u} \cdot \boldsymbol{g}_\alpha^*) + (\nabla \cdot \boldsymbol{T}) \cdot \boldsymbol{g}_\alpha^* \quad = \quad \boldsymbol{S} \cdot \boldsymbol{g}_\alpha^* \,, \qquad \alpha = 1, 2$$

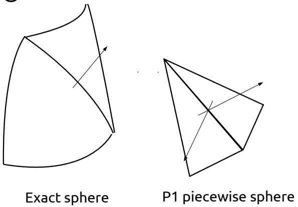
2) As before: time derivative in 2D, right-hand side in 3D

e.g. $\alpha = 1$, $$\frac{\partial}{\partial t}(hu^1 \underbrace{\boldsymbol{g}_1^* \cdot \boldsymbol{g}_1^*}_{=1} + hu^2 \underbrace{\boldsymbol{g}_2^* \cdot \boldsymbol{g}_1^*}_{=0}) + \ldots = \frac{\partial}{\partial t}(hu^1) + \ldots$$

NEW: Mixed 2D/3D formulation with exact map

Improvements:

- Exact geometrical representation of the sphere: no need for high order elements
- Local mass matrices are block-diagonal: no coupling between flux components
- Continuity of the normals at elements edges:
  - Uniquely reference system for Riemann Problem
  - Trivial extension to continuous finite elements



Discontinuous Galerkin in weak form

Some notation

- momentum flux $\boldsymbol{T} = h\boldsymbol{u}\boldsymbol{u} + P\boldsymbol{I}$, with pressure $P = \frac{1}{2}gh^2$
- topography and Coriolis source terms: $\boldsymbol{S} = gh\nabla b + \Omega \boldsymbol{k} \times h\boldsymbol{u}$
- numerical flux $\boldsymbol{T}^{\Upsilon} \cdot \boldsymbol{n} = \boldsymbol{H}(\mathrm{U}_L, \mathrm{U}_R)$

Momentum equation for spherical element $\mathcal{K}$:

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathcal{K}} h\boldsymbol{u}_h \cdot \underbrace{\boldsymbol{g}^*_\alpha \varphi_i}_{\boldsymbol{v}_i}\, d\boldsymbol{x} + \int_{\partial\mathcal{K}} \boldsymbol{T}^{\Upsilon}_h \cdot \boldsymbol{n} \cdot \boldsymbol{g}^*_\alpha \varphi_i \, ds \quad &- \quad \int_{\mathcal{K}} \boldsymbol{T}_h : \nabla\left(\boldsymbol{g}^*_\alpha \varphi_i\right) d\boldsymbol{x} \\ &= \quad \int_{\mathcal{K}} \boldsymbol{S}_h \cdot \boldsymbol{g}^*_\alpha \varphi_i \, d\boldsymbol{x}\end{aligned}$$

On Well Balancing

For the lake at rest state $(h + b = const,\ \boldsymbol{u} = 0) \Rightarrow \boldsymbol{T}_h = P_h \boldsymbol{I}$.
The DG method reduces to:

$$\int_{\partial\mathcal{K}} P_h^{\Upsilon} \boldsymbol{I} \boldsymbol{g}_\alpha^* \varphi_i \cdot \boldsymbol{n}\, ds - \int_{\mathcal{K}} P_h \boldsymbol{I} : \nabla \left(\boldsymbol{g}_\alpha^* \varphi_i\right)\, d\boldsymbol{x} = -\int_{\mathcal{K}} g h_h \nabla b \cdot \boldsymbol{g}_\alpha^* \varphi_i\, d\boldsymbol{x}$$

Previous work on well balanced DG:

- Cartesian well-walanced DG with Lax-Friedrich's Flux [Xing and Shu, 2005]
- 3D well-balanced DG with Lax-Friedrich's Flux + strong form [Bonev et al, 2018]

On Well Balancing

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Pb. with hybrid approach: projection on the tangent basis involves non-algebraic functions in all of the above integrals

- Well balanced fluxes of [Xing and Shu, 2005] necessary but not enough
- Exact integration of the metric terms impossible: lack of discrete analogs of differential relations allowing to prove well balanced (e.g. pass from weak to strong form)

Well balanced is lost

Well Balanced Discontinuous Galerkin

First remark: the strong form of the system is better suited for well-balancedness as discussed in [Bonev et al, 2018]:

$$\int_{\partial\mathcal{K}} \left(P_h^{\curlyvee} - P_h\right) \boldsymbol{I}\boldsymbol{g}_\alpha^* \varphi_i \cdot \boldsymbol{n}\, ds \;+\; \int_{\mathcal{K}} (\nabla \cdot P_h \boldsymbol{I}) \cdot \boldsymbol{g}_\alpha^* \varphi_i \, d\boldsymbol{x} = -\int_{\mathcal{K}} g h_h \nabla b_h \cdot \boldsymbol{g}_\alpha^* \varphi_i \, d\boldsymbol{x}$$

This is equivalent to add to the weak form the integration by parts error (high order well-balanced correction)

$$\epsilon^K = \int_{\mathcal{K}} P_h \boldsymbol{I} : \nabla \left(\boldsymbol{g}_\alpha^* \varphi_i\right) \, d\boldsymbol{x} - \int_{\partial\mathcal{K}} P_h \boldsymbol{I} \boldsymbol{g}_\alpha^* \varphi_i \cdot \boldsymbol{n}\, ds + \int_{\mathcal{K}} (\nabla \cdot P_h \boldsymbol{I}) \cdot \boldsymbol{g}_\alpha^* \varphi_i \, d\boldsymbol{x}$$

Well Balanced Discontinuous Galerkin

Second remark: a well balanced numerical flux verifies $P_h^\Upsilon = P_h$ in the lake at rest case. This means that

- The surface term vanishes:

$$\int_{\partial\mathcal{K}} \left(P_h^\Upsilon - P_h\right) \boldsymbol{I} \boldsymbol{g}_\alpha^* \varphi_i \cdot \boldsymbol{n}\, ds = 0$$

- For well balanced to be verified, the volume terms must verify

$$\nabla \cdot P_h \boldsymbol{I} = -g h_h \nabla b_h$$

on the lake at rest state $h + b =$const.

Well Balanced Discontinuous Galerkin

We can achieve this condition in two ways

- Approach 1 : write $P_h = P(h_h)$ and compute divergence as

$$\nabla \cdot P_h \boldsymbol{I} = g h_h \nabla h_h$$

For the the lake at rest $h + b = \eta =$const. , hence

$$\nabla \cdot P_h \boldsymbol{I} = -g h_h \nabla b_h$$

Well balanced recovered as in [Bonev et al, 2018] for any quadrature strategy

Well Balanced Discontinuous Galerkin

We can achieve this condition in two ways

- Approach 2: Set $P_h = g\dfrac{(h^2)_h}{2}$, allowing direct intepolation of the fluxes, and compute

$$gh_h \nabla b_h = g\eta_h \nabla b_h - \nabla \cdot \left(g\frac{b^2}{2}\boldsymbol{I}\right)_h$$

On the lake at rest

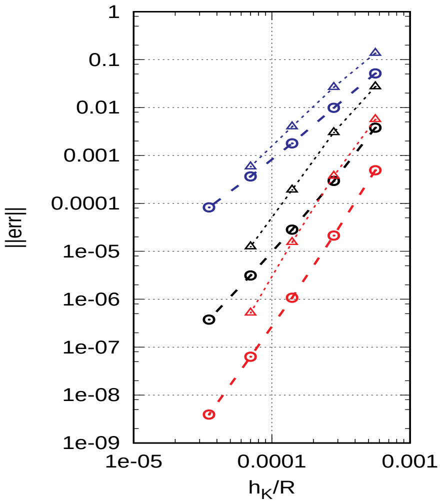
$$\begin{aligned}\nabla \cdot P_h \boldsymbol{I} = & \sum_\sigma \frac{h_\sigma^2}{2}\nabla\varphi_\sigma = \sum_\sigma \frac{\eta_0^2 - 2\eta_0 b_\sigma + b_\sigma^2}{2}\nabla\varphi_\sigma \\ & = -\eta_0 \nabla b_h + \nabla \cdot \left(g\frac{b^2}{2}\boldsymbol{I}\right)_h = gh_h \nabla b_h\end{aligned}$$

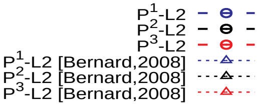
Well balanced recovered for any quadrature strategy

Implementation used here

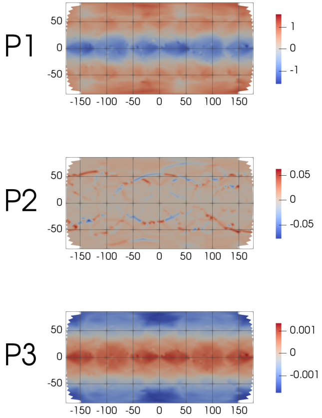
- Riemann Problem solved in 3D with velocity rotation/change of basis to pass from 2D to 3D
- Strong form for pressure, approximation $P_h := P(h_h)$ for WB
- Rotated lat-lon for elements in polar regions with element based flagging exploiting DG setting
- Entropy viscosity for shock capturing [Guermond, 2009]

W92 tests: zonal flow





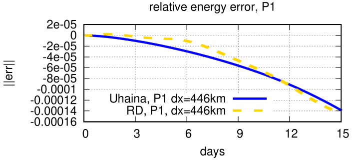
Error for $h_K = 446\ km$

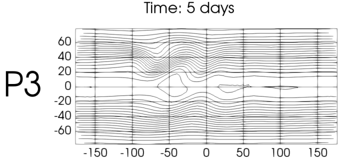


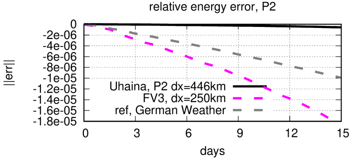
W92 tests: zonal flow over mountain

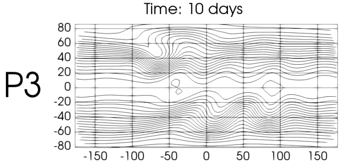
Mass is conserved to machine accuracy.

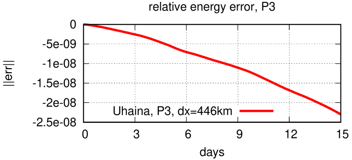
Energy conservation: relative error in left column

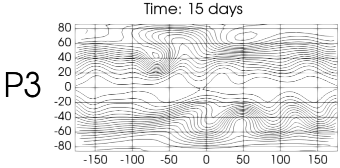








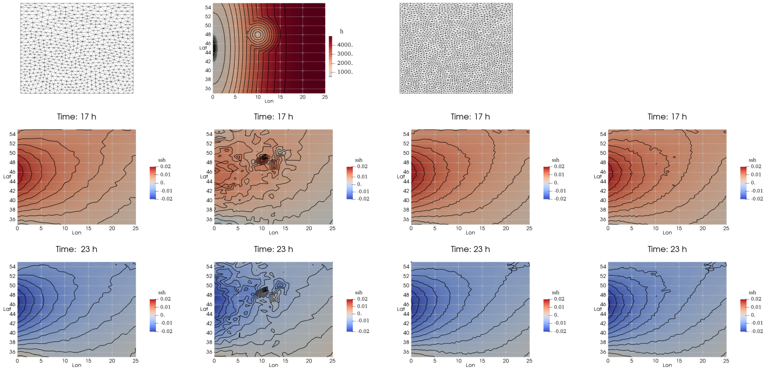




Well Balancing test

North boundary: inlet $\rightarrow$ small amplitude ($A = 0.01\,m$) M2 tide.
West boundary: wall. East/South boundary: outflow.

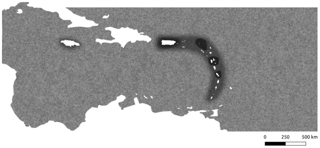
$h_K = 100\,km$		$h_K = 50\,km$	

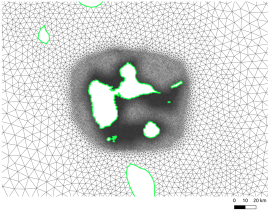


WB DG	weak DG	WB DG	weak DG

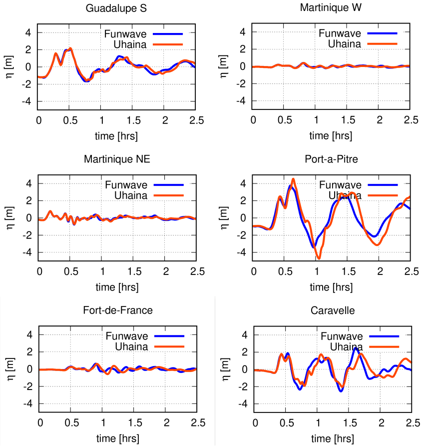
Realistic tsunami example

- The initial tsunami waveform is obtained from a random rupture of the Lesser Antilles subduction zone. Randomness is associated with the heterogeneity of the fault slip which is computed by a Karhunen-Loeve expansion.
- The mesh has variable mesh size $h_K = 10\,km - 300\,m$.
- $CFL = 0.5$, only P1 tested, entropy viscosity is active
- Validation against `FUNWAVE` on a structured grid with $\Delta x = 800m$.





Realistic tsunami example



Conclusions and Questions

These developments have been done within the free-surface code `Uhaina` (INRIA, UPPA, IMB, IMAG, BRGM) [Filippini et al., 2018] which relies on the `Aerosol` HPC finite element library (INRIA, UPPA).

- Improvement of the mixed 2D/3D form of [Bernard et al.,2008] in terms of accuracy and implementation simplicity.
- Well-balancing.
- Validation against a realistic tsunami.

Perspectives:

- Effect of inexact quadrature on: accuracy, well balanced
- Extension to CG